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Existence of Solutions for $\psi-$Hilfer Fractional Differential Inclusions with Multi-Point Boundary Conditions

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Abstract: In this paper, we investigate the following problem

$$
{}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}\left({}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}\xi(s)\right)\in L(s,\xi(s)),
$$

$$
\xi(\alpha)=0,\xi(\beta)=\sum_{i=1}^{m}\omega_i\xi(\theta_i),\ s\in S=[\alpha,\beta],
$$

where ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}$, ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}$ denote the $\psi-$ Hilfer fractional derivative of order $\phi_1,\ \phi_2$ respectively. $\phi_1,\ \phi_2\in(0,1)$, $\eta_1,\ \eta_2\in(0,1)$, $\omega_i\in\mathrm{R}^+$, L is a multivalued map on $[\alpha,\beta]\times R$.By means of the multi-valued fixed point theorems, sufficient conditions for the existence of solutions for the $\psi-$ Hilfer fractional differential inclusions with multi-point boundary conditions are presented. We give an example to show the effectiveness of the main theorem.

Keywords: $\psi-$ Hilfer fractional differential inclusions; existence; fixed point theorem; multi-point boundary conditions

1. Introduction

Fractional calculus is a hot research field because of its wide and useful applications in real worlds, such that physics, fluid dynamics, engineering, electromagnetism, chemistry, and so on [1].

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Recently, some attention has been focused on the generalized fractional derivative with respect to another function ψ. Almeida in [2] first introduced $\psi-$Caputo fractional derivative. Tayloy's Theorem, Fermat's Theorem, semigroup law, etc. were studied. From then on, lots of definitions of that kind with respect to another function ψ have been proposed. In 2019, $\psi-$Riesz-Caputo derivative was defined by Yang and Bai in [3]. The authors studied the following problem involving $\psi-$Riesz-Caputo derivative:

$$ {}^{\mathrm{RC}}\mathrm{D}_{\alpha^+}^{q,\ \psi}(\xi(s)) \in L(s,\xi(s)),\ \xi(\alpha)+\xi(\beta)=0,\ \xi'(\alpha)+\xi'(\beta)=0. $$

Where $\alpha<\beta, 1<q<2, \psi\in C^2([\alpha,\beta]), \psi'(s)>0.$ The existence of solutions results were presented. In 2018, Sousa and Oliveira gave a new definition of the $\psi-$Hilfer factional derivative in [4]. The convergence properties and some results concerning fractional calculus were investigated. The key advantage of using the $\psi-$Hilfer fractional derivative to construct a model is that multiple differential operators can be used by appropriately selecting the parameter values and the function ψ [5].

In [6], Elkhateeb and Latha Maheswari et al. considered the following problem:

$$ \begin{aligned} & {}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}\left({}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}\xi(s)+m(\xi(s))\right) \\ & = n(s,\xi(s),\ \xi(\lambda s), {}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\alpha,\ \beta;\ \psi}\xi(s)), \\ & \xi(\alpha)=0,\ \ \xi(\beta)=\sum_{i=1}^{m}\omega_i\xi(\theta_i), \end{aligned} \tag{1} $$

where ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}$, ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}$ denote the $\psi-$Hilfer fractional derivative of order $\phi_1,\ \phi_2$ respectively. $\phi_1,\ \phi_2>\alpha,\ \ \phi_1,\ \phi_2,\ \alpha\in(0,1)$, $\eta_1,\ \eta_2,\ \beta\in[0,1]$, $\omega_{\mathrm{i}}\in\mathrm{R}^+,\ \theta_{\mathrm{i}}\in(a,b),\ \ m,n$ are continuous functions on a Banach space. The authors presented some results about existence of solutions.

Motivated by [6], we study the following $\psi-$Hilfer fractional differential inclusions:

$$ {}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}\left({}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}\xi(s)\right)\in L(s,\xi(s)), $$

$$\xi(\alpha)=0,\ \xi(\beta)=\sum_{i=1}^{m}\omega_i\xi(\theta_i),\ s\in S=[\alpha,\beta], \tag{2}$$

where ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}$, ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}$ denote the $\psi-$Hilfer fractional derivative of order $\phi_1,\ \phi_2$ respectively. $\phi_1,\ \phi_2\in(0,1)$, $\eta_1,\ \eta_2\in(0,1)$, $\omega_i\in\mathrm{R}^+$, L is a multivalued map on $[\alpha,\beta]\times R$.It is noticed that we generalize the single value result to the multivalued one. Three sufficient conditions for the existence of solutions are given. In order to cover the gap that there were few papers concerning $\psi-$ Hilfer fractional differential inclusions, we are willing to do our research.

2. Preliminaries

First of all, we recall some preliminaries about fractional calculus [1]and multi-valued maps [7-8].

Definition 1. Let $\phi\in(n-1,n)$, and $n\in\mathrm{N}$, \$ if $\phi\in(n-1,n)$, $\alpha<\beta$, $\alpha,\ \beta\in\mathrm{R}$ and $h,\psi\in C^n([\alpha,\beta],R)$, $\psi(s)$ is increasing and $\psi'(s)\neq 0$, for all $s\in[\alpha,\beta]$, then the $\psi-$Hilfer fractional derivative ${}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi,\ \eta;\ \psi}$, of order ϕ of a function h and type $0\le\eta\le 1$ is defined

$$ {}^{\mathrm{H}}\mathrm{D}_{\alpha^+}^{\phi,\ \eta;\ \psi}h(s)=I_{\alpha^+}^{\eta(n-\phi),\psi}\left(\frac{ds}{\psi'(s)}\right)^n I_{\alpha^+}^{(1-\eta)(n-\phi),\psi}h(s), \tag{3}$$

where $\mathrm{n}=[\phi]+1$, $[\phi]$ is the integer part of the real number $[\phi]$.

Lemma 1. Let $h\in C([\alpha,\beta],R)$, $0<\phi_1,\ \phi_2<1$, $0\le\eta_1,\ \eta_2\le 1$, $\omega_i\in\mathrm{R}^+$, $\theta_i\in(\alpha,\beta)$, $\gamma_1=\phi_1+\eta_1(1-\phi_1)$, $\alpha\ge 0$ and $\omega\neq 0$, then the solution of the following problem

$$^{H}D_{\alpha^+}^{\phi_1,\ \eta_1;\ \psi}\left(^{H}D_{\alpha^+}^{\phi_2,\ \eta_2;\ \psi}\xi(s)\right)=h(s),$$

$$\xi(\alpha)=0,\ \xi(\beta)=\sum_{i=1}^{m}\omega_i\xi(\theta_i) \tag{4}$$

satisfies the equation

$$\xi(s)=I_{\alpha^+}^{\phi_1+\phi_2;\psi}h(s)+\frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}[\sum_{i=1}^{m}\omega_i I_{\alpha^+}^{\phi_1+\phi_2;\psi}h(\theta_i)-I_{\alpha^+}^{\phi_1+\phi_2;\psi}h(\beta)],$$

where

$$\omega=\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}-\sum_{i=1}^{m}\omega_i\frac{(\psi(\theta_t)-\psi(\alpha))^{\gamma_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}.$$

Proof. Taking $m=0$, in Lemma 7 in [6], we get the result immediately.

Lemma 2.[7] Assume that (Y, d) is a complete metric space. Suppose that $M:Y\rightarrow P_{cl}(Y)$ is a contraction, then $\mathrm{Fix}M\neq\Phi$.

Lemma 3. [8] Let A be a Banach space, B a closed convex subset of A, and $0\in E$ an open subset of B with $0\in E$. If $H:\overline{E}\rightarrow P_{c,cv}(B)$ is a upper semicontinuous compact map. Then either

(i) H has a fixed point in $\overline{E}$, or

(ii) there exists a $u\in\partial E$, and $\eta\in(0,1)$ satisfying $u\in\eta H(u)$.

3. Main results

3.1 The Lipschitz case

(A_1) L is compact multivalued maps on $[\alpha,\beta]\times R$, such that for every

$[\alpha,\beta]\times \mathrm{R}$, L（•， ξ） is measurable.

(A_2) For almost all $s\in[\alpha,\beta]$ such that

$$d(L(s,\xi),L(s,\bar{\xi}))\leq \rho(s)\,|\,\xi-\bar{\xi}\,|,\ \ \xi,\bar{\xi}\in \mathrm{R},$$

where $\rho\geq 0$, L-integrable and $d(0,L(s,0))\leq \rho(s).$ For convenience, denote

$$\varpi:=\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}+\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}$$
$$\times[\sum_{i=1}^{m}\omega_i\frac{(\psi(\theta_t)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2+1)}+\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}].$$

Theorem 1. Suppose that $(A_1)-(A_2).$ Then problem (2) has at leas a solution on S, provided that

$$\varpi\,\|\,\rho\,\|<1.$$

Proof. Denote the multivalued continuous operator T as follows, if $\tau\in \mathrm{S}_{\mathrm{L},\ \xi}$:

$$\mathrm{T}\xi\,(s)=\{h\in([\alpha,\beta],R):h(s)=\int_{\alpha}^{s}\frac{\psi'(t)(\psi(s)-\psi(t))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau(t)dt$$

$$+\frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}\times[\sum_{i=1}^{m}\omega_i\int_{\alpha}^{\theta_i}\ \frac{\psi'(s)(\psi(\theta_i)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau(s)ds$$

$$-\sum_{i=1}^{m}\omega_i\int_{\alpha}^{\beta}\ \frac{\psi'(s)(\psi(\beta)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau(s)ds\}.$$

We divide the proof into two parts.

Part 1. If $\{\mathrm{u_n}\}_{\mathrm{n}\geq 0}\in T(\xi)$ and $\mathrm{u_n}\to \mathrm{u}$ $(n\to\infty)$. Then u is a continuous function, and for each $s\in[\alpha,\beta],$ there exists $\tau_n\in S_{L,\xi}$ satisfying

$$u_n(s)=\int_\alpha^s \frac{\psi'(t)(\psi(s)-\psi(t))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_n(t)dt+\frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}$$

$$\times[\sum_{i=1}^m \omega_i\int_\alpha^{\theta_i} \frac{\psi'(s)(\psi(\theta_i)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_n(s)ds$$

$$-\sum_{i=1}^m \omega_i\int_\alpha^{\beta} \frac{\psi'(s)(\psi(\beta)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_n(s)ds],$$

By L has compact values, there exists a subsequence $\{\tau_n\}$ converging to $\tau\in \mathrm{L}^1([\alpha,\beta],R)$. we have $u_n(s)\to u(s)$. Thus, $u\in T(\xi)$.

Part 2. Suppose that $\xi,\overline{\xi}$ are continuous functions and $h_1\in T(\xi)$. There exists $\tau_1(s)\in \mathrm{L}(s,\xi(s))$, then we have

$$h_1(s)=\int_\alpha^s \frac{\psi'(t)(\psi(s)-\psi(t))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_1(t)dt+\frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}$$

$$\times[\sum_{i=1}^m \omega_i\int_\alpha^{\theta_i} \frac{\psi'(s)(\psi(\theta_i)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_1(s)ds$$

$$-\sum_{i=1}^m \omega_i\int_\alpha^{\beta} \frac{\psi'(s)(\psi(\beta)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_1(s)ds],$$

Define the multivalued operator Ω by

$$\Omega(s)=\{\ell\in R:|\tau_1(s)-\ell(s)|\le\rho(s)|\xi(s)-\overline{\xi}(s)|\}.$$

As the multivalued operator $\Omega\bigcap \mathrm{L}(s,\overline{\xi}(s))$. is measurable. Then there exits a measurable selection $\tau_2(s)$ such that $\tau_2\in \mathrm{L}(s,\overline{\xi}(s))$, we get

$$|\tau_1(s)-\tau_2(s)|\le\rho(s)|\xi(s)-\overline{\xi}(s)|.$$

Define $h_2(t)$ as $h_1(t)$ replacing $\tau_1(s)$ with $\tau_2(s)$, and one has

$$\| h_1 - h_2 \| \leq \varpi \| \rho \| \| \xi - \bar{\xi} \| .$$

Interchanging ξ and $\bar{\xi}$ yields

$$d(L(s,\xi), L(s,\bar{\xi})) \leq \varpi | \rho \| .$$

By the assumption inequality, T is a contraction. Thus, T has a fixed point that is a solution to problem (2) by virtue of Lemma 2.

3.2. The Caratheodory case

(A_3) L is Caratheodory multivalued map which has nonempty compact and convex values on $[\alpha,\beta]\times R$;

(A_4) If $\delta:[0,\infty)\to[0,\infty)$ is a continuous nondecreasing function and l is a positive integral function on $[\alpha,\beta]$ such that $\| L(s,\xi) \| := \sup\{ | \tau | :$

$\tau \in L(s,\xi)\} \leq l(s)\delta(\| \xi \|), \quad (s,\xi)\in[\alpha,\beta]\times R.$

Theorem 2. Assume that $(A_1)-(A_2)$ hold. If there exists a constant $N>0$, satisfying

$$\frac{N}{\delta(N)\ \varpi \| l \|} > 1,$$

Proof. Consider the operator defined as T. All the assumptions of Lemma 3 are satisfied concerning T.

Step 1. As $S_{L,\xi}$ is convex, T is convex for each continuous function ξ.

Step 2. For a positive $\Delta>0$, let $B_\Delta = \{\xi \in C([\alpha,\beta], R) : \| \xi \| \leq \Delta\}$ be a bounded ball, then for $h \in T(\xi), \xi \in B_\Delta$, there exists $\tau \in S_{L,\xi}$, for $s\in[\alpha,\beta]$, we have

$$\| h \| \leq \delta(\| \Delta \|)[\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}+\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\gamma_1+\phi_2)}$$

$$\times[\sum_{i=1}^{m}\omega_i\frac{(\psi(\theta_t)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2+1)}+\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}]\| l \|.$$

Step 3. Let s_1, $s_2 \in [\alpha,\beta]$, and $s_1 < s_2$, $\xi \in \mathrm{B}_\Delta$, where B_Δ is a bounded set in $\mathrm{C}([\alpha,\beta],R)$ for $\mathrm{h} \in \mathrm{T}(\xi), \xi \in \mathrm{B}_\Delta$, we have

$$| h(s_1)-h(s_2) | \leq \frac{(2(\psi(s_2)-\psi(s_1))^{\phi_1+\phi_2}+\psi(s_2)-\psi(\alpha))^{\phi_1+\phi_2}-\psi(s_1)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}$$

$$\times[\sum_{i=1}^{m}\omega_i\frac{(\psi(\theta_t)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}+\frac{(\psi(\beta)-\psi(\alpha))^{\phi_1+\phi_2}}{\Gamma(\phi_1+\phi_2+1)}]\delta(\| \Delta \|)\| l \| \to 0, (s_1 \to s_2).$$

which means T is completely continuous. Thus, T maps bounded set into equicontinuous sets.

step 4. Set $\xi_n \to \xi_*$, $h_n \to T(\xi_n)$, and $h_n \to h_*$, Then, we have

$$h_n(s)=\int_\alpha^s\frac{\psi'(t)(\psi(s)-\psi(t))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_n(t)dt+\frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}$$

$$\times[\sum_{i=1}^{m}\omega_i\int_\alpha^{\theta_i}\frac{\psi'(s)(\psi(\theta_i)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_n(s)ds$$

$$-\sum_{i=1}^{m}\omega_i\int_\alpha^{\beta}\frac{\psi'(s)(\psi(\beta)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau_n(s)ds].$$

And we obtain h_* as h_n replacing τ_n with τ_*.Thus, we show that there exists $f_* \in S_{L,\xi_*}$. The continuous linear thoperator Ξ: $\mathrm{L}^1([\alpha,\beta],R) \to C([\alpha,\beta],R)$

Is as follows:

$$f \to \Xi(\tau)(s) = \int_\alpha^s \frac{\psi'(t)(\psi(s)-\psi(t))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau(t)dt + \frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}$$

$$\times[\sum_{i=1}^m \omega_i \int_\alpha^{\theta_i} \frac{\psi'(s)(\psi(\theta_i)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau(s)ds$$

$$-\sum_{i=1}^m \omega_i \int_\alpha^{\beta} \frac{\psi'(s)(\psi(\beta)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)}\tau(s)ds].$$

Notice that

$$\| h_n(s)-h_*(s) \| \le \int_\alpha^s \frac{\psi'(t)(\psi(s)-\psi(t))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)} | \tau_n(t)-\tau_*(t) | dt$$

$$+\frac{(\psi(s)-\psi(\alpha))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)} \times[\sum_{i=1}^m \omega_i \int_\alpha^{\theta_i} \frac{\psi'(s)(\psi(\theta_i)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)} | \tau_n(s)-\tau_*(s) | ds$$

$$+\sum_{i=1}^m \omega_i \int_\alpha^{\beta} \frac{\psi'(s)(\psi(\beta)-\psi(s))^{\phi_1+\phi_2-1}}{\Gamma(\phi_1+\phi_2)} | \tau_n(s)-\tau_*(s) | ds],$$

as $n \to \infty$. Moreover, we have $h_n(s) \in \Xi(S_{L,\tau_n})$. By $\tau_n \to \tau_*$, we get h_* for some $\tau_* \in S_{L,\xi_*}$, which means T has a closed graph.

Step 5. It follows from step2 that

$$\frac{\| \xi \|}{\delta(N)\ \varpi \| l \|} \le 1,$$

There exist N with $\| \xi \| \ne \mathrm{N}$. Denote $\mathrm{U} = \{\xi$ is continous on $[\alpha,\beta]: \| \xi \| < N\}$. T is upper semicontinuous and completely continuous. By the definition of U, there is no $\xi \in \partial\mathrm{U}$ such that $\xi \in \eta \mathrm{T}(\xi)$ for some $\eta \in (0,1)$. Hence, we have (ii) of Lemma 3 is not true and there exists a fixed point $\xi \in \overline{\mathrm{U}}$, which is a solution of problem (2). The proof is completed.

4. Application

Example 1. Consider the $\psi-$ Hilfer fractional differential inclusion

$$
\begin{gathered}
{}^{\mathrm{H}}\mathrm{D}_{0}^{\frac{1}{2},\frac{1}{10};\sin t}\left({}^{\mathrm{H}}\mathrm{D}_{\alpha^{+}}^{\frac{7}{10},\frac{2}{5};\sin t}\xi(s)\right)\in L(s,\xi(s)),\\
\xi(0)=0,\quad \xi(1)=\sum_{i=1}^{3}\left(\frac{-i}{i+3}\right)^{i+1}\xi\left(\frac{i}{5}\right), s\in[0,1].
\end{gathered}
\tag{5}
$$

$$
f\to L(s,\xi(s)):=\left[\frac{|\xi|^{5}}{|\xi|^{5}+3}+s+3,\frac{|\xi|}{|\xi|+1}+2\right], |f|\le 5,
$$

$$
L(s,\xi(s)):=\sup\{|\tau|:\tau\in L(s,\xi(s))\}\le 5:=l(s)\delta(|\xi|).\ l(s)=1,\delta(|\xi|)=5,
$$

then we have $\frac{\mathrm{N}}{5\varpi}>1,$

$$
\varpi=\frac{(\sin(1)-\sin(0))^{\frac{6}{5}}}{\Gamma(11/5)}+\frac{\sin(1)-\sin(0))^{\frac{23}{100}}}{\Gamma(5/4)|w|}
$$

$$
\times\left[\sum_{i=1}^{3}\left(\frac{-i}{i+3}\right)^{i+1}\frac{(\sin(\frac{i}{5})-\sin(0))^{\frac{6}{5}}}{\Gamma(22/10)}+\frac{(\sin(1)-\sin(0))^{\frac{6}{5}}}{\Gamma(22/10)}\right]
$$

that is $\mathrm{N}>2.78041.$ By Theorem 2, $\psi-$ Hilfe fractional differential inclusion (5) has at least one solution on $[0,1].$

5. Conclusion

The main aim of this paper is to generalize the recent single value problem to the multivalued one. We study the $\psi-$ Hilfefractional differential inclusions with multi-valued conditions. Existence of solutions to the problem is discussed. Sufficient conditions of existence of solutions results are presented and an example involving function $\sin x$ is given to illustrate our main results. It is noticed that if we take $m = 0$ and the right hand of (1) the multi-valued function L, then (1) becomes the problem (2). The results are new and contribute to the existence results about this issue.

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