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# Calculation of Settlement in Sands Using a Non-Linear Constitutive Equation

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Abstract. In this article a procedure is presented in order to calculate the settlement of a foundation in sand. It uses a nonlinear differential constitutive equation that is integrated in the domain of the confinement pressure over the soil; we obtain then a constitutive equation that takes into account the variation of confinement pressure due to the presence of an engineering construction. This procedure is in advance of other methods which not consider the increase of compression over the soil, especially when the increments of stresses are high. Moreover, in the constitutive equation we use very few mechanical properties which do no change with the variation of confinement pressure.

Keywords. Deformation of sand, constitutive equation, footing settlement.

# 1. Introduction

Nowadays there have been great advances for the computing of deformations in granular soils. It has been possible to calibrate theories with field observations [6]. Another advance consists in the publication of constitutive equations that use mechanisms at macroscopic and microscopic levels [2]. Numerical solutions, employing the finite element method ([5], [4], [9]) and other procedures that are located in the frame of the critical state theory [8], are also used.

Nevertheless, there is an aspect that is not contemplated in the foregoing studies; it consists in the fact that the soil deformation modulus  $E_s$ , that is used in some of these studies, is a function of the confinement pressure over the soil. At the same time, this compression is affected by the presence of the engineering construction, because the stresses originated by this construction increase the confinement pressure. An additional problem consists in that sometimes the difference between the initial stress due to the soil weight is too much low respect to the stress due to the engineering construction, so it is senseless to take an average of  $E_s$  between these two values of stresses.

In order to solve the problem, in this article we present a constitutive differential equation, that is further integrated in the confinement pressure domain, which at the same time gives an expression that makes it possible to compute the deformation of a stratum of cohesionless soil, taking into account the aforementioned factors.

In the next paragraphs we present an example to illustrate this situation.

In order to estimate deformation of a stratum of soil, we can use Hooke's law

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$$\varepsilon_z = \frac{1}{E_s} \left[ \sigma_z - \nu \left( \sigma_x + \sigma_y \right) \right] \tag{1}$$

$$\Delta \delta_z = \frac{\Delta z_o}{E_s} \left[ \sigma_z - \nu \left( \sigma_x + \sigma_y \right) \right] \tag{2}$$

where  $\Delta \delta_z$  is the deformation of the stratum,  $E_s$  is the deformation modulus of soil,  $\Delta z_o$  is the thickness of the stratum and  $\sigma_z$ ,  $\sigma_x$ ,  $\sigma_y$  are the stress increments due to the presence of an engineering structure.

We can estimate the soil modulus  $E_s$  with the formula of Janbu [3]

$$E_s = E_{so} + K p_a \left(\frac{\sigma_{3m}}{p_a}\right) \tag{3}$$

In which  $E_{so}$  is the initial deformation modulus when the soil has a certain cementation,  $p_a$  is the atmospheric pressure (101.3 kPa), K and n are soil parameters that are a function of material density, and  $\sigma_{3m}$  is the average confinement stress over a soil element.

 $\sigma_{3m}$  is the sum of the pressure due to the soil weight plus the increment of stress due to the loads of the structure.

Due to the weight of the soil

$$\sigma_{3o} = \frac{1 + 2K_o}{3} p_{vo} \tag{4}$$

where  $K_o$  is the pressure coefficient at rest and  $p_{vo}$  is the vertical stress due to the weight of soil.

Due to the loads of the structure, the increment of stress confinement is

$$\Delta\sigma_3 = \frac{\sigma_z + \sigma_x + \sigma_y}{3} \tag{5}$$

where  $\sigma_z$ ,  $\sigma_x$  y  $\sigma_y$  are the stress increments originated by the loads of a structure.

$$\sigma_{3f} = \sigma_{3o} + \Delta \sigma_3$$
  
$$\sigma_{3m} = \frac{\sigma_{3o} + \sigma_{3f}}{2} = \sigma_{3o} + \frac{\Delta \sigma_3}{2}$$

In engineering practice sometimes the final confinement pressure  $\sigma_{3f}$  is very much higher than initial pressure  $\sigma_{3o}$ , so it becomes senseless to make an average between these values.

As an example, let us take the reinforced concrete footing of Figure 1. The footing has a width of 1.7 m and a length of 2 m. *N* is the number of blows corresponds to the standard penetration test (SPT).

The calculations are shown in Table 1; we see that in stratum 1, the confinement stress varies from 7.185 to 151.57 kPa. Does it make sense to average  $E_s$  between these two magnitudes? Something alike happens in strata 2 and 3.

So it is necessary to consider the variation of the confinement stress, as it is presented in the next section.



Rock

Figure 1. Footing overlying sandy strata.

Stra-	Κ	$\sigma_{3o}$	$\sigma_z$	$\sigma_x$	$\sigma_y$	$\sigma_{3f}$	$\sigma_{3m}$	п	$\Delta \delta_z$
tum									
		kPa	kPa	kPa	kPa	kPa	kPa		mm
1'	517.5	7.185	196.71	117.05	119.40	151.57	79.38	0.590	0.85
2	653.5	10.448	180.11	53.09	47.82	104.12	57.28	0.577	1.28
3	576	15.240	134.84	15.81	11.33	69.23	42.24	0.584	1.82
								$\delta_z =$	3.95

Table 1. Settlement calculation using Janbu's formula.

#### 2. Constitutive equation to calculate the deformation of a cohesionless soil

Let us consider a soil element at a depth z, subjected to stresses due to weight of the soil and to the loads of a structure. Assuming that the thickness  $\Delta z_o$  of the element is small enough, so that the relation between the horizontal stress increment and the vertical stress increment is a constant, we can write

$$a_{1} = \frac{\sigma_{x}}{\sigma_{z}} = constant \quad ; \quad a_{2} = \frac{\sigma_{y}}{\sigma_{z}} = constant$$

$$\sigma_{x} = a_{1}\sigma_{z} \tag{6}$$

$$\sigma_{y} = a_{2}\sigma_{z} \tag{7}$$

Substituting Eqs. (6) and (7) in Hooke's law (Eq. (1)):

$$\varepsilon_{z} \approx \frac{1}{E_{s}} \sigma_{z} \left[ 1 - \nu \left( a_{1} + a_{2} \right) \right]$$

$$\varepsilon_{z} \approx \frac{1}{E_{s}} \left( f \sigma_{z} \right)$$
(8)

$$f = 1 - \nu \left(a_1 + a_2\right) = 1 - \upsilon \left(\frac{\sigma_x + \sigma_y}{\sigma_z}\right) \tag{9}$$

Confinement stress over the element is:

$$\sigma_3 = \sigma_{30} + \Delta \sigma_3 \tag{10}$$

$$\Delta\sigma_3 = \frac{\sigma_z + \sigma_x + \sigma_y}{3} \tag{11}$$

Replacing Eqs. (6) and (7) in Eq. (11):

$$\Delta \sigma_3 = \frac{\sigma_z + a_1 \sigma_z + a_2 \sigma_z}{3} = \frac{1 + a_1 + a_2}{3} \sigma_z$$

$$\Delta \sigma_3 = c \sigma_z \tag{12}$$

$$c = \frac{1}{3} + \frac{1}{3}(a_1 + a_2) = \frac{1}{3} + \frac{1}{3}\left(\frac{\sigma_x + \sigma_y}{\sigma_z}\right)$$
(13)

Substituting in Eq. (10):

$$\sigma_3 = \sigma_{3o} + c \sigma_z \tag{14}$$

Let us apply differential stress increments to the element (Figure 2). With the previous expressions, we can write a general constitutive equation, in which the vertical strain of the element is directly proportional to the stress increments in Hooke's law (Eq. (1)) and inversely proportional to the confinement pressure (Eqs. (10) and (11)), so [1]:

$$d\varepsilon_{z} = \frac{\frac{1}{A} \frac{\left[d\sigma_{z} - \upsilon \left(d\sigma_{x} + d\sigma_{y}\right)\right]}{p_{a}}}{\left[\frac{\sigma_{3o} + \frac{\sigma_{z} + \sigma_{x} + \sigma_{y}}{3}}{p_{a}}\right]^{s}}$$
(15)



Figure 2. Stresses over an element of soil.

We see that, as it happens in nature, the vertical strain of the soil is a direct function of the stress increments and an inverse function of the confinement pressure over the soil element (Figure 2).

In Eq. (15), A is the rigidity modulus of soil and s is an exponent; both of them are functions of the density of the material.  $p_a$  = atmospheric pressure = 101.3 kPa.

Furthermore

$$\begin{bmatrix} d\sigma_z - \upsilon \left( d\sigma_x + d\sigma_y \right) \end{bmatrix} = d\sigma_z \left[ 1 - \upsilon \left( \frac{d\sigma_x}{d\sigma_z} + \frac{d\sigma_y}{d\sigma_z} \right) \right] = f \, d\sigma_z$$
$$\sigma_{3o} + \frac{\sigma_z + \sigma_{x+} \sigma_y}{3} = \sigma_{3o} + \Delta \sigma_3 = \sigma_{3o} + c \, \sigma_z$$

Eq. (15) is then

$$d\varepsilon_z = \frac{1}{A p_a^{1-s}} \frac{f \, d\sigma_z}{(\sigma_{3o} + c \, \sigma_z)^s} \tag{16}$$





Figure 3. Deformations of an element.

Let us integrate this expression. As we increase the vertical stress from 0 to  $\sigma_z$ , the height of the element decreases from initial height  $\Delta z_o$  to final height  $\Delta z_f$  (Figure 3), so

$$\int_{\Delta z_{0}}^{\Delta z_{f}} \frac{d(\Delta z)}{\Delta z} = -\frac{1}{Ap_{a}^{1-s}} \int_{0}^{\sigma_{z}} \frac{f d\sigma_{z}}{(p_{c0} + c\sigma_{z})^{s}}$$
$$\ln \Delta z]_{\Delta z_{0}}^{\Delta z_{f}} = -\frac{f}{cAp_{a}^{1-s}} \frac{(p_{co} + c\sigma_{z})^{1-s}}{1-s} \Big]_{0}^{\sigma_{z}}$$
$$\frac{\Delta z_{f}}{\Delta z_{o}} = \exp\left\{-\frac{f\left[(p_{co} + c\sigma_{z})^{1-s} - (p_{co})^{1-s}\right]\right]_{0}^{\sigma_{z}}}{cAp_{a}^{1-s}}\right\}_{0}^{\sigma_{z}}$$
(17)

In Figure 3

$$\Delta z_f = \Delta z_o + \Delta w_f$$

$$\frac{\Delta z_f}{\Delta z_o} = 1 + \frac{\Delta w_f}{\Delta z_o}$$

$$\Delta w_f = \left(\frac{\Delta z_f}{\Delta z_o} - 1\right) \Delta z_o \tag{18}$$

Replacing Eq. (18) in Eq. (17)

$$\Delta w_f = \left\{ \exp\left\{ -\frac{f\left[ (\sigma_{3o} + c \sigma_z)^{1-s} - (\sigma_{3o})^{1-s} \right]}{(1-s) c A p_a^{1-s}} \right\} - 1 \right\} (\Delta z_o)$$
(19)

The value of  $\Delta w_f$  in Eq. (19) is always negative, so, in order to have a positive magnitude of the deformation of the element, we write:

$$\Delta \delta z = -\Delta w_f$$

Expression (19) is then [1]

$$\Delta \delta_z = \left\{ 1 - \exp\left\{ -\frac{f\left[ (\sigma_{3o} + c \sigma_z)^{1-s} - (\sigma_{3o})^{1-s} \right]}{(1-s) c A p_a^{1-s}} \right\} \right\} (\Delta z_o)$$
(20)

With Eq. (20) we calculate vertical deformation of a soil element with initial thickness  $\Delta z_o$ , subjected to stress increments  $\sigma_z$ ,  $\sigma_x$  y  $\sigma_y$ , originated by the loads of a building. In cohesionless soils we find experimentally that exponent  $s\approx 0.5$ .

Another advantage of using a constitutive equation, is that it contains very few mechanical properties, which allows us to make a statistical analysis of these properties.

Considering data of settlements of engineering constructions, in which the geometry of the foundation, the subsoil stratigraphy and the number of blows of standard penetration tests (SPT) are known, a statistical analysis of the mechanical property A was made. The results of this analysis are presented in the following paragraphs.

Mean rigidity modulus  $A_m$  is obtained from the number of blows (SPT), with:

$$A_m = 26.25 N^{1.125} \tag{21}$$

The estimation of an unfavorable magnitude of modulus A is accomplished using the concept of statistical prediction, which infers such unfavorable value employing the random variable t of Student. So,

An unfavorable magnitude of modulus A is computed as

$$A = A_m C \tag{22}$$

In which

$$C = \exp\left[-0.784 t_{\alpha} \sqrt{1.00758 + 0.0152 (\ln N - 2.976)^2}\right]$$
(23)

 $t_{\alpha}$  is a *t* of Student, whose values as a function of the level of confidence  $\alpha$ , are indicated in Table 2.

Confidence	$t_{\alpha}$
level a	
%	
2.5	1.978
5	1.657
10	1.288
15	1.041
20	0.844
25	0.676
30	0.526
40	0.254
50	0

**Table 2.** Values of Student's variable  $t_a$ .

#### 3. Example

Calculate the settlement of the reinforced concrete footing of Figure 1, using the no linear procedure of the foregoing section, taking a confidence level  $\alpha = 20\%$ . The footing has a width of 1.7 m and a length of 2 m.

Table 3 shows the computation of deformations of the three strata of soil. Stresses due to the weight of earth and due to the vertical loads of the structure are calculated at the middle of the height of each stratum, and are the same of Table 1.

					··· 1	,		
Stra- tum	A	p <sub>vo</sub>	Ko	v	σ30	С	f	$\Delta \delta_z$
		kPa			kPa			mm
1'	504.89	12	0.398	0.285	7.185	0.734	0.658	1.02
2	665.95	18	0.371	0.270	10.448	0.520	0.848	1.36
3	573.39	25.85	0.384	0.278	15.240	0.400	0.944	1.80
							$\delta_z =$	4.18

**Table 3.** Settlement calculation with non linear procedure;  $\alpha = 20\%$ .

We obtained a settlement of the footing of  $\delta_z = 4.18$  mm, with  $\alpha = 20\%$ . Appling a similar procedure, the mean settlement (with  $\alpha = 50\%$ ) results 2.15 mm.

For illustration purposes, we present the evaluation of deformation of stratum 1: Coefficient  $K_o$  is [7]

$$K_o = (1 - sen \phi)(OCR)^{sen \phi} = (1 - sen 37^{\circ})(OCR)^{sen 37^{\circ}} = 0.398$$

Poisson ratio v is computed by

$$\nu = \frac{K_o}{1 + K_o} = \frac{0.398}{1 + 0.398} = 0.285$$

$$\begin{split} &A_m = 26.25(25)^{1.125} = 981.32\\ &\text{With } \alpha = 20\%, \, t_\alpha = 0.844\\ &C = \exp\left[-0.784(0.844)\,\sqrt{1.00758 + 0.0152\,(\ln 25 - 2.976)^2}\right] = 0.5145\\ &A = A_m C = 981.32(0.5145) = 504.89\\ &f = 1 - 0.285 \bigg(\frac{117.05 + 119.40}{196.71}\bigg) = 0.658\\ &c = \frac{1}{3} + \frac{1}{3}\frac{117.05 + 119.40}{196.71} = 0.734\\ &\Delta \delta_z = \left\{1 - \exp\left\{-\frac{0.658\left[(7.185 + 0.734(196.71))^{1-0.5} - (7.185)^{1-0.5}\right]\right\}\right\}(0.3) = 0.001018\,m \end{split}$$

### 4. Conclusions

- a) In view that in a soil element the confinement pressure varies during the loads applied by an engineering construction, it is presented a differential constitutive equation for the calculation of the deformation of the element, in which strain is directly proportional to the stress increments and inversely proportional to confinement pressure (Eq. (15)).
- b) The differential constitutive equation is integrated in the interval of variation of the stresses originated by the loads of the structure, which allows us to find an expression for the computation of the deformation of a sand stratum. In this expression (Eq. (20)) the mechanical properties A and s of the soil are independent of the applied loads and of the confining pressure over the element.
- c) An example for the calculation of settlement of an isolated footing overlying three strata of sandy soils is included.

## References

- A. Demeneghi, Cálculo del asentamiento de un cimiento en arena, XXIV Reunión Nal Mec Suelos, Vol. 2, Aguascalientes, Ags, Soc Mex Mec Suelos (2008), 301-305.
- [2] H.L. Fang, A state-dependent multi-mechanism model for sands, Géotechnique 53, No. 4 (2003), 407-420.
- [3] N. Janbu, Soil compressibility as determined by oedometer and triaxial tests, *European Conf Soil Mech Found Eng*, Wiesbaden, Germany, Vol. 1 (1963), 19-25.
- [4] J. Lee and R. Salgado, Estimation of footing settlement in sand, *International Jour Geomechanics*, Vol. 2, No. 1 (2002), 1-28.
- [5] J. Lee, J. Eun, M. Prezzi and R. Salgado, Strain influence diagrams for settlement estimation of isolated and multiple footings in sand, *Jour Geotech Geoenviron Eng*, Vol. 134, No. 4 (2008), 417-427.

- [6] M. Maugeri, F. Castelli, M.R. Massimino and G. Verona, Observed and computed settlements of two shallow foundations on sand. *Jour Geotech and Geoenviron Eng*, Vol. 124, No. 7 (1998), 595-605.
- [7] P.W. Mayne and F.H. Kulhawy, K<sub>0</sub>-OCR relationships in soil, *Jour Geot Eng Div*, ASCE, Vol. 108, N° GT6 (1982), 851-872.
- [8] R. Salgado, P. Bandini and A. Karim, Shear strength and stiffness of silty sand, *Jour Geotech Geoenviron Eng*, Vol. 126, No. 5 (2000), 451-462.
- [9] N. Yamamoto, M.F. Randolph and I. Einav, Simple formulas for the response of shallow foundations on compressible sands, *International Jour Geomechanics*, Vol. 8, No. 4 (2008), 230-239.