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Existence of Solutions for Fractional Hybrid Differential Inclusions with Tree-Point Boundary Hybrid Conditions

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Abstract: In this paper, the existence of solutions for fractional hybrid differential inclusions with tree-point boundary hybrid conditions is investigated:

$$^{C}D_{0^+}^{p}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)\in L(s,\xi(s)),0<p\le 2,$$

$$\alpha_1\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=0}+\beta_1\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a}=\gamma_1,$$

$$\alpha_2\,{}^{C}D_{0^+}^{q}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=r}+\beta_2\,{}^{C}D_{0^+}^{q}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a}=\gamma_2,0<r<a,$$

where $D_{0^+}^{p}$ and $D_{0^+}^{q}$ denotethe Caputo fractional derivative of order p,q respectively. $0<q\le 1,\alpha_i,\beta_i,\gamma_i, i=1,2,$ such that $\alpha_1+\beta_1\ne 0$, $\alpha_2 r^{1-q}+\beta_2 a^{1-q}\ne 0$, $m\in C([0,a]\times R,R)$, $n\in C([0,a]\times R,R\setminus\{0\})$, $L\in[0,a]\times R\to P(R)$, is a multivalued map. By means of the multi-valued hybrid fixed point theorems, we present sufficient conditions for the existence of solutions for the fractional hybrid differential inclusions with three-point boundary hybrid conditions. An illustrative example is given to show the effectiveness of our main result. We generalize the single known results to the multi-valued ones.

Keywords: fractional hybrid differential inclusions; existence of solutions; fixed point theorem; tree-point boundary hybrid conditions

1. Introduction

Fractional calculus is a generalization of the integer order of ordinary differentiation and integration to arbitrary non-integer one, which is of great use as description of

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things' hereditary property [1]. Recently some attentions have been paid to fractional differential equations [2]. It can be applied to many fields such as engineering, electromagnetism, chemistry, electromagnetism and so on. In 2010, Dhage and Lakshmikantham [3] introduced the definition of hybrid differential equations. They study the integer order of differential equations:

$$\frac{\mathrm{d}}{\mathrm{ds}}\left[\frac{\xi(s)}{u(s,\xi(s))}\right]=v(s,\xi(s)),$$
$$\xi(s_0)=\xi_0\in R.$$

Where u,v are continuous functions on $[s_0,s_0+a], a>0.$ Some inequalities were given to investigate the existence of extremal solutions and a comparison result. The next year, Zhao, Sun, et al. in [4] generalized this kind of differential equation of integer order to the fractional Riemann-Liouville type ones:

$$\mathrm{D}^{\varepsilon}\left[\frac{\xi(s)}{u(s,\xi(s))}\right]=v(s,\xi(s)),0<\varepsilon<1,$$
$$\xi(s_0)=\xi_0\in R.$$

The existence of extremal solutions were presented and a comparison principle was proved. Since then, some authors have focused on the study of hybrid fractional equations [5]. In 2016, Ahmad et al. in [6] investigated a class of hybrid Caputo fractional integro-differential inclusions with nonlocal conditions.

$${}^{C}\mathrm{D}^{\gamma}\left[\frac{\xi(s)-\sum_{i=1}^{m}\mathrm{I}^{\gamma_i}m_i(s,\xi(s))}{u(s,\xi(s))}\right]=v(s,\xi(s)),0<\varepsilon<1,$$
$$\xi(0)=\mu(\xi),\xi(1)=a,$$

where ${}^{C}\mathrm{D}^{\gamma}$ is Caputo derivative, I^{γ_i} is the Riemann-Liouville fractional integral of order γ_i, $i=1,2,...,m, f\neq 0$ is a continuous function, L a multivalued map, $\mathrm{m_i}$ and μ are also continuous functions and a is a constant. Existence of solutions to the above problem is given.

In [7], Derbazi, Hammouche, Benchohra and Zhou considered the fractional hybrid differential equations:

$$
\begin{aligned}
&{}^{C}D_{0^+}^{p}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)=L(s,\xi(s)),0<p\le 2,\\
&\alpha_1\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=0}+\beta_1\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a}=\gamma_1,\\
&\alpha_2\,{}^{C}D_{0^+}^{q}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=r}+\beta_2\,{}^{C}D_{0^+}^{q}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a}=\gamma_2,0<r<a,
\end{aligned}
\tag{1}
$$

where $D_{0^+}^{p}$ and $D_{0^+}^{q}$ denotethe Caputo fractional derivative of order p,q respectively. $0<q\le 1$, $\alpha_i,\beta_i,\gamma_i, i=1,2,$ such that $\alpha_2 r^{1-q}+\beta_2 a^{1-q}\neq 0$, m,n,L are continuous functions, with $n\neq 0$. The authors gave the existence of solutions results by some fixed-point theorems.

There are many different kinds of fractional derivatives and integral definitions, such as Caputo type, Riemann-Liouville type, Hadamard type, Atangana-Baleanu type, Hilfer type et al. Because of the relationship between fractional Caputo derivative(integral) and Riemann-Liouville fractional derivative (integral), most of the hybrid fractional differential equations and inclusions in resent research work are concerning the Caputo and Riemnn-liouville type. We are inspired by the above mentioned works. Considering the problem

$$
\begin{aligned}
&{}^{C}D_{0^+}^{p}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)\in L(s,\xi(s)),0<p\le 2,\\
&\alpha_1\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=0}+\beta_1\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a}=\gamma_1,\\
&\alpha_2\,{}^{C}D_{0^+}^{q}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=r}+\beta_2\,{}^{C}D_{0^+}^{q}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a}=\gamma_2,0<r<a,
\end{aligned}
\tag{2}
$$

Where L is a multivalued map. We present the sufficient conditions for the existence of solutions to the problem (2). Best to our knowledge, problem (2) was never be concerned, so we are willing to cover this gap.

2. Preliminaries

We give some definitions of fractional calculus [1-2]and multi-valuedmaps [8-9].

Definition 1 Let $\varsigma>0$, and $n=[\varsigma]+1$, if $h\in C^{n}([a,b])$, then the Caputo fractional derivative of order ς is defined

$$ {}^{C}D_{0^+}^{\varsigma} h(s) = \frac{1}{\Gamma(n-\varsigma)} \int_0^s (s-t)^{(n-\varsigma-1)} h^{(n)}(t)dt $$

exists almost everywhere on $[a,b]$ ($[\varsigma]$ is the integer part of ς.)

Definition 2. The Riemann-Liouville fractional integral of order $\varsigma > 0$ for a continuous function $h:[0,\infty) \to R$ is defined as

$$ I_{0^+}^{\varsigma} h(s) = \frac{1}{\Gamma(\varsigma)} \int_0^s (s-t)^{\varsigma-1} h(t)dt $$

Lemma 1. [7] Let $\kappa \in C([0,a], R)$, then the solution of the boundary value problem

$$ \begin{aligned} &{}^{C}D_{0^+}^{p}\left(\tfrac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right) = \kappa(s), 0 < p \le 2, \\ &\alpha_1\left(\tfrac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=0} + \beta_1\left(\tfrac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a} = \gamma_1, \\ &\alpha_2\, {}^{C}D_{0^+}^{q}\left(\tfrac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=r} + \beta_2\, {}^{C}D_{0^+}^{q}\left(\tfrac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=a} = \gamma_2, 0 < r < a, \end{aligned} \tag{3} $$

satisfies the equation

$$ \begin{aligned} \xi(s) = n(s,\xi(s)) \times \Big[&\int_0^s \frac{(s-t)^{p-1}}{\Gamma(p)} \kappa(s)ds - \frac{\beta_1}{\alpha_1+\beta_1} \int_0^a \frac{(a-s)^{p-1}}{\Gamma(p)} \kappa(s)ds + \frac{\gamma_1}{\alpha_1+\beta_1} \\ &+ \frac{\alpha_2 \int_0^r \frac{(r-s)^{p-q-1}}{\Gamma(p-q)} \kappa(s)ds + \beta_2 \int_0^a \frac{(a-s)^{p-q-1}}{\Gamma(p-q)} \kappa(s)ds - \gamma_2}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q} + \beta_2 a^{1-p})} \\ &\times (\beta_1 a - (\alpha_1+\beta_1)s)\Gamma(2-q) \Big] + m(s,\xi(s)). \end{aligned} \tag{4} $$

For convenience, denote $P_{cp,cv}(X)$ is a set, where all the elements are in $P(X)$, which is convex and compact; $P_{cl}(X)$ is a set, where all the elements are in $P(X)$, which is closed.

Lemma 2. [10] Suppose that Q is a Banach algebra, M, N: $\mathrm{Q} \to \mathrm{Q}$ are two single-valued mapas and F is a multi-valued compact and convex operator such that:

(a) M and N are Lipschitzian, which has two Lipschitz constants q_1 and q_2, respectively;

(b) F is a upper semi-continuous multivalued map;

(c) $q_1 \mathrm{H} + q_2 < \frac{1}{2}$, $\mathrm{H} = \| \bigcup \mathrm{F}(\mathrm{Q}) \|$. Then, either

(i) the operator $\varsigma \in \mathrm{M}\varsigma \mathrm{F}\varsigma + \mathrm{N}\varsigma$ has a solution or

(ii) $\mathrm{E} = \{\varsigma \in \mathrm{X}: \mu\varsigma \in \mathrm{M}\varsigma \mathrm{F}\varsigma + \mathrm{N}\varsigma, \ \mu > 1\}$ is unbounded.

3. Main results

We assume that the following conditions hold:

(A_1) L is a compact and convex L^1 -Caratheodory multi-valued map.

(A_2) If $\tau: [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function and $\omega \in \mathrm{L}([0, a], \mathrm{R}^+)$ such that for each $(s, \xi) \in [0, a] \times \mathrm{R}$.

$$\| L(s, \xi) \| := \sup\{\eta \in \mathrm{L}(\mathrm{s}, \xi)\} \le \omega(\mathrm{s}) \tau(\| \xi \|),$$

(A_3) If $\mathrm{m}: [0, a] \times \mathrm{R} \to \mathrm{R} \setminus \{0\}$ is continuous and there is a function $\phi > 0$ such that

$$| \mathrm{m}(\mathrm{s}, \xi) - \mathrm{m}(\mathrm{s}, \bar{\xi}) | \le \phi(\mathrm{s}) | \xi - \bar{\xi} |, \xi, \bar{\xi} \in \mathrm{R}. \psi$$

(A_4) The function $\mathrm{n}: [0, a] \times \mathrm{R} \to \mathrm{R}$ is continuous and there exists a positive function ψ, such that for $s \in [0, a], \xi, \bar{\xi} \in \mathrm{R}$.

$$|\mathrm{n}(\mathrm{s},\xi)-\mathrm{n}(\mathrm{s},\bar{\xi})|\le\psi(\mathrm{s})|\xi-\bar{\xi}|,\ \xi,\bar{\xi}\in\mathrm{R}.$$

Theorem 1. Suppose that conditions $(A_1)-(A_4)$ hold. Assume that there exists a positive constant $\mathrm{M}>0$ such that

$$\|\psi\|\mathrm{K}_1+\|\phi\|<\frac{1}{2},$$

where

$$\mathrm{K}_1:=\tau(\mathrm{M})\,\omega\left(\frac{a^p}{\Gamma(p+1)}+\frac{|\beta_1|a^p}{|\alpha_1+\beta_1|}\right)+(|\alpha_2|r^{p-q}+|\beta_2|a^{p-q})$$

$$\times\left(\frac{|\beta_1|a+(|\alpha_1|+|\beta_1|a)\Gamma(2-q)}{|\alpha_1+\beta_1|(|\alpha_2|r^{p-q}+|\beta_2|a^{p-q})\Gamma(p-q+1)}\right)+\frac{|\gamma_1|}{|\alpha_1+\beta_1|},$$

then problem (2) has at least a solution in $[0,a]$.

Proof. Let $\mathrm{A}\xi(s)=n(s,\xi(s))$, $s\in[0,a]$ and B be a multi-valued map on $[0,a]$, for $\eta\in\mathrm{S}_{\mathrm{L},\xi}$:

$$B(\xi)=\{h\in C([0,a],R):h(s)=I_{0^+}^{p}\eta(s)-\frac{\alpha_1}{\alpha_1+\beta_1}I_{0^+}^{p}\eta(a)+\frac{\gamma_1}{\alpha_1+\beta_1}$$

$$+\frac{(\beta_1a-(\alpha_1+\beta_1s)\Gamma(2-q)(\alpha_2I_{0^+}^{p-q}\eta(r)+\beta_2I_{0^+}^{p-q}\eta(a)-\gamma_2)}{(\alpha_1+\beta_1)(\alpha_2r^{1-q}+\beta_2a^{1-q})\Gamma(p-q+1)},$$

$$\mathrm{C}\xi(s)=m(s,\xi(s)),\ s\in[0,a].$$

We shall prove that all the conditions of Lemma 2 are satisfied.

Step 1. B is compact and convex. In fact, B is the composition $B=K\circ S_{L,\xi}$ and $K:L([0,a],R)\to X$ is continuous defined by

$$K\xi(s)=\int_0^s \frac{(s-t)^{p-1}}{\Gamma(p)}\eta(t)dt-\frac{\beta_1}{\alpha_1+\beta_1}\int_0^a \frac{(a-s)^{p-1}}{\Gamma(p)}\eta(s)ds+\frac{\gamma_1}{\alpha_1+\beta_1}$$

$$+\frac{\alpha_2\int_0^r \frac{(r-s)^{p-q-1}}{\Gamma(p-q)}\eta(s)ds+\beta_2\int_0^a \frac{(a-s)^{p-q-1}}{\Gamma(p-q)}\eta(s)ds-\gamma_2}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-p})}$$

$$\times(\beta_1 a-(\alpha_1+\beta_1)s)\,\Gamma(2-q).$$

First of all, the composition operator $K\circ S_{L,\xi}$ has compact values on X. If we show that $\{K\eta_n\}$ is an equicontinuous sequence, and it is uniformly convergent. In fact, let $s_1<s_2$ it follows that

$$|K\eta_n(s_2)-K\eta_n(s_1)|\le\Big|\int_0^{s_1}\frac{(s_2-s)^{p-1}-(s_1-s)^{p-1}}{\Gamma(p)}\eta_n(s)ds$$

$$-\int_{s_1}^{s_2}\frac{(s_2-s)^{p-1}}{\Gamma(p)}\eta_n(s)ds\Big|$$

$$+\Big|\frac{(\beta_1 a-(\alpha_1+\beta_1)|s_2-s_1|)\Gamma(2-q)}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-q})\Gamma(p-q)}+(|\alpha_2|\int_0^r (r-s)^{p-q-1}|\eta_n(s)|ds$$

$$+|\beta_2|\int_0^a (a-s)^{p-q-1}|\eta_n(s)|ds+|\gamma_2|)\to 0,\ \text{as}\ s_1\to s_2.$$

Thus, using the Arzela-Ascolitheorem, we have that $\{K\eta_n\}$ is an equicontinuous sequence.

Secondly, B is convex valued. Let $h_1,h_2\in B(\xi)$. Then there exist η_1, $\eta_2\in S_{G,\xi}$ such that for any $\gamma\in[0,1]$, we have

$$\gamma h_1(s)+(1-\gamma)h_2(s)\le\int_0^s \frac{(s-t)^{p-1}}{\Gamma(p)}[\gamma\eta_1(t)+(1-\gamma)\eta_2(t)]dt$$

$$-\frac{\beta_1}{\alpha_1+\beta_1}\int_0^a \frac{(a-s)^{p-1}}{\Gamma(p)}[\gamma\eta_1(s)+(1-\gamma)\eta_2(s)]ds$$

$$+\frac{(\beta_1 a-(\alpha_1+\beta_1)s)\Gamma(2-q)}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-q})\Gamma(p-q)}(\alpha_2\int_0^r (r-s)^{p-q-1}[\gamma\eta_1(s)+(1-\gamma)\eta_2(s)]ds$$

$$+\beta_2\int_0^a (a-s)^{p-q-1}[\gamma\eta_1(s)+(1-\gamma)\eta_2(s)]ds-\gamma_2)+\frac{\gamma_1}{\alpha_1+\beta_1}$$

Since $L(s,\xi(s))$ is convex, $\gamma\eta_1(s)+(1-\gamma)\eta_2(s)\in L(s,\xi(s))$ for all $s\in[0,a]$ and so $\gamma h_1(s)+(1-\gamma)h_2(s)\in S_{G,\xi}$ is convex. Thus, the proof of step 1 is completed.

Step 2. A and C are Lipschitz functions on X. Let $\nu,\bar{\nu}\in X$. According to (A_4), for $s\in[0,a]$, we get

$$|A\nu(s)-A\bar{\nu}(s)|=|n(s,\nu(s))-n(s,\bar{\nu}(s))|\le\psi(s)|\nu(s)-\bar{\nu}(s)|\le\|\psi\|\|\nu-\bar{\nu}\|.$$

Based on (A_3), we obtain

$$|C\nu(s)-C\bar{\nu}(s)|=|m(s,\nu(s))-m(s,\bar{\nu}(s))|\le\phi(s)|\nu(s)-\bar{\nu}(s)|\le\|\phi\|\|\nu-\bar{\nu}\|.$$

Step 3. B is completely continuous and upper semi-continuous on X. Suppose that there exists a constant $\vartheta>0$, such that $\|\xi\|<\vartheta$, for all $\xi\in S$. Next, we shall show that $B(S)$ is a uniformly bounded and equicontinuous set in X If $\xi\in B(S)$, there exists a $\eta\in S_{G,\xi}$ such that

$$|\xi(s)|\le\tau(\vartheta)\|\omega\|(\frac{a^p}{\Gamma(p+1)}+\frac{|\beta_1|a^p}{|\alpha_1+\beta_2|\Gamma(p+1)})$$

$$+\frac{(|\beta_1|a+|\alpha_1+\beta_1|)\Gamma(2-q)}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-q})\Gamma(p-q+1)}(|\alpha_2|r^{p-q}+|\beta_2|a^{p-q})$$

$$+\frac{|\gamma_2|(|\beta_1|a+|\alpha_1+\beta_1|)\Gamma(2-q)}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-q})\Gamma(p-q+1)}+\frac{|\gamma_1|}{|\alpha_1+\beta_1|}:=K_1.$$

Therefore, $\|\xi\|<K_1$. As the same the discussion, we have that $B(S)$ is equicontinuous. We have known that B is completely continuous and upper semicontinuous, and next we will prove that B has a closed graph. Assume that $\{\xi_n\}$ is a sequence in X such that $\xi_n\to\xi_*$. Suppose that $y_n\in B\xi_n$ is a sequence such that and $y_n\to y^*$. It suffices to have $y^*\in B\xi^*$. As $y_n\in B\xi_n$, then we have a $\xi_n\in S_{G,\xi_n}$,we have

$$\|y_n(s)-y^*(s)\|\le|\int_0^s\frac{(s-t)^{p-1}}{\Gamma(p)}[\eta_n(s)-\eta(s)]ds-\frac{\beta_1}{\alpha_1+\beta_1}$$

$$\times\int_0^a\frac{(a-s)^{p-1}}{\Gamma(p)}[\eta_n(s)-\eta(s)]ds+\frac{(\beta_1 a-(\alpha_1+\beta_1)s)\Gamma(2-q)}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-q})\Gamma(p-q)}$$

$$\times(\alpha_2\int_0^r(r-s)^{p-q-1}[\eta_n(s)-\eta(s)]ds+\beta_2\int_0^a(a-s)^{p-q-1}[\eta_n(s)-\eta(s)]ds\|\to 0$$

We have that $K\circ S_{L,\xi}$ is closed. Thus, $y_n\in K(S_{G,\xi})$.

Step 4. Obviously, we get $K_1q_1+q_2<\frac{1}{2}$.

Step 5. The condition (i) of Lemma 2 is true. For some $\mu>1$, setting ξ is any solution to (2) such that $\mu\xi\in A\xi B\xi+C\xi$ .Then we have

$$\xi(s)=\delta n(s,\xi(s))[I_{0^+}^{p}\eta(s)-\frac{\alpha_1}{\alpha_1+\beta_1}I_{0^+}^{p}\eta(a)+\frac{\gamma_1}{\alpha_1+\beta_1}$$

$$+\frac{(\beta_1 a-(\alpha_1+\beta_1 s)\Gamma(2-q)(\alpha_2 I_{0^+}^{p-q}\eta(r)+\beta_2 I_{0^+}^{p-q}\eta(a)-\gamma_2)}{(\alpha_1+\beta_1)(\alpha_2 r^{1-q}+\beta_2 a^{1-q})\Gamma(p-q+1)}]+\delta m(s,\xi(s))$$

Where $\delta=\frac{1}{\mu}<1$. We have

$$|\xi(s)|=(\|\psi\|\|\xi\|+N_0)K_1+\|\phi\|\|\xi\|+M_0.$$

Where $N_0=\sup_{s\in[0,a]}n(s,0),\ M_0=\sup_{s\in[0,a]}m(s,0),$ then we have

$$\|\xi\|\le\frac{N_0K_1+M_0}{1-\|\psi\|-\|\phi\|}.$$

That is the set $E=\{\xi\in X,\mu\xi\in A\xi B\xi+C\xi,\mu>1\}$ is bounded, which means the condition (i) of Lemma 2. is true. By Lemma 2, we conclude that (2) has at least one solution on $[0,a]$.

4. Application

Example1 Consider the fractional hybrid inclusion:

$$\begin{aligned}
&{}^{C}D_{0^+}^{\frac{3}{2}}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)\in L(s,\xi(s)),\\
&\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=0}+2\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=1}=\frac{1}{2},\\
&3\,{}^{C}D_{0^+}^{\frac{1}{2}}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=\frac{1}{2}}+\frac{1}{4}\,{}^{C}D_{0^+}^{\frac{1}{2}}\left(\frac{\xi(s)-m(s,\xi(s))}{n(s,\xi(s))}\right)_{s=1}=1,
\end{aligned}\qquad (5)$$

$$n(s,\xi(s))=\frac{\sqrt{\pi}e^{-2\pi s}\cos(\pi s)}{7\pi+15e^{s}}\frac{\xi(s)}{1+\xi(s)}+\frac{s}{10},$$

$$m(s,\xi(s)) = \frac{s^2}{10}(\frac{1}{2}\xi(s) + \sqrt{\xi^2 + 1}),\ s \in [0,1].$$

$$\xi \to L(s,\xi(s)) := [e^{-s}\cos^2\xi(s), \frac{e^{-2s}}{\sqrt{8+s}}\sin\xi(s)],\ \xi \in \mathrm{R}.$$

$$\| L(s,\xi(s)) \| := \sup\{|\eta| : \eta \in L(s,\xi(s))\} \le w(s)\tau(|\xi|) \le |\xi|.$$

$$w(s) = e^{-2s}, \tau(|\xi|) = |\xi|,\quad \phi(s) = \frac{s^2}{100}, \psi(s) = \frac{\sqrt{\pi}e^{-2\pi s}}{7\pi + 15e^s}.$$

then we have $\| w \| = 1, \| \phi \| = \frac{1}{10}, \| \psi \| = \frac{\sqrt{\pi}}{(7\pi + 15)^2}, N_0 = \frac{1}{10}, M_0 = \frac{1}{10}.$

That is $\mathrm{M} > 2.5071.$ By Theorem 1, We conclude that the problem (5) has at least one solution on $[0,1]$.

5. Concluding remarks

The main aim of this paper is to generalize the recent single value problem to the multivalued one. We study the fractional hybrid differential inclusions with tree-point boundary hybrid conditions. Existence of solutions to the problem is discussed. Sufficient conditions of existence of solutions results are presented by a hybrid fixed point theorem of Schaefer type for three operators and an example is given to illustrate our main result. It is noticed that if we take the right hand of the problem (1) L as the multi-valued function, then problem (1) becomes the problem (2).

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