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A Gale-Shapley View of Unique Stable Marriages

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Abstract. Stable marriage of a two-sided market with unit demand is a classic problem that arises in many real-world scenarios. In addition, a unique stable marriage in this market simplifies a host of downstream desiderata. In this paper, we explore a *new* set of sufficient conditions for *unique stable matching* (USM) under this setup. Unlike other approaches that also address this question using the structure of preference profiles, we use an algorithmic viewpoint and investigate if this question can be answered using the lens of the *deferred acceptance* (DA) algorithm without actually running the algorithm. Our results yield a set of sufficient conditions for USM (viz., MaxProp and MaxRou) and show that these are disjoint from the previously known sufficiency conditions like *sequential preference* and *no crossing*. We provide a characterization of MaxProp that makes it efficiently verifiable (without using DA), and shows the gap between MaxProp and the entire USM class.

1 Introduction

The stable marriage problem considers a two-sided market where agents of each side (e.g., men) are assumed to have a linear preference over the other side (e.g., women) and matches are one-toone, i.e., each agent has a single demand. Stability asks for a pairing between these agents such that there does not exist any pair of a man and a woman who would like to abandon the current matching and mutually prefer a marriage among themselves. Gale and Shapley [10] proved that such a stable matching always exists and is obtained via a computationally simple algorithm called deferred acceptance (DA). However, there could be multiple stable matchings and it raises questions on which one to pick. The stable matching problem is very well studied in the literature and several useful results exist related to DA and its variants. For instance, the questions regarding the maximum [21] or average number of stable matchings [30], complexity of counting stable marriages [18], matching with incomplete lists [20], indifferences [26], heterogeneous jobs and workers [7], and many more, have already been investigated. See Iwama and Miyazaki [19], Manlove [25] for a comprehensive survey on the stable matching problem and Roth [34] for a survey of DA-type algorithms.

In this context, uniqueness of stable matching [8, 6] has a very important place. First, since the actual pairings of men and women are a stable matching based on their *reported* preferences, a normative goal is to ensure that it is indeed their *actual* preferences, i.e., the stable matching algorithm is *strategyproof*. However, it is known that DA is not strategyproof for a non-proposer [11] unless there is

a unique stable matching. Though a unique stable matching is not sufficient for strategyproofness [33] except in the incomplete information setup [9], it is a property from which further structures of strategyproofness can be obtained. We define the class of preference profiles where the set of stable matchings is a singleton as *unique stable matching* (USM) in this paper.

The second reason why USM is desirable is the anti-symmetry of the preferences of men and women over the stable matchings. It is known that between two different stable matchings μ_1 and μ_2 , if μ_1 is at least as preferred as μ_2 by all men, then μ_2 must be at least as preferred as μ_1 by all women, i.e., men and women have exactly opposite preferences over the stable matchings [12]. Hence, finding a stable matching that is *unbiased* to any side of the market is often challenging. A considerable amount of research effort has been put to find a *fair* compromise between the two extremes (see, e.g., Klaus and Klijn [23], Tziavelis et al. [37], Brilliantova and Hosseini [4]). However, the question of bias also does not appear in the USM class since there is exactly one stable matching.

Finally, unique stable matchings have appeared in many real-world matching markets, e.g., in the US National Resident Matching Program [35], Boston school choice [29], online dating [16], and the Indian marriage market [3].

In this paper, we aim to understand the internal structure of the USM class using a DA algorithmic lens.

1.1 Our contributions

The main contributions of this paper are as follows (illustrated graphically in Figure 1).

- We view the USM problem using the number of proposals and rounds in the classic Gale-Shapley DA algorithm, and introduce two new conditions, *men-proposing max-proposals* or m-MaxProp and *men-proposing max-rounds* or m-MaxRou (similarly w-MaxProp and w-MaxRou for the women-proposing versions), defined w.r.t. men (women)-proposing DA. These properties identify those preference profiles where a men (women)-proposing DA algorithm takes maximum possible number of proposals or rounds respectively. We show the mutual relationship of these two properties in Theorem 2 when the number of men (or women) |M|(= |W|) = n ≥ 3. We show that each of these conditions is *sufficient* for USM (Theorem 3).
- We characterize the class MaxProp in Theorem 4 and show that these conditions are efficiently verifiable (Theorem 6) without requiring any appeal to DA.
- Prominent existing sufficient conditions for USM, the sequential preference condition (SPC [8]) and the no crossing condition

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Figure 1: The above two figures illustrate the sub-structures of the USM class for $n \ge 3$ and n = 2 respectively. For n = 3, however, the gap between MaxProp and MaxRou is empty as they are equivalent. (This gap is non-empty for $n \ge 4$.) The dashed lines and the shaded regions denote the new sub-structures of USM that are contributions of this paper. We also characterize the class MaxProp and provide the complexity of verification. In Figure 1b, the fact USM = SPC was known from Eeckhout [8]. We provide a more direct proof of this fact.

(NCC [6]), are disjoint from the new sufficient conditions proposed in this paper for $n \ge 3$ (Theorem 7). Hence, our results make the internal sub-structure of the USM class outside NCC and SPC clearer. In particular, the men and women proposing versions of the MaxProp class turn out to be disjoint as well (Theorem 8).

• For n = 2, we show that the classes m-MaxProp and m-MaxRou (similarly w-MaxProp and w-MaxRou) coincide, and so do SPC and NCC. Also, m-MaxProp and w-MaxProp are contained within SPC for n = 2. We also provide a direct proof of the fact that for n = 2, USM and SPC are equivalent, a result originally proved by Eeckhout [8]. However, we also point out an inconsistency in the claim of SPC being necessary for USM for n = 3 [8] through Example 3. Interestingly, for n = 2, the classes m-MaxProp and w-MaxProp have an overlap. Due to space limitations, we prove these results in the full version of this paper [13].

1.2 Related works

Several works have focused on finding sufficient conditions for USM, the two most well-known of these being the sequential preference condition [8] and the no crossing condition [6]. Others include the co-ranking condition [24], the acyclicity condition [32], the universality condition [17], oriented preferences [31], aligned preferences [27], uniqueness consistency [22], α -reducibility [1], and iterative α -reducibility [28]. These results provide structural insight into the types of preference profiles that lead to uniqueness in stable matchings. Refer to Section 3 for a detailed discussion on previously known sufficient conditions for USM.

It is noteworthy that nearly all of the sufficient conditions listed above are either restrictions or generalizations of the sequential preference condition (SPC) or the no crossing condition (NCC). In contrast, our work unveils two sufficient conditions that have no overlap at all with SPC or NCC, as we prove later.

Finding a necessary condition has also been investigated, and there are two prominent approaches. The first one uses the idea of α -reducibility, proposed originally by Alcalde [1]. A marriage problem satisfies α -reducibility if every sub-population of men and women has a *fixed pair* (a pair of man and woman who prefer each other the most). Clark [6] shows that this condition is both necessary and sufficient for the existence of a unique stable matching in *every sub-*

population of men and women. If USM is considered only for the full population, α -reducibility is sufficient *but not necessary*.

A different approach to this problem uses the idea of acyclicity, originally proposed by Chung [5]. Acyclicity implies that if the agents point to their most preferred partners, then the resulting directed graph should not have any directed cycle. While Romero-Medina and Triossi [32] show that it is a sufficient condition for USM, the necessity condition using this method is explored recently by Gutin et al. [15]. Gutin et al. [15] use the acyclicity on a reduced graph that they define as the normal form. The idea of normal form is used for submatching markets by Irving and Leather [18], and Balinski and Ratier [2]. Gutin et al. [15] claim that the difficulty in finding a necessary condition for USM in these approaches was that the acyclicity property was being used on the complete preference profile, while the entire preference profile may not be relevant for a unique stable matching. Using the idea of normal form, they prune the preferences where an agent can never match with certain partners in any stable matching. This acyclicity on a normal form turns out to be necessary and sufficient for USM [15].

Our approach differs considerably from those discussed above, in the way our conditions are defined. Instead of looking at the USM class through the preference structures of the players, we view it using the DA algorithm and its execution over a profile. Our results consider the maximum number of proposals made by the agents and the number of rounds in DA, and provide the extra structures that yield a clearer view of the space between the currently known sufficient conditions and the USM class (Figure 1). It shows that certain properties of an algorithm can also help clarify the structure of USM, without even running that algorithm.

2 Preliminaries

Consider a two-sided unit-demand matching market, where the two sides are represented, WLOG, by men and women respectively. The agents of each side are expressed as two equi-cardinal finite sets, denoted by M and W, |M| = |W| = n, respectively. The sets share no common agents, i.e., $M \cap W = \emptyset$. All men have strict preferences over all women and vice versa. Individual preferences, denoted \succ_i for agent i, are assumed to be complete, transitive, and anti-symmetric. The notation $m_i \succ_{w_k} m_j$ denotes $w_k \in W$ prefers $m_i \in M$ over $m_j \in M$, and similarly, $w_i \succ_{m_k} w_j$ denotes $m_k \in M$ prefers $w_i \in W$ over $w_j \in W$. The preference profile is denoted by $\succ := \{\succ_i : i \in M \cup W\}$. The set of all complete, transitive, and anti-symmetric preference profiles in this setup is denoted by \mathcal{P} . A *matching* and several other definitions in this setting follow Gale and Shapley [10].

Definition 1 (Matching). A matching in \succ is a mapping μ from $M \cup W$ to itself such that for every man $m \in M, \mu(m) \in W$, for every woman $w \in W, \mu(w) \in M$, and for every $m, w \in M \cup W, \mu(m) = w$ if and only if $\mu(w) = m$.

The above definition says that each man is matched to exactly one woman and vice-versa. To define stability of a matching, we need the definition of *blocking pair* as given below.

Definition 2 (Blocking Pair). A pair $(m, w), m \in M, w \in W$ is a blocking pair of a matching μ in \succ if $m \succ_w \mu(w)$ and $w \succ_m \mu(m)$.

Informally, the above definition means that the pair (m, w) both prefer each other over their currently matched partners. This leads to the definition of stable matching as follows.

Definition 3 (Stable Matching). A matching μ in \succ is stable if it does not have any blocking pair.

Gale and Shapley [10] showed that for any preference profile \succ , a stable matching always exists and can be found via the *deferred acceptance* (DA) algorithm. The working principle of this algorithm is the following. The algorithm comes in two versions based on whether the men or the women are the proposers. In every round of the menproposing DA algorithm, each unmatched man proposes to his favorite woman that has not rejected him already. Each woman, in that round, receives the proposals and *tentatively* accepts the most favorite man that has proposed to her and rejects the rest. The rejected men go to the next round and repeat this activity. The algorithm stops when no man remains unmatched. A formal representation is given in Algorithm 1.

Algorithm 1: (Men-proposing) Deferred Acceptance (DA) **Input:** men $M = \{m_1, ..., m_n\}$, women $W = \{w_1, ..., w_n\}$, and preferences $\succ = \{\succ_i : i \in M \cup W\}$ **Output:** a stable matching μ 1 for $i \in M \cup W$ do **2** | $\mu(i) \leftarrow \emptyset$ **3 while** $\exists m \in M$ such that $\mu(m) = \emptyset$ **do** // This is a round for $m_i \in M$ such that $\mu(m_i) = \emptyset$ do 4 // This is a proposal $w \leftarrow$ highest woman in \succ_{m_i} to whom m_i has not 5 proposed yet if $\exists m_i \in M$ such that $\mu(m_i) = w$ then 6 7 if $m_i \succ_w m_j$ then $\mu(m_i) \leftarrow w, \, \mu(w) \leftarrow m_i$ 8 9 $\mu(m_j) \leftarrow \emptyset$ else 10 $\mu(m_i) \leftarrow w, \mu(w) \leftarrow m_i$ 11 12 return μ

The following two facts about DA will be used frequently throughout this paper. The proofs are deferred to the full version [13]. **Fact 1.** In the men-proposing DA algorithm, there exists a woman $w \in W$ who receives exactly one proposal.

Fact 2. In the men-proposing DA algorithm, the maximum possible number of proposals is $n^2 - n + 1$, and the maximum possible number of rounds is $n^2 - 2n + 2$. Both the bounds are achievable, i.e., there exists a preference profile $\succ \in \mathcal{P}$ where the above numbers are attained.

Although DA always converges to a stable matching, it is also known that men-proposing DA and women-proposing DA converge to men and women optimal stable matchings respectively, which could be quite different. There is a hierarchy among the stable matchings from the men and women points of view as given by the following result.

Theorem 1 (Gale and Sotomayor [12]). *If for any two distinct stable matchings* μ_1 *and* μ_2 *in* \succ *, if each man finds* μ_1 *at least as preferred as* μ_2 *, then every woman will find* μ_2 *at least as preferred as* μ_1 .

The subclass of \mathcal{P} where the set of stable matchings is a singleton is defined as the *unique stable matching* (USM) class. In USM, the men and women proposing DA reach the same stable matching. Because of the various satisfactory properties exhibited by this class as discussed in Section 1, there had been various attempts to characterize the structures of the preference profiles in USM. In the following section, we introduce two prominent sufficient conditions for USM.

Remark. There are certain necessity results of USM as well, using ideas like α -reducibility [6] and *acyclicity* using a *normal form* of the preferences [15]. However, in this paper, our objective is to view it from a DA algorithmic perspective and we discuss how our results can be applicable without running DA and even in domains with partial preferences (Section 5).

3 Current State-of-the-art Sufficient Conditions

Although there have been various sufficient conditions proposed for USM [32, 14, 31, e.g.], the *sequential preference condition* (SPC, [8]) and *no crossing condition* (NCC, [6]) provide a deeper structural view of the preference profiles of the agents that gives rise to USM.

Definition 4 (Sequential Preference Condition). A preference profile \succ satisfies sequential preference condition (SPC) if there exists an ordering of men, m_1, m_2, \ldots, m_n , and women, w_1, w_2, \ldots, w_n , such that

- 1. man m_i prefers w_i over $w_{i+1}, w_{i+2}, \ldots, w_n$, and
- 2. woman w_i prefers m_i over $m_{i+1}, m_{i+2}, \ldots, m_n$.

Eeckhout [8] showed that SPC is sufficient for uniqueness of stable matching. It is, however, not necessary for $n \ge 3$ as we show in the example below.

Example 1 (USM but not SPC). *Consider the following preference profile.*

This does not satisfy SPC, since SPC needs at least one pair of man and woman that rank each other at the top. However, the menproposing DA yields the matching where m_i is matched with w_i , i = 1, 2, 3, which is the men-optimal matching. However, in this case, it is women-optimal as well since each woman gets her top preference. By Theorem 1, this profile has an unique stable matching, i.e., it belongs to USM.

Later, Clark [6] defined the following refinement that implies SPC.

Definition 5 (No Crossing Condition). A preference profile \succ satisfies no crossing condition (NCC) if there exists an ordering (m_1, m_2, \ldots, m_n) of M and an ordering (w_1, w_2, \ldots, w_n) of W, such that if i < j and k < l, then

1. $w_l \succ_{m_i} w_k \Rightarrow w_l \succ_{m_j} w_k$, and 2. $m_j \succ_{w_k} m_i \Rightarrow m_j \succ_{w_l} m_i$.

This condition implies that if the men and women are lined up in that given order and any pair of men (or women) are asked to point to his (or her) favorite partner among a pair of potential partners, their pointers cannot cross each other. Though NCC implies SPC, the converse is not true for $n \ge 3$ [6]. These sufficient conditions for $n \ge 3$ are shown on the LHS of Figure 1a. SPC and NCC, however, become identical with USM for n = 2 as we discuss later.

Following the discovery of SPC and NCC, various other sufficient conditions for USM have been proposed, the majority of them being either restrictions or generalizations of the former two conditions. For instance, the co-ranking condition of Legros and Newman [24] and the universality condition of Holzman and Samet [17] are both contained within NCC, while the uniqueness consistency condition of Karpov [22] relaxes SPC, and the oriented preferences of Reny [31] and aligned preferences of Niederle and Yariv [27] generalize SPC to many-to-one markets. Even the α -reducibility condition of Alcalde [1] lies between NCC and SPC, and its iterative version from Park [28] coincides with SPC itself.

Since the SPC and NCC conditions clearly take center stage in the current state-of-the-art in sufficient conditions for USM, we limit the comparison of our new conditions to only these two. Example 1 shows that there exists unexplored space in USM \cap SPC. We provide additional structure to that space in this paper.

4 Our Results

This paper considers the USM problem from the DA perspective. We first define two new conditions that we prove to be sufficient for USM. The definitions deal with the number of proposals women get in men-proposing DA and the number of rounds involved. In the rest of the paper, WLOG, we use men-proposing DA whenever we consider DA. However, analogous definitions and results hold for a symmetrically opposite women-proposing version as well. Fact 2 prompts us to define the following two classes of preferences.

4.1 MaxProposals and MaxRounds

These two classes of preferences are defined as follows.

Definition 6 (MaxProp and MaxRou). A preference profile \succ satisfies

- 1. MaxProp, if the proposers make $n^2 n + 1$ proposals in DA, and
- 2. MaxRou, if the proposing process in DA happens for $n^2 2n + 2$ rounds.

Note that, the above two classes are critically dependent on the proposing side. We will denote the classes where the maximum number of proposals (and rounds) are coming from the men-proposing DA as m-MaxProp (and m-MaxRou) respectively. The womenproposing versions of the classes will be denoted as w-MaxProp and w-MaxRou respectively. In the rest of the paper, WLOG, we will imply the men-proposing versions of MaxProp and MaxRou respectively when we refer to them and prove their properties. The results for the women-proposing versions are identical and are skipped. However, in Section 6.2, we show that the classes m-MaxProp and w-MaxProp are disjoint for $n \ge 3$. Interestingly, these two classes partially overlap for n = 2, and we discuss it in Section 6.3. Our first result shows the relationship between the classes MaxProp and MaxRou.

Theorem 2. If a preference profile \succ satisfies MaxRou, then \succ also satisfies MaxProp.

Proof. WLOG, assume men-proposing DA in this case. Suppose a preference profile \succ satisfies MaxRou. This implies that if we run the men-proposing DA algorithm, it would take $n^2 - 2n + 2$ rounds to terminate. We make the following observations directly from the algorithm.

- The first round involves *n* proposals as nobody is matched yet, i.e., each man makes a proposal.
- Each round (except the last one) must see at least one man getting rejected, else the termination criterion of the algorithm is met, and thus, every round (except the first one) has at least one proposal.

Hence, the total number of proposals in \succ is $\ge n + n^2 - 2n + 1 = n^2 - n + 1$. By Fact 2, we know that the number of proposals is at most $n^2 - n + 1$. Hence, the number of proposals in \succ must be $= n^2 - n + 1$. Therefore \succ satisfies MaxProp.

The converse of the above theorem is not true for $n \ge 4$ as the following example shows. For n = 3, MaxProp implies MaxRou and we prove this in the full version [13].

Example 2 (MaxProp but not MaxRou for $n \ge 4$). Consider the following preference profile involving four men and four women.

$m_1:$	$w_1 \succ w_2 \succ w_3 \succ w_4$	$w_1:$	$m_2 \succ m_3 \succ m_4 \succ m_1$
m_2 :	$w_3 \succ w_2 \succ w_1 \succ w_4$	w_2 :	$m_3 \succ m_4 \succ m_1 \succ m_2$
m_3 :	$w_3 \succ w_1 \succ w_2 \succ w_4$	w_3 :	$m_4 \succ m_1 \succ m_2 \succ m_3$
$m_4:$	$w_1 \succ w_2 \succ w_3 \succ w_4$	$w_4:$	$m_1 \succ m_2 \succ m_3 \succ m_4$

In this example, two men $(m_1 \text{ and } m_3)$ get rejected in the first round of DA. Both these men propose in the next round and it is easy to check that the number of proposals for this profile is $n^2 - n + 1 = 13$. However, MaxRou requires one proposal per round (thereby increases the number of rounds to the maximum). But here we have two proposals in round 2, which fails MaxRou.

We now state an important lemma which will be used in the following subsections to prove several properties of MaxProp. The result gives structure to the proposals observed in profiles satisfying MaxProp.

Lemma 1. WLOG, let w_n be the woman who receives exactly one proposal in men-proposing DA on \succ . If $\succ \in$ MaxProp, then all men $m \in M$ propose to all women in $W \setminus \{w_n\}$.

Proof. Since $\succ \in$ MaxProp, we have $n^2 - n + 1$ proposals. Since w_n receives exactly one proposal, the other n - 1 women receive a total of $n^2 - n$ proposals. No woman can receive more than n proposals (since there are n men). Hence, the only way n - 1 women can receive $n^2 - n$ proposals is if each woman in $W \setminus \{w_n\}$ receives n proposals. Thus, all $m \in M$ must propose to all $w \in W \setminus \{w_n\}$. \Box

Notice that, if a woman receives proposals from all men, she is always assigned to her most preferred man according to the menproposing DA. Hence, the following corollary is immediate from the lemma above.

Corollary 1. If $\succ \in$ MaxProp, all women except the one who gets exactly one proposal, get matched with their most preferred men. Formally, if w_n is the woman who gets exactly one proposal, then for all $i \in \{1, ..., n - 1\}$, $\mu(w_i) \succ_{w_i} m_j$ or $\mu(w_i) = m_j$ for all $j \in [n]$.

4.2 MaxProp implies USM

In this section, we prove one of the major results of this paper that provides a new sufficient condition of USM.

Theorem 3. If a preference profile \succ satisfies MaxProp, then \succ is in USM.

Proof. Suppose a preference profile \succ satisfies MaxProp. We show that the (men-optimal) output of men-proposed DA algorithm (say μ) is also women-optimal. It is also known that the men(women)-optimal stable match is unique [36]. Then, together with Theorem 1, μ would be the unique stable matching.

Let w_n be the woman who receives exactly one proposal. By Corollary 1, all other women are matched with their first preferences.

Suppose, there is another stable matching $\mu' \neq \mu$ on the same profile \succ , which is more preferable than μ for women. Then, for all $i \in [n]$, either $\mu'(w_i) \succ_{w_i} \mu(w_i)$ or $\mu'(w_i) = \mu(w_i)$, and for some $j \in [n], \mu'(w_j) \succ_{w_i} \mu(w_j)$.

However, $w_1, w_2, \ldots, w_{n-1}$ are already matched to their first preferences by μ . So, $\mu'(w_i) = \mu(w_i)$ for $i = 1, \ldots, n-1$, and $\mu(w_n)$ has to be the only man remaining who has to be matched to w_n even in μ' . Hence, $\mu = \mu'$, which is a contradiction. Thus, μ is women-optimal, and is the unique stable matching.

However, the converse of the previous theorem is not true. The following example shows that MaxProp is not necessary for USM. In fact, this example does not satisfy SPC either.

Example 3 (USM but neither MaxProp nor SPC). *Consider the following preference profile.*

$m_1:$	$w_1 \succ w_3 \succ w_2$	w_1 :	$m_2 \succ m_1 \succ m_3$
m_2 :	$w_2 \succ w_1 \succ w_3 ;$	w_2 :	$m_3 \succ m_1 \succ m_2$
m_3 :	$w_1 \succ w_2 \succ w_3$	w_3 :	$m_1 \succ m_2 \succ m_3$

Since there is no pair of man and woman (m, w) that prefers each other the highest, it is not SPC. The men-proposing DA takes 6 proposals, while the maximum number of proposals is $3^2 - 3 + 1 = 7$. Hence, this profile does not satisfy MaxProp. However, the menoptimal matching (obtained via men-proposing DA) results in all women receiving their most preferred men, which is women-optimal as well. Therefore, this profile belongs to USM.

From Theorems 2 and 3, the following corollary is immediate.

Corollary 2. If a preference profile \succ satisfies MaxRou, then \succ is in USM.

5 A Characterization of MaxProp

In this section, we find the conditions on preference profiles that are necessary and sufficient for MaxProp. We also show that these certifications of belonging to MaxProp can be done efficiently without involving the DA algorithm. We begin with a few structural properties of MaxProp.

Lemma 2. If a preference profile $\succ \in \mathcal{P}$ satisfies MaxProp, then there must be a woman $w \in W$ who is the least preferred woman for each $m \in M$.

Proof. We prove this result via contradiction. WLOG, suppose woman w_n is the woman who receives exactly one proposal (by Fact 1) when men-proposing DA is run on \succ . Suppose there is a man m_i who does not have w_n as his last preference. Let m_i prefer w_n over some woman w_j , $j \neq n$. Then by Lemma 1 (as \succ satisfies MaxProp), m_i must propose to w_j , and since he prefers w_n over w_j , he must propose to w_n before w_j . But, w_n gets exactly one proposal and never rejects the man that proposes her. So m_i cannot propose to w_j after proposing to w_n , since it requires w_n to reject m_i under DA to make that happen. Hence, we reach a contradiction.

Note that the above lemma claims existence of a woman who is least preferred by every man if the profile satisfies MaxProp. In the proof, we have identified that woman as the woman who receives exactly one proposal in DA.

Lemma 3. Suppose, a preference profile \succ satisfies MaxProp. WLOG, w_n be the woman who is every man's last preference in \succ , and m_n get matched with w_n in men-proposing DA. Then for each $i \in \{1, ..., n-1\}$, w_i 's first preference is some m_j $(j \neq n)$, and m_j 's penultimate preference is w_i .

Proof. From Lemma 2, we know that the woman w_n who is every man's last preference in \succ also receives exactly one proposal in menproposing DA. By Lemma 1 (as \succ satisfies MaxProp), each woman $w_i \in W \setminus \{w_n\}$ gets proposed by every man in M. This implies that she finally gets matched with her most preferred man. Since m_n gets matched with w_n , w_i 's first preference must be some m_i $(j \neq n)$.

Again using Lemma 1, m_j proposes to all (n-1) women in $W \setminus \{w_n\}$, and he makes his last proposal to the woman who is finally matched with him, i.e., w_i . Since, m_j 's least preferred woman is w_n , w_i must be m_j 's penultimate preference in \succ .

Using these results, we will now state a set of conditions that are necessary and sufficient for MaxProp. These conditions also identify the additional structure needed for a preference profile in USM to satisfy MaxProp.

Theorem 4. A preference profile \succ satisfies MaxProp (m-MaxProp, WLOG) if and only if there exists an ordering m_1, \ldots, m_n of M and an ordering w_1, \ldots, w_n of W satisfying the following three conditions:

- 1. w_n is the least preferred woman for each $m_i \in M, i = 1, ..., n$.
- 2. For each $i \in \{1, ..., n-1\}$, w_i 's first preference is m_i , and m_i 's penultimate preference is w_i .
- 3. For each $k \in \{1, ..., n-1\}$, the second preference of w_k is from $\{m_{k+1}, m_{k+2}, ..., m_n\}$.

Before proving this theorem, we make the following observation on condition 3.

Observation 1. Let the second preference of any woman w_{ℓ} be denoted by $s(w_{\ell})$. Define G to be the digraph on vertices $\{1, 2, ..., n-1\}$, with an edge from *i* to *j* if $s(w_i) = m_j$. Then, there exists an ordering of men and women satisfying condition 3 of Theorem 4 if and only if G is acyclic.

The above observation is immediate from the insights that (1) an ordering of men and women satisfies condition 3 of Theorem 4 if and only if it gives a topological ordering for G, and (2) a directed graph has a topological ordering if and only if it is acyclic. We are now ready to prove Theorem 4.

Proof of Theorem 4. (\Rightarrow) : Consider a preference profile \succ that satisfies MaxProp. Since \succ satisfies MaxProp, conditions 1 and 2 of this theorem follow from Lemmas 2 and 3 respectively. We will prove condition 3 by showing that the digraph G as defined in Observation 1 is acyclic. Suppose not. Then, G must have at least one directed cycle C involving at least two vertices. Denote the set of vertices in this cycle as V(C). We will show that there exist two different stable matchings, which contradicts that \succ satisfies MaxProp (since MaxProp implies USM by Theorem 3). Construct a matching μ' as follows. For each edge $i, j \in V(C)$ such that a directed edge exists from i to j in $G, \mu'(w_i) = m_j$. For all the remaining women w_i , where $i \in N \setminus V(C), \mu'(w_i) = m_i$. Note that μ' is a stable matching, because of the following reasons.

- None of the women w_i, where i ∈ C, can form a blocking pair. The only better match the woman w_i can get is to be matched with her first preference m_i (since she is currently matched to her second preference and condition 2 says that her top preference is m_i). But that man m_i has w_i as the penultimate preference (condition 2) and w_n as the last preference (condition 1), and is currently matched with neither of them under μ'. So, m_i does not find this a profitable deviation.
- None of the remaining women w_i , where $i \in N \setminus V(C)$, can form a blocking pair either, since $\mu'(w_i) = m_i$, i.e., they have been matched with their most preferred men (condition 2), with the exception of w_n , who cannot form a blocking pair as she is every man's last preference (condition 1).

However, $\mu(w_i) = m_i$ is also a stable matching, as each w_i gets matched with her most preferred man m_i (except w_n who cannot form a blocking pair due to condition 1). Clearly, $\mu \neq \mu'$, since in μ' , at least two women between $1, \ldots, (n-1)$ are matched with their second most preferred men. Thus, we have found two distinct stable matchings μ and μ' for \succ , which gives us a contradiction to USM (and therefore MaxProp).

(\Leftarrow): Consider a preference profile \succ satisfying all three conditions of this theorem. Pick any stable matching μ on \succ .

First, note that $\mu(w_n) = m_n$, i.e., w_n has to be matched with m_n in every stable matching on \succ . This is because if w_n is matched with $m_i \in M \setminus \{m_n\}$ then (m_i, w_i) forms a blocking pair: m_i 's least preferred woman is w_n (condition 1) and w_i 's most preferred man is m_i (condition 2).

We will prove that $\mu(w_i) = m_i$ for all *i*. Suppose not. Let k be largest such that $\mu(w_k) \neq m_k$. This implies that for all $i \in \{k + 1, k + 2, ..., n\}$, we have $\mu(w_i) = m_i$. Therefore, w_k is matched with neither (a) her first nor (b) her second preference. This is because, (a) condition 2 says that m_k is w_k 's most preferred man, and (b) the second preference of w_k i.e. $s(w_k)$ is from $\{m_{k+1}, m_{k+2}, ..., m_n\}$ (by condition 3) but they are matched with $\{w_{k+1}, w_{k+2}, ..., w_n\}$ respectively (by assumption that k is the largest). But then, w_k can form a blocking pair with $m' := s(w_k)$ that is her second preference, as m' has been matched with his least or penultimate preferences, and would prefer w_k over $\mu(m')$, and we reach a contradiction.

Thus $\mu(m_i) = w_i, \forall i \in N$, is *the* unique stable matching for \succ , and hence the men-proposed DA algorithm must arrive at this match-

ing. According to this algorithm, each man m_i starts with proposing to his most preferred woman and proposes to the next woman in his preference profile every time he gets rejected, until he reaches his penultimate woman w_i (except for m_n , who proposes until he reaches his last preference w_n). Each m_i for $i \in \{1, \ldots, n-1\}$ proposes (n-1) times, and m_n proposes n times, adding up to a total of $(n-1)(n-1) + n = n^2 - n + 1$ proposals. Thus, the preference profile \succ satisfies MaxProp.

This concludes both directions of the proof.

Theorem 4 gives the necessary and sufficient conditions of Max-Prop in the form of three conditions. It is worth asking how critical each of the conditions is. In the full version [13], we provide three examples to show that each of these conditions is tight.

Note that if at any intermediate stage of the men-proposing Gale-Shapley algorithm, k men propose, it can lead to at most k rejections. Only the men who get rejected in a round may propose in the next round. Hence, the following observation is immediate.

Observation 2. If there are k men who propose in a particular round, then at most k men (not necessarily the same men) can propose in all subsequent rounds.

To characterize the distinction between the preference profiles that are MaxProp and MaxRou, we provide the following result that characterizes MaxRou using one additional structural property. The proof is deferred to the full version of this paper [13].

Theorem 5. A preference profile \succ satisfies MaxRou if and only if it satisfies the following conditions

- *1.* \succ satisfies MaxProp (thus, there exists a woman w_n who is least preferred by each man (Lemma 2)), and
- 2. each woman in $W \setminus \{w_n\}$ is a top preference of some man.

Now, a naive way to check if a preference profile \succ satisfies Max-Prop (MaxRou) is to run the DA algorithm and check if it achieves the maximum number of proposals (rounds). This would take $\mathcal{O}(n^2)$ time. But, using the characterization of MaxProp (MaxRou), i.e., Theorem 4 (Theorem 5), we can do much better. Define the following decision problems isMaxProp(\succ) and isMaxRou(\succ) as the problems to determine if \succ satisfies MaxProp and MaxRou respectively.

Theorem 6. For any preference profile \succ , isMaxProp(\succ) and isMaxRou(\succ) can both be checked in $\mathcal{O}(n)$ time.

Proof. First, we consider isMaxProp(\succ). Clearly, condition 1 and 2 of Theorem 4 can be checked in $\mathcal{O}(n)$ time. Now, we know that whether a directed graph G(V, E) is acyclic can be checked in $\mathcal{O}(|V| + |E|)$ time. Consider that graph G, defined in Observation 1, has n - 1 vertices and at most n - 1 edges. Thus, condition 3 can also be checked in $\mathcal{O}(n)$ time. Hence, the first part of this theorem is proved.

For isMaxRou(\succ), note that condition 2 of Theorem 5 can also be checked in $\mathcal{O}(n)$ time. Hence, combining this with the first part of this theorem, we conclude that isMaxRou(\succ) is checkable in $\mathcal{O}(n)$ time.

Discussions. These results help us understand the MaxProp and MaxRou conditions (and thereby USM) better.

(i) The structures look only at partial preferences. The result says we need to know only the top *two* preferred alternatives of one side (say women), the bottom *two* (top one and bottom *two*,

for MaxRou) preferred alternatives of the other side (say men), and be agnostic about the preferences at the other positions. Therefore, we can apply this result on domains with partial preferences as long as the preferences at these positions are known. From a practical viewpoint, depending on the applications, such profiles may show up in practice.

(ii) From the structure given by Theorem 4 (or Theorem 5), it is possible to count what fraction of preference profiles satisfy MaxProp (or MaxRou).

6 Position of MaxProp in the USM space

In this section, we analyze the position of MaxProp (and MaxRou) in the class of all preference profiles satisfying USM, relative to known structures in this space.

6.1 MaxProp is disjoint from SPC for $n \ge 3$

In this section, we address the relative positions of the SPC and MaxProp classes within the space of USM. We show that these two classes are disjoint.

Theorem 7. For $n \ge 3$, there does not exist any preference profile $\succ \in \mathcal{P}$ that satisfies both SPC and MaxProp.

Proof. Suppose, there exists a preference profile \succ that satisfies both the SPC and MaxProp. By definition of SPC, there exists an ordering of men and women such that

- 1. man m_i prefers w_i over $w_{i+1}, w_{i+2}, \ldots, w_n$, and
- 2. woman w_i prefers m_i over $m_{i+1}, m_{i+2}, \ldots, m_n$.

Hence, m_1 will be proposing to only w_1 , who will never reject him, as he is her top preference. Thus, m_1 makes only one proposal. Since MaxProp holds, we know there are a total of $n^2 - n + 1$ proposals to be made. Hence, the remaining n - 1 men make $n^2 - n$ proposals, which means each man makes $(n^2 - n)/(n - 1) = n$ proposals. Since in the men-proposed deferred acceptance algorithm, no man proposes to the same woman twice, each woman has to receive a proposal from all (n - 1) men, i.e., each woman receives $\ge n - 1$ proposals. Thus, there is no woman who receives exactly one proposal, and this contradicts Fact 1. Hence we have the theorem.

Discussions. This result naturally implies that for $n \ge 3$, the classes SPC and MaxRou, NCC and MaxProp, as well as NCC and MaxRou are mutually disjoint (see Figure 1 for an illustration).

6.2 m-MaxProp and w-MaxProp are disjoint for $n \ge 3$

In this section, we show that the MaxProp classes generated by menproposing and women-proposing DA are disjoint when there are at least *three* agents on each side of the market.

Theorem 8. For $n \ge 3$, there does not exist any preference profile $\succ \in \mathcal{P}$ that satisfies both m-MaxProp and w-MaxProp.

Proof. Suppose there exists a preference profile $\succ \in \mathcal{P}$ satisfying both m-MaxProp and w-MaxProp. Consider the men-proposing DA algorithm on \succ . Since \succ satisfies m-MaxProp, by Corollary 1, each $w \in W \setminus \{w_n\}$ is matched with her most preferred man, where w_n is the woman receiving exactly one proposal.

Using Theorem 3, we also know that \succ satisfies USM, i.e., menproposing DA and women-proposing DA arrive at the same matching. Hence, women-proposing DA on \succ yields a matching in which each $w \in W \setminus \{w_n\}$ is matched with her most preferred man, by making only one proposal. The remaining woman w_n can make at most n proposals. Thus, women-proposing DA on \succ can have at most $1 \times (n-1) + n = 2n - 1$ proposals.

Further, \succ satisfies w-MaxProp, which means women-proposing DA on \succ involves $n^2 - n + 1$ proposals (Fact 2). In order for this to happen on \succ , it must hold that $n^2 - n + 1 \leq 2n - 1$, or $n^2 - 3n + 2 \leq 0$. However, we know that for $n \geq 3$, $n^2 - 3n + 2 > 0$. Hence, we have a contradiction.

Therefore, for $n \ge 3$, there is no $\succ \in \mathcal{P}$ satisfying both m-MaxProp and w-MaxProp.

6.3 The curious case of n = 2

When the number of agents in each side is two, the structure of these spaces looks very different. The classes MaxProp and MaxRou become identical, while SPC and NCC become identical with USM. Quite surprisingly, MaxProp becomes a subset of SPC. Moreover, unlike the $n \ge 3$ case, here the m-MaxProp and w-MaxProp classes overlap partially.

Collecting all these results, the space of these conditions is graphically shown in Figure 1b. Our results are derived from the following theorems. All proofs are deferred to the full version of the paper [13].

Theorem 9 (MaxProp = MaxRou). For n = 2, every preference profile \succ satisfying MaxProp also satisfies MaxRou.

Theorem 10 (SPC = NCC). For n = 2, every preference profile \succ satisfying SPC also satisfies NCC.

Theorem 11 (SPC = USM). For n = 2, preference profile \succ satisfies SPC if and only if it is in USM.

Theorem 12 (MaxProp \subset SPC). For n = 2, every preference profile \succ satisfying MaxProp also satisfies SPC.

It is known that for n = 2, SPC also becomes necessary for USM [8]. In the full version [13], we show a direct proof of this fact. Note that Eeckhout [8] claims SPC to be necessary for USM even for n = 3, which is not true. As we show in Example 3, there are profiles for n = 3 that belong in USM but not SPC.

In the full version [13], we also show the relative structures of m-MaxProp and w-MaxProp for the n = 2 case, proving that they overlap partially (unlike the case of $n \ge 3$).

7 Conclusions and Future Work

We considered the USM problem from a Gale and Shapley [10] *deferred acceptance algorithmic* perspective. The properties like Max-Prop and MaxRou that count the number of proposals and rounds respectively in this algorithm yield novel insights into the structure of USM. Both the MaxProp and MaxRou properties are computationally easy to verify (Theorem 4) without invoking the DA algorithm. In addition, these conditions carve out a different and unexplored sub-space of USM (see Figure 1). The variation of these spaces for n = 2 and $n \ge 3$ is interesting.

As a future plan, we would like to see if any algorithmic property (of not necessarily DA) can explain the whole of the USM class and if there exists an efficient (better than DA) algorithm that can identify USM. We would also like to explore the generalization of our results to the setting of many-to-one matchings.

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Ethics Statement

This paper adheres to the principles of research ethics, research integrity, and social responsibility. The research conducted in this paper is purely theoretical, and no human or animal subjects were involved in the research. Therefore, there were no ethical concerns related to the treatment of participants or the use of personal data. Also, designing algorithms for stable marriage is quite a standard goal both in computer science and economics. Therefore the results presented in this paper do not consider anything that adversely impacts society.

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