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# Conformal Circular Mapping for Unbounded Multiple Connected Regions

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**Abstract.** Multiple connected regions bounded by circles are crucial from the point of view of analyzing physical problems and reducing the amount of computation. However, finding a conformal mapping function that maps a multiple connected region to circular domain is challenging. Koebe's iterative method provides a theoretically feasible path for the conformal mapping of a multiple connected region to a circular region. In this study, a numerical implementation of Koebe's iterative method is accomplished using the charge simulation method, and an algorithm for conformal circular mapping of unbounded multiple connected regions is proposed. Through numerical experiments, this paper successfully verifies the effectiveness of the proposed algorithm.

**Keywords.** Circular mapping, Numerical conformal mapping, Koebe's iterative method, Charge simulation method

## 1. Introduction

The circular mapping allows the original problem to be solved in a simpler geometrical environment by transforming the original region into a circular domain using conformal mapping [1-4]. For example, the Kirchhoff-Routh path function [1] can be solved using circular mapping to make the original flow problem easier to solve. The Kirchhoff-Routh path function is an important concept in fluid dynamics, which provides a method for analyzing the fluid flow characteristics in a flow field containing multiple obstacles. Therefore, implementing an algorithm for mapping the conformal mapping of a multi-connected region into a circular domain is crucial.

The theory of numerical methods for computing mapping functions has been continuously improved [5-7]. Among them is the numerical conformal mapping computational method based on the charge simulation method proposed by Amano [7]. The method does not require numerical integration in the solution process and uses the principle of maximum mode to evaluate the error. It is fast and easy to evaluate the computational accuracy and has become a mainstream method for solving conformal mapping functions.

However, conformal mapping is a strong constraint requiring the mapping function to satisfy local and global conformal conditions throughout the domain, but in the case of multiple connected regions, each boundary carries its boundary conditions ,which must be satisfied during the mapping process. Therefore, we consider introducing

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iterative methods in the conformal mapping process. Koebe's iterative method [8,9] is used in Complex Analysis and Geometric Function Theory to find the optimal solution through a stepwise approximation, which allows each step to be adjusted according to the previous step's results. It is favorable to improve the mapping function step by step to achieve higher accuracy, and the method shows good stability, which means that the iterative process can converge to the correct solution, which is essential to ensure the reliability of the algorithm [10].

Therefore, in this study, Koebe's iterative method based on the charge simulation method for the computation of conformal mapping is introduced to provide an efficient solution for the conformal mapping from an unbounded multiple connected region to a circular domain, and a theoretical refinement of the numerical conformal mapping algorithm based on the charge simulation method is carried out. The effectiveness of the method is finally demonstrated by numerical experiments.

## 2. Conformal Mapping Based on the Charge Simulation Method

### 2.1. The Circular Mapping Function

**Theorem 1.** Let $D$ be a region of connectivity $n>1$ in the extended complex plane such that $\infty \in D$. Then there exist a unique circular domain $\Omega$ of connectivity $n$ and a unique one-to-one analytic function $f(z)$ such that $f(D)=\Omega$ ,see Figure 1, and $f$ in $D$ satisfying

$$f(z)=z+O(\frac{1}{z}) \tag{1}$$

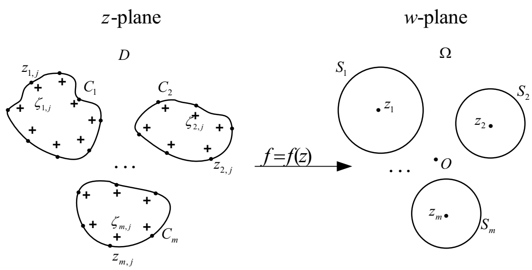


**Figure 1.** Conformal mapping of unbounded multiple connected region to circular domain

For unbounded $D$ , each iteration of Koebe's method requires computing the conformal mapping from an unbounded simply connected region onto the exterior unit disk $n$ times.

Consider a conformal mapping $f=f(z)$ from the exterior of region $D$ consisting of a closed smooth Jordan curve $C$ in the plane to the exterior of the unit circle $S$ , where $z$ is a point on region $D$ and its boundary $C$ . Then, one can uniquely determine the function of such a mapping by a normalization condition

$$f(\infty)=\infty, f^{'}(\infty)>0 \tag{2}$$

which expands the function Loran at infinity as

$$f(z)=\gamma^{-1}z+a_0+a_1z^{-1}+a_2z^{-2}+\cdots \tag{3}$$

taking any point $z_0$ in the region enclosed by $C$, then at $z \to \infty$ there is

$$\log \frac{f(z)}{z - z_0} = -\log \gamma + g(z) + \mathrm{i}h(z) \tag{4}$$

where $g(z)$ and $h(z)$ are a pair of conjugate harmonic functions within the region $D$ and $g(\infty) = 0, h(\infty) = 0$, hence the conformal mapping function can be expressed as

$$f(z) = (z - z_0)e^{-\log\gamma + \mathrm{g}(z) + \mathrm{ih}(z)} \tag{5}$$

the $g(z)$ satisfying the Laplace equation. Therefore, under normalization conditions, $g(z)$ is a solution to the Dirichlet potential field problem that satisfies

$$\begin{cases} \nabla^2 g(z) = 0, z \in D \\ g(z) = \log \gamma - \log | z - z_0 |, z \in C \\ g(\infty) = 0 \end{cases} \tag{6}$$

reduced to the problem of finding the solution of the Dirichlet potential field problem, $g(z)$ and the conjugate summation function $h(z)$ and the radius $\gamma$.

### 2.2. Charge Simulation Method

With $z_0$ taken as the origin, approximate $f, g, h, \gamma$ by $F, G, H, \Gamma$, respectively, while approximating the function $g(z) + \mathrm{i}h(z)$ by a linear combination of the complex logarithmic functions, expressed as follows

$$G(z) + \mathrm{i}H(z) = -\sum_{j=1}^{n} Q_j \left( \log | z - \zeta_j | + \mathrm{i} \arg(z - \zeta_j) \right), z \in \bar{D} \tag{7}$$

$G(z)$ and $H(z)$ are the primary combination of the real and imaginary parts of the complex logarithmic potential $\log(z - \zeta_j)$, respectively, with the charge $Q_j$ as a coefficient to be determined, discretizing the boundary into $n$ constraints, $z_i, i = 1, ..., n$, $\zeta_j, j = 1, ..., n$ which are simulation charge points configured inside the curve $C$. From Eq.(6), the approximation function $G(z)$ should satisfy the boundary conditions

$$\sum_{j=1}^{n} Q_j \log | z - \zeta_j | = \log | z - z_0 | - \log \Gamma \tag{8}$$

Secondly, according to the infinity point, it satisfies condition $g(\infty) = 0, h(\infty) = 0$, when $z \to \infty$, there is

$$\lim_{z \to \infty} \sum_{j=1}^{n} Q_j \log(z - \zeta_j) = 0 \tag{9}$$

Moreover, since the imaginary part of Eq.(7) is single-valued in the region $\bar{D}$, equivalently

$$\int_c \mathrm{d}H(z) = \int_c \mathrm{d}\sum_{j=1}^{n} Q_j \arg(z-\zeta_j) = 2\pi\sum_{j=1}^{n} Q_j = 0 \tag{10}$$

Associate the Eq.(9) and Eq.(10) to get the constraint equation

$$\sum_{j=1}^{n} Q_j \log|z_i-\zeta_j| + \log\Gamma = \log|z_i - z_0|, i=1,...,n \tag{11}$$

Solve to obtain the values of the charge $Q_j, j=1,...,n$ and the approximate mapping radius $\Gamma$ and finally construct the approximate mapping function

$$F(z) = (z-z_0)e^{-\log\Gamma+G(z)+iH(z)} \tag{12}$$

$H(z)$ is not only single-valued on $\bar{D}$ but also required to be continuous. Since the principal values of the logarithmic function $\log(z-\zeta_j)$ are used in the calculations, and $\operatorname{Arg} z$ has a $2\pi$ discontinuity on the negative semiaxis, a simple substitution of $\arg$ for $\operatorname{Arg}$ would cause a discontinuity, so a point $\zeta_0$ inside the boundary $C$ is taken, and Eq.(10) is united to convert Eq.(7) into the mathematically equivalent continuous form

$$G(z)+iH(z) = \sum_{j=1}^{n} Q_j \log(z-\zeta_j) - \sum_{j=1}^{n} Q_j \log(z-\zeta_0) = \sum_{j=1}^{n} Q_j \log\frac{(z-\zeta_j)}{(z-\zeta_0)} \tag{13}$$

## 3. Koebe's Iterative Method Based on the Charge Simulation Method

This paper presents an efficient method for constructing a circular domain $\Omega$, define the initial region $D^{0,0}=D$, the initial boundary $C_i^{0,0}, i=1,2,...,m$, and $k=0,1,2,...$ as the number of external iterations. The conformal mapping function $\omega: D^{0,0}\to D^{m,k}$ maps the multi-connected region $D^{0,0}$ to the circular domain $D^{m,k}=\Omega$, whose complement in the circular domain $D^{m,k}$ is the circle $\{D_1^{m,k}, D_2^{m,k},..., D_3^{m,k}\}$.

Region $D_i^{i-1,k}$, enclosed by curve $C_i^{i-1,k}$, is mapped to the unit circle $D_i^{i,k}$, and $C_i^{i-1,k}$ denotes the $i$ boundary curve after the $i-1$ loop in the $k$ iteration. The region $D_j^{i-1,k}$ enclosed by the remaining $m-1$ curves $C_j^{i-1,k}$ is mapped accordingly to $D_j^{i,k}, j\neq i$, i.e., $D_j^{i,k}=\omega^{i,k}(D_j^{i-1,k}), j=1,..,i,..,m, i=1,....,m$, and the number of iterations $k$ is increased when $i=m$.

Loop each boundary in turn, and all boundaries looped once are called an external iteration. When one external iteration $k$ is finished, the mapping function is written as

$$\hat{\omega}^k = \omega^{1,k}\circ\omega^{2,k}\circ\cdots\circ\omega^{m,k} \tag{14}$$

at this point, function $\hat{\omega}^k$ does not satisfy the normalisation condition, so function $\hat{\omega}^k$ is expanded by Laurent at $\infty$

$$\hat{\omega}^k(z) = bz + c_0 + c_1 z^{-1} + \cdots \tag{15}$$

the constants $b$ and $c_0$ can be computed using the following equations

$$b=-\frac{1}{2\pi i}\int_C \frac{\hat{\omega}^k(z)}{z-\alpha}\frac{dz}{z-\alpha}, c_0=-\frac{1}{2\pi i}\int_C [\hat{\omega}^k(z)-bz]\frac{dz}{z-\alpha} \tag{16}$$

$\alpha$ is any point outside the region $D$. Define the correction function

$$f_k=\frac{z-c_0}{b} \tag{17}$$

the conformal mapping function after one external iteration $k$ is obtained

$$\omega^k=\omega^k(z)=f_k\circ\hat{\omega}^k(z) \tag{18}$$

The convergence of Koebe's iterative method is proved in the [10].

**Theorem 2.** For unbounded region $D$, there exists a constant $c>0$ such that for $i=1,2,3,...$ and for all $z\in C$

$$\left\|\omega^k(z)-\omega^{k-1}(z)\right\|_\infty \le cp^k \tag{19}$$

$p=p'^4, p'^4=\max\limits_{1\le i,j\le m,i\ne j}\left|r_i+r_j/z_{0i}+z_{0j}\right|, z_{0i}=z_{0i}^{m,k-1}, r_i=\sum_{j=1}^n\left|z_i^{m,k-1}(j)-z_{0i}^{m,k-1}\right|/n$. It is clear that Koebe's method always converges since $0<p<1$.

The above steps are repeated, and when the convergence condition for mapping function $\omega^k$ is reached, the conformal mapping function $\omega=\omega^1\circ\omega^2\circ\cdots\circ\omega^k$ is obtained. The iterative procedure is shown in Figure 2.

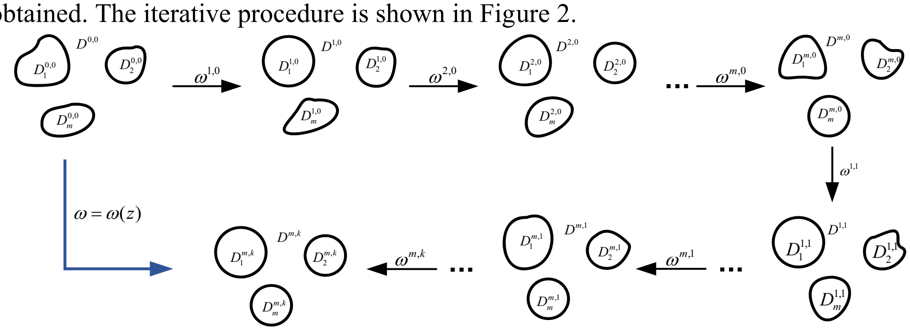

**Figure 2.** Illustration of Koebe's iterative method

The steps of the algorithm are as follows.

**Algorithm.** Koebe's iterative method based on the charge simulation method

**Input:** $n,\varepsilon,Max,z_{ij},\zeta_{i0},\zeta_{ij}(i=1,...,m; j=1,...,n),k=0$

**Computing:** $q_i=A_i/B_i, \left\|\omega^k-\omega^{k-1}\right\|_\infty, r_i=\sum_{j=1}^n\left|z_i^{m,k-1}(j)-z_{0i}^{m,k-1}\right|/n$

**step1** while $\left\|\omega^i-\omega^{i-1}\right\|_\infty \ge p^k$ do

**step2** for $i=1:m$

**step3** $q_i=A_i/B_i$

**step4** if $i=m$; return $\hat{\omega}^i=\hat{\omega}^k, k+1$; end if

**step5** end for

**step6** $b_k=\frac{1}{2\pi i}\int\hat{\omega}^k(z)/(z-z_{k0})^2\mathrm{d}z;\quad c_k=\frac{1}{2\pi i}\int(\hat{\omega}^k(z)-bz)/(z-z_{k0})\mathrm{d}z;$

**step7** $\omega^k = \hat{\omega}^k((z - c_k)/b_k)$

**step8** end while

**Output:** $\omega^k, \varepsilon$

## 4. Numerical Example

A numerical example is carried out in the environment of MATLAB 2018b to consider an unbounded multiple connected region consisting of closed smooth Jordan curves. The experiment is also carried out for orange-shaped boundary curves.

For the elliptic boundary, the bound points $z_j, j = 1,...,n$ of the boundary and charge points $\zeta_j, j = 1,...,n$ are configured using the Joukowski transform in [11].

$$\zeta_j = J\left(\frac{\rho_2}{q}\mathrm{e}^{\mathrm{i}\theta_j}\right), z_j = J\left(\rho_2 \mathrm{e}^{\mathrm{i}\theta_j}\right), J(z) = \rho_1\left(z + \frac{1}{z}\right) \tag{20}$$

where $\rho_1 = \sqrt{(a^2 - 1)}/2, \rho_2 = (a+1)/(a-1)$, the constraint points and charge points are uniformly distributed on the boundary and outside the problem domain.

**Example.** Introduces an orange-shaped boundary with equation
$C_6 : z = z_{06} + 2r(\theta)\mathrm{e}^{\mathrm{i}\theta},\ r(\theta) = \sqrt{\cos 2\theta + \sqrt{\cos^2 2\theta + 1.1^4 - 1}}.$
the remaining elliptical boundary with equation
$Cm : (x - \mathrm{Re}\, z_{0m})^2 / a_m^2 + (y - \mathrm{Im}\, z_{0m})^2 / b_m^2 = 1,\ m = 1,2,...,5$
where $z_{01} = 3\text{-i}, z_{02} = 0.5\text{-}0.5\text{i}, z_{03} = 3\text{+}2\text{i}, z_{04} = 0.5\text{+}2\text{i}, z_{05} = \text{-}0.5\text{+i}, z_{06} = 5\text{+i}$, $a_2 = 1.7$, $a_3 = 1.5, a_4 = 0.8, a_1 = a_5 = 1, b_1 = b_3 = 0.2, b_2 = b_5 = 0.3, b_4 = 0.1$.

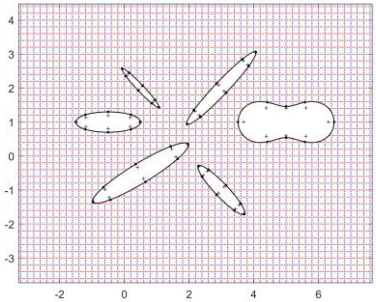

Figure 3. Original domain $D$

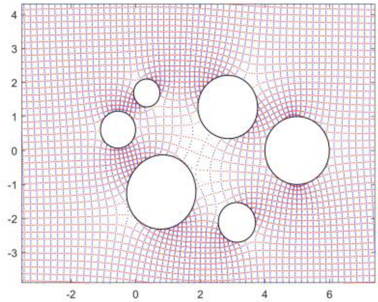

Figure 4. Circular mapping results

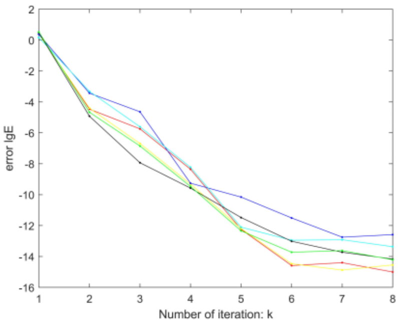


**Figure 5**. Numerical conformal mapping errors

Figure 3 shows the boundary of the problem domain, part of the grid, and the distribution of constraints and charge points, '·' represents the constraints, '+' represents the charge points. Figure 4 shows the image of the problem domain after the conformal mapping. Figure 5 shows the error for each boundary, the logarithm of the base of 10 is taken as the vertical coordinate and the number of iterations is taken as the horizontal coordinate. From the error plot, the method in this paper successfully realizes the circular mapping of unbounded multiple connected regions and achieves good mapping results within a small number of iterations.

## 5. Conclusion

This paper presents a high-precision numerical implementation of Koebe's iterative method based on the charge simulation method for computing the conformal mapping from unbounded multiple connected regions to circular domains. As verified by numerical example, it can be seen that the method of this thesis is universally feasible both in boundary shape and connectivity dimension, and a small number of iterations can obtain high accuracy mapping results. In the future, we can consider the study of bounded multiple connected region circle mapping and other forms of region conformal mapping and apply our numerical method to the computation of problems in hydrodynamics and electromagnetism.

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