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Approximate Mechanism Design for Facility Location with Multiple Objectives

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Abstract. We identify strategy proof mechanisms for facility location that simultaneously approximate well *both* the maximum distance from the nearest facility *and* the minimum utility of any agent. Somewhat surprisingly, while the deterministic MEDIAN and the randomized ENDORAV mechanisms perform optimally with respect to approximating the maximum distance, neither perform optimally with respect to approximating the minimum utility. With deterministic mechanisms for locating a single facility, we prove that the MIDORNEAREST mechanism is optimal with respect to approximating both the maximum distance and the minimum utility. By comparison, the MEDIAN mechanism has an unbounded approximation ratio for approximating the minimum utility. With randomized mechanisms for locating a single facility, we construct the first mechanism that is optimal with respect to approximating the minimum utility. For deterministic and randomized mechanisms locating two or more facilities, we identify strategy proof mechanisms that are within a constant factor of optimal with respect to both objectives.

1 Introduction

In many mechanism design problems, a fundamental tension exists between truthfulness and optimality [22, 15, 14]. Returning the optimal solution can give agents an incentive to mis-report in order to influence the outcome in their favour. For example, when scheduling independent tasks on unrelated parallel machines, no mechanism can both be strategy proof and minimize the make-span [19, 6]. One response to this tension is to keep strategy proofness but return approximate instead of optimal solutions. For example, for the paradigmatic social choice problem of locating a facility on a line, Procaccia and Tennholtz [20] prove that the MEDIAN mechanism, which locates the facility at the median agent, is strategy proof and 2-approximates the optimal maximum distance any agent needs to travel, and that no deterministic and strategy proof mechanism can provide a better approximation. Many other researchers have since subsequently explored how well strategy proof mechanisms can approximate the optimality of the total or maximum distance agents travel in more complex settings such as with more facilities, other metrics (e.g. circles and Euclidean space), and constraints such as capacity limits on the number of agents served by a facility.

In a recent survey of mechanism design for facility location problems, this setup – strategy proof mechanisms which approximate well the distance that agents travel – has been so frequently explored that Chan *et al,* [4] called it the "classic setting" for approximate mechanism design. Approximability results in this classic setting have identified strategy proof mechanisms for locating one or more facilities which approximate well the distances agents travel to be served. For example, the unique deterministic and strategy proof mechanism for locating two facilities on the line with a bounded approximation ratio for either the optimal maximum or total distance is the ENDPOINT mechanism, and no deterministic and strategy proof mechanism has a bounded approximation ratio for three or more facilities [10].

In more complex metrics than the line, the picture for strategy proof mechanisms with good approximation performance is bleaker. Even on a simple star metric, there exists no deterministic and strategy proof mechanism with a bounded approximation ratio for two facilities and just three agents [10]. One of our contributions is to show that it is premature to suppose that strategy proofness and good approximation ratios are often incompatible for the facility location problem. If we change the objective from optimizing distances travelled to the utility of agents, good approximation ratios can be achieved without sacrificing strategy proofness.

One main contribution, however, addresses a fundamental shortcoming of the classic setting. Maximizing the distance of agents from the nearest facility focuses on problems where agents are close to facilities and distances are small. To achieve good approximation ratios, a mechanism must return high quality solutions on such problems. This ignores problems which are arguably more challenging where some agents are necessarily some distance from the nearest facility. We therefore propose extending the classic setting to overcome this shortcoming. In particular, we initiate the study of egalitarian facility location where we look to approximate well *both* the maximum distance and the minimum utility.

Our more extended analysis considers both high quality solutions where all distances are small and utilities are large, as well as low quality solutions where some distances are large and utilities are small. Note that Han, Jerrett, and Anshelevic [12] have previously studied mechanisms for facility location that simultaneously approximate well multiple objectives. However, their work is limited to exploring multiple *distance based* objectives (e.g. approximating simultaneously both the maximum and sum of distances). It does not consider, as we do, utility based objectives such as the minimum utility. Their study therefore suffers from the limitations identified here that approximating distance based objectives alone focuses on settings where distances are small, and ignores challenging instances where distances are necessarily large.

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1.1 Results Overview

We provide a comprehensive account of strategy proof mechanisms for egalitarian facility location that optimize multiple objectives simultaneously. In particular, our results identify deterministic and randomized strategy proof mechanisms with good approximation ratios with respect to both the objectives of minimizing the maximum distance and maximizing the minimum utility. Our results demonstrate that the landscape of strategy proof mechanisms with good approximation ratios is more complex than that identified when considering just a single objective of distance. For a single facility, our results are optimal. For multiple facilities, our results are asymptotically tight and within a constant factor of optimal.

Surprisingly, with respect to the worst off agent, neither the deterministic MEDIAN nor the randomized ENDORAV mechanisms always perform well. For locating a single facility with a deterministic mechanism, we prove that the MIDORNEAREST mechanism is optimal with respect to optimizing both maximum distance and minimum utility. By comparison, the MEDIAN mechanism has an unbounded approximation ratio of the optimal minimum utility. We also close an open problem by showing that the MIDORNEAREST is optimal with respect to minimum happiness, a closely related objective to minimum utility. For locating a single facility with a randomized mechanism, we construct the first known mechanism that is optimal with respect to approximating the minimum utility.

2 Optimizing egalitarian welfare

The egalitarian or Rawlsian rule picks the outcome maximizing the minimum utility of any agent. To define utility in the facility location problem, we assume agents and facilities are on a finite interval and not the infinite real line. We suppose the interval this is $[0, 1]$ but it could be $[a, b]$ if we normalise by $b - a$. Supposing agents and facilities lie on a finite interval is interesting for multiple reasons. In practice, agents and facilities are often limited to a finite region due to physical constraints or to limit misreports. When locating electric charging stations in a a warehouse, robots and the charging stations might have to be in the warehouse. When locating warehouses in a distribution network, we might be limited to major intersections. When locating a school, we might have to be within the boundaries of the town. When locating ferry stops on a river, the stops must be locations on the river. And when setting a thermostat in a classroom, the temperature must be within the limits of the boiler. There are thus many settings where agents are limited to a finite interval. Restricting agents to a finite interval also limits how much agents can misreport their location to influence the outcome. A finite interval has been used in several other recent studies (e.g. [2, 16, 17, 13]).

Utilities have previously been used in other closely related settings such as obnoxious facility location where they are a natural way to model such problems (e.g. [5]). In mixed and hybrid problems where certain facilities may be obnoxious to some or all agents, utilities again provide a more consistent modelling tool (e.g. [7]). Utilities have also been studied in operation research when the focus is typically on goals such as optimisation and robustness rather than mechanism design. For example, they are naturally used in multi-objective problems where we combine distance goals with other (utility) objectives. When we consider both distances and utilities, you might be concerned that our analysis will now put too much importance on instances where some agents are far away from any facility and have small utility. To achieve good approximation ratios, don't we now have to ensure that such agents have an utility close to the optimal minimum utility? Have we not just exchanged one problem (approximating distances close to zero) with another (approximating utilities close to zero)? And won't this mean that when distances are large and utilities small, a small change in the location of the facility has an outsized influence on the approximation ratio?

This is not the case. In fact, the opposite is true. Such concerns are an issue when considering distances but not when considering utilities. With distances, the optimal facility location can result in maximum distances that are small or even zero. To achieve good approximation ratios, we must approximate such instances well (and this means that good, even bounded, approximation ratios of the optimal maximum distance are sometimes hard to achieve). With utilities, on the other hand, optimal minimum utilities are never small. Indeed, for any facility location problem, the minimum utility of any agent given an optimal facility location is at least $1/2$. Therefore optimal minimum utilities are never close to zero and, as we will show, approximation ratios are not dominated by the challenge of approximating well small utility values.

3 Facility location problem

In a facility location problem, we need to decide where to locate one or more facilities to serve a set of agents. We consider n agents located at x_1 to x_n with $x_i \in [0, 1]$. Without loss of generality, we suppose $x_1 \leq \ldots \leq x_n$. A deterministic mechanism f locates the m facilities at y_1 to y_m. Formally, $f(\langle x_1, \ldots, x_n \rangle) = \langle y_1, \ldots, y_m \rangle$. Agents are served by the nearest facility. A randomized mechanism returns a lottery over such deterministic outcomes. We let d_i be the distance of agent i to their nearest facility: $d_i = \min_j |x_i - y_j|$. As in [2], since agents and facilities are on the interval $[0, 1]$, we define u_i, the utility of agent i as $1 - d_i$. With randomized mechanisms, we consider the expected value of distances and utilities.

A simple fairness property is anonymity. A mechanism is *anonymous* iff permuting the order of the agents does not change the outcome. Formally f is anonymous iff for any permutation σ, we have $f(\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)} \rangle) = f(\langle x_1, \ldots, x_n \rangle)$. Another important property is resistance to manipulation. A mechanism is *strategy proof* iff no agent can mis-report her location and reduce her (expected) distance to the nearest facility. Formally f is strategy proof iff for any agent i, it is not the case that there exists x'_i with $\min_{y_j \in f(\langle x_1, \ldots, x_n \rangle [x'_i/x_i])} |x_i - y_j| < d_i$ where $[x'_i/x_i]$ substitutes x'_i for x_i in the vector input $\langle x_1, \ldots, x_n \rangle$. We will consider how well strategy proof mechanisms approximate objectives like the optimal maximum distance or minimum utility. A mechanism has an approximation ratio ρ for a maximization (minimization) objective iff the objective it returns is at least $1/\rho$ (at most ρ) times the optimal.

We consider a number of anonymous and strategy proof mechanisms. Many are based on the median function $median(z_1, \ldots, z_p)$ which returns z_i where $|\{j | z_j < z_i\}| < \lceil p/2 \rceil$ and $|\{j | z_j > z_i\}| \leq \lfloor p/2 \rfloor$. For example, the GENMEDIAN mechanism locates a facility at $median(x_1, \ldots, x_n, z_1, \ldots, z_{n-1})$ where the $n - 1$ parameters z_1 to z_{n-1} represent "phantom" agents. A classic result about voting rules due to Moulin shows that this mechanism is strategy proof [18]. The LEFTMOST (RIGHTMOST) mechanism is an instance of GENMEDIAN with parameters $z_i = 0$ ($z_i = 1$ for $i \in [1, n)$, locating the facility at the leftmost (rightmost) agent. The MEDIAN mechanism is an instance of GENMEDIAN with parameters $z_i = 0$ for $i \leq \lfloor n/2 \rfloor$ and 1 otherwise, locating the facility at the median agent. The MIDORNEAREST mechanism is an instance of GENMEDIAN with parameters $z_i = 1/2$ for $i \in [1, n)$. It locates the facility either at $1/2$ if $x_1 \leq 1/2 \leq x_n$, otherwise at the agent nearest to $1/2$.

An interesting class of GENMEDIAN mechanisms is the class of PERCENTILE mechanisms which, given parameter p, locate the facility at $x_{1+\lfloor p(n-1)\rfloor}$ (see, for instance, [21]). Any PERCENTILE mechanism is an instance of GENMEDIAN with parameters $z_i = 0$ or 1. The LEFTMOST mechanism is a PERCENTILE mechanism with $p = 0$, while the MEDIAN mechanism is an instance with $p = {}^1\!/_2$. The PERCENTILE mechanism extends to two or more facilities in the obvious way. For instance, given parameter p_1 and p_2, the PERCENTILE mechanism locates one facility at $x_{1+\lfloor p_1(n-1)\rfloor}$ and the second at $x_{1+\lfloor p_2(n-1)\rfloor}$. The ENDPOINT mechanism is a PERCENTILE mechanism with $p_1 = 0$ and $p_2 = 1$, locating facilities at the leftmost and rightmost agents. We denote a PERCENTILE mechanism for three or more facilities as an ENDPOINT mechanism iff the parameters include 0 and 1.

We propose two new mechanisms for locating two facilities that can be viewed as cousins of the MIDORNEAREST mechanism for locating a single facility. The THIRDORNEAREST mechanism mechanism locates one facility at ${}^1\!/_3$ if $x_1 < {}^1\!/_3$ and x_1 otherwise, and the other facility at ${}^2\!/_3$ if $x_n > {}^2\!/_3$ and x_n otherwise. Similarly the QUARTERORNEAREST mechanism locates one facility at ${}^1\!/_4$ if $x_1 < {}^1\!/_4$ and x_1 otherwise, and the other facility at ${}^3\!/_4$ if $x_n > {}^3\!/_4$ and x_n otherwise.

We also consider randomized mechanisms that return a lottery over solutions. For a single facility, the randomized ENDORAV mechanism selects the leftmost agent with probability ${}^1\!/_4$, the midpoint between the leftmost and rightmost agent with probability ${}^1\!/_2$, and the rightmost agent otherwise (i.e. with probability ${}^1\!/_4$). This is strategy proof [20]. For two facilities, the ENDSORAV mechanism (named "Mechanism 2" in [20]) works as follows. Let $mid = {}^{(x_1+x_n)}\!/_2$, $l = \max_i\{i \mid x_i \leq mid\}$, $r = \min_i\{i \mid x_i \geq mid\}$, $\Delta = \max(x_l - x_1, x_n - x_r)$. Then the mechanism constructs the following probability distribution of facility locations: x_1 and x_n with probability ${}^1\!/_2$, $x_1 + \Delta$ and $x_n - \Delta$ with probability ${}^1\!/_6$, and $x_1 + {}^\Delta\!/_2$ and $x_n - {}^\Delta\!/_2$ otherwise (i.e. with probability ${}^1\!/_3$). This gives a solution with a maximum distance of ${}^{5\Delta}\!/_6$ in expectation compared to an optimal of ${}^\Delta\!/_2$. This mechanism is strategy proof.

Finally, for any number m of facilities, the randomized EQUALCOST mechanism [9] has three steps. In the first step, it computes an optimal covering of all agent locations with m disjoint intervals $[\alpha_i, \alpha_i + p]$ which minimizes the interval length p. WLOG we assume that $\alpha_i < \alpha_{i+1}$. In the second step, it constructs a random bit uniformly $z \in [0, 1]$. In the third and final step, it places a facility at $\alpha_i + zp$ for i odd, and $\alpha_i + (1-z)p$ for i even. The EQUALCOST mechanism is strategy proof and has a bounded approximation ratio for the maximum distance for any number of facilities. We recall that a bounded approximation ratio cannot be achieved with any deterministic mechanism and three or more facilities.

4 Classic setting

Starting with Procaccia and Tennholtz [20], studies of strategy proof mechanisms for facility location have mostly focused on the "classic setting" of approximating the total and maximum distance. For example, Ferraioli, Serafino and Ventre [8] consider what verification conditions can be applied to reports of agents to ensure good approximation bounds on facility location problems. However, their analysis is limited to total and maximum distance. As a second example, Aziz *et al.* [1] and Walsh [24] consider the impact of capacity constraints on the design of strategy proof mechanism for facility location but again limit their analysis to the "classic setting" of approximating the total and maximum distance. As a third example, Golomb and Tzamos [11] identify tight additive approximation guarantees for maximum and total distance when locating a single facility on the line, and almost tight additive approximation guarantees with multiple facilities.

For a single facility, the MEDIAN mechanism returns the optimal total distance, and 2-approximates the maximum distance, and no other deterministic and strategy proof mechanism can do better [20]. For two facilities, the ENDPOINT mechanism is the only deterministic and strategy proof mechanism with a bounded approximation ratio for the total or maximum distance [10]. For three or more facilities, no deterministic and strategy proof mechanism has a bounded approximation ratio for the total or maximum distance [10]. Randomized mechanisms can do better. For instance, the randomized EQUALCOST mechanism is strategy proof and has an approximation ratio of 2 for maximum distance, and of n for total distance with any number of facilities [9].

Aziz *et al.* [3] consider utilities in facility location problems, identifying the unique strategy proof mechanism that satisfies unanimity and proportional fairness. Their focus is on fairness objectives like proportionality while our focus is on egalitarian solutions. Walsh [23] has also looked at optimizing the minimum utility for facility location, but on a different and more specialized problem where facilities are limited in their location (e.g. facilities must be located at discrete integer points). Unlike here, this study does not identify mechanisms that optimize well *both* the maximum distance and the minimum utility. Results on approximation ratios in [23] also do not apply here as the space of mechanisms when facilities are limited in their location is smaller. We also study here classes of mechanisms not considered in [23] such as randomized mechanisms.

While our results are limited to the 1-d setting, they are interesting for a wide variety of reasons. The 1-d problem models several real world settings such as locating power stations along a river or distribution warehouses along a highway. There are also non-geographical settings that can be viewed as 1-d problems (e.g. choosing the temperature for a classroom, or selecting a committee of people with different political views). In addition, we can use the 1-d problem to solve more complex problems (e.g. decomposing the 2-d rectilinear problem into a pair of 1-d problems). Finally, the 1-d problem is the starting point to consider more complex metrics (e.g., trees and networks) and provides bounds on solutions for these more complex settings (e.g. lower bounds on the 1-d setting map onto the 2-d problem).

5 Deterministic mechanisms

We begin our study of strategy proof mechanisms that approximate well both the maximum distance and minimum utility of any agent with deterministic and strategy proof mechanisms.

5.1 Single facility

Procaccia and Tennenholtz [20] prove that any PERCENTILE mechanism (such as MEDIAN or LEFTMOST) 2-approximates the optimal maximum distance, and no deterministic and strategy proof mechanism can do better. In fact, it is not difficult to show that any GENMEDIAN mechanism (not just any PERCENTILE mechanism) 2-approximates the optimal maximum distance. PERCENTILE mechanisms do not perform as well at approximating the optimal minimum utility. Indeed, their approximation ratio of the optimal minimum utility is unbounded. To see this, consider a PERCENTILE mechanism on a problem with one agent at 0, and another agent at 1. We can, however, identify mechanisms that do better. In particular, the MIDORNEAREST mechanism is optimal with respect to approximating both the maximum distance and the minimum utility.

Theorem 1. *The* MIDORNEAREST *mechanism* $^3/_2$*-approximates the optimal minimum utility, and* 2*-approximates the optimal maximum distance. No deterministic and strategy proof mechanism has smaller approximation ratios for either objective.*

Proof. The MIDORNEAREST mechanism is a GENMEDIAN mechanism so, like all GENMEDIAN mechanisms, 2-approximates the optimal maximum distance. To compute the approximation ratio of the minimum utility, we consider three cases. In the first case, $x_1 \leq {}^1/_2 \leq x_n$ and the facility is located at $^1/_2$. We suppose $1-x_n \leq x_1$. The case $1-x_n > x_1$ is symmetric. The minimum utility is $^3/_2 - x_n$ units, compared to an optimal of $1 - {}^{(x_n - x_1)}/_2$ units. The approximation ratio is therefore $(1 - {}^{(x_n - x_1)}/_2)/({}^3/_2 - x_n)$. The worst case for this ratio is when $x_1 = {}^1/_2$ and $x_n = 1$, and the approximation ratio is $^3/_2$. In the second case, $x_n < {}^1/_2$. The minimum utility is $1 - (x_n - x_1)$ units. The optimal minimum utility is $1 - {}^{(x_n - x_1)}/_2$ units. Define $f(z) = {}^{1 - z}/_{1 - 2z}$ where $z = {}^{x_n - x_i}/_2$. For $z \in [0, {}^1/_4]$, $f(z)$ takes a maximum of $^3/_2$ at $z = {}^1/_4$, corresponding to $x_n = {}^1/_2$ and $x_1 = 0$. The approximation ratio is therefore $^3/_2$ at best. The third case, with $x_1 > {}^1/_2$ is symmetric.

To show that no deterministic and strategy proof mechanism has a smaller approximation ratio for the minimum utility, suppose such a mechanism exists. Consider two agents at $x_1 = 0$ and $x_2 = 1$. Suppose the facility is located at $^1/_2 + \epsilon$ for $\epsilon \geq 0$. The case of $\epsilon < 0$ is dual. Suppose the second agent reports $x_2 = {}^1/_2 + \epsilon$. The optimal minimum utility is $^3/_4 - {}^\epsilon/_2$ units. To achieve an approximation ratio of less than $^3/_2$, the minimum utility must be greater than $^1/_2 - {}^\epsilon/_3$ units. The facility must therefore be in $[0, {}^1/_2 + {}^\epsilon/_3)$. Therefore if agents are at $x_1 = 0$ and $x_2 = {}^1/_2 + \epsilon$, the second agent has an incentive to mis-report their location as $x_2 = 1$. □

Our results for a single facility are summarized in Table 1. The MIDORNEAREST mechanism stands out as achieving the best approximation ratios possible. Note that these ratios are optimal as they match the lower bounds on the approximation ratio that can be achieved by deterministic mechanisms that are strategy proof.

mechanism \ measure	maximum distance	minimum utility
lower bound	2	**3/2**
MIDORNEAREST	**2**	**3/2**
MEDIAN	2	∞
PERCENTILE	2	∞

Table 1: Summary of approximation ratios achieved by different strategy proof and deterministic mechanisms for the single facility location problem. **Bold** for results proved here.

5.2 Two facilities

With a single facility, no PERCENTILE mechanism bounds the approximation ratio of the optimal minimum utility. With two facilities, PERCENTILE mechanisms do slightly better. In particular, an unique PERCENTILE mechanism has a bounded approximation ratio of the optimal maximum distance or minimum utility.

Theorem 2. *The only* PERCENTILE *mechanism for locating two facilities with bounded approximation ratio of the optimal minimum utility is the* ENDPOINT *mechanism which* $^3/_2$*-approximates it.*

Proof. Consider k agents at 0 and $n - k$ at 1 with $k < n$. The solution with optimal minimum utility has facilities at 0 and 1, with a minimum utility of 1 unit. The only PERCENTILE mechanism that guarantees for any n and k that facilities are always located at 0 and at 1, and so have a minimum utility that is not zero units, is the ENDPOINT mechanism. With this mechanism, the worst case for the approximation ratio occurs when agents are at 0, $^1/_2$ and 1, and the ratio is $^3/_2$. □

Recall that the only deterministic and strategy proof mechanism for two facilities with a bounded approximation ratio for the optimal maximum distance is the ENDPOINT mechanism [10]. Surprisingly, there are multiple deterministic and strategy proof mechanisms besides the ENDPOINT mechanism that bound the approximation ratio for the optimal minimum utility. There are even mechanisms with better ratios than the ENDPOINT mechanism.

Theorem 3. *When locating two facilities, the* THIRDORNEAREST *mechanism* $^3/_2$*-approximates the optimal minimum utility. The* QUARTERORNEAREST *mechanism* $^4/_3$*-approximates the optimal minimum utility.*

Proof. The worst case has agents at 0 and 1 when the optimal minimum utility is 1. The THIRDORNEAREST and QUARTERORNEAREST mechanisms return solutions with minimum utility $^2/_3$, and $^3/_4$ respectively, giving approximation ratios of $^3/_2$ and $^4/_3$ respectively. □

Finally, we provide a lower bound on the best possible ratio for approximating the minimum utility.

Theorem 4. *Any deterministic and strategy proof mechanism for two facilities has an approximation ratio of the optimal minimum utility of at least* $^{10}/_9$.

Proof. By contradiction. Suppose there exists a strategy proof mechanism with a smaller approximation ratio. Consider agents at $^1/_6$, $^1/_3$ and 1. The optimal solution locates facilities at $^1/_4$ and in the interval $[{}^{11}/_{12}, 1]$. To meet the approximation ratio, the leftmost facility must be in the interval $[{}^1/_6, {}^{41}/_{120})$. There are two cases. In the first case, the leftmost facility is in the interval $[{}^1/_4, {}^{41}/_{120})$. Suppose the agent at $^1/_6$ mis-reports their location as 0. The optimal solution now locates facilities at $^1/_6$ and in the interval $[{}^5/_6, 1]$. To meet the approximation ratio, the leftmost facility must be in the interval $({}^1/_{12}, {}^1/_4)$. This puts it closer to the agent at $^1/_6$ than previously. Hence, the agent at $^1/_6$ has an incentive to misreport their location as 0. In the second case, the leftmost facility is in the interval $[{}^1/_6, {}^1/_4)$. Suppose the agent at $^1/_3$ mis-reports their location as $^1/_2$. The optimal solution now locates facilities at $^1/_3$ and in the interval $[{}^5/_6, 1]$. To meet the approximation ratio, the leftmost facility must be in the interval $({}^1/_4, {}^5/_{12})$. This puts it closer to the agent at $^1/_3$ than previously. Hence, the agent at $^1/_3$ has an incentive to misreport their location as $^1/_2$. □

Results about the performance guarantees achieved by strategy proof mechanisms for locating two facilities on the line are summarized in Table 2. No mechanism dominates.

5.3 Three or more facilities

With a single facility, no PERCENTILE mechanism bounds the approximation ratio of the optimal minimum utility. With two facilities, there is an unique PERCENTILE mechanism with a bounded approximation ratio of the optimal minimum utility. With three or more

mechanism \ measure	maximum distance	minimum utility
lower bound	2	**10/9**
ENDPOINT	2	**3/2**
PERCENTILE, ¬ENDPOINT	∞	$\boldsymbol{\infty}$
THIRDORNEAREST	∞	**3/2**
QUARTERORNEAREST	∞	**4/3**

Table 2: Summary of approximation ratios achieved by different strategy proof and deterministic mechanisms for the two facility location problem. **Bold** for results proved here.

facilities, a family of PERCENTILE mechanisms have a bounded approximation ratio of the optimal minimum utility. Specifically, any ENDPOINT mechanism which locates one facility at the leftmost agent, another at the rightmost agent, and other facilities at these or other percentiles has a bounded approximation ratio.

Theorem 5. *With three or more facilities,* PERCENTILE *mechanisms that are not* ENDPOINT *mechanisms have an unbounded approximation ratio of the optimal minimum utility, while* ENDPOINT *mechanisms 2-approximate the optimal minimum utility.*

Proof. Consider k agents at 0 and $n-k$ at 1 with $k \leq n$. The solution with optimal minimum utility has facilities at 0 and 1, with a minimum utility of 1 unit. The only PERCENTILE mechanisms that guarantee for any n and k facilities are at both 0 and 1, and thus a minimum utility that is not zero units, are ENDPOINT mechanisms. Let p be the smallest non-zero parameter of such an ENDPOINT mechanism, and $n = \lceil 2/p \rceil$. Suppose there is one agent at 0, another at $1/2$ and n agents at 1. The optimal minimum utility is 1 unit but the mechanism returns a solution with minimum utility of $1/2$ unit. The approximation ratio is therefore 2. □

Finally, we give a lower bound on the approximation ratio.

Theorem 6. *Any deterministic and strategy proof mechanism for m facilities ($m \geq 2$) has an approximation ratio of the optimal minimum utility of at least $\frac{8m-1}{8m-2}$.*

Proof. Suppose there exists a mechanism with a smaller ratio α. Consider agents at $0, 1/m, \ldots 1$. The solution with optimal minimum utility puts one facility at the midpoint between two agents. We suppose this is at $1/2m$. The other cases are similar. To meet the approximation ratio, the mechanism must locate the leftmost facility at $b \in (1/4m, 3/4m)$. Consider what happens if the agent at $1/m$ reports their location as b. We argue that, to meet the approximation ratio, the facility must be located strictly to the left of b. The most problematic case is when $b = 1/4m$. Then the optimal minimum utility puts the leftmost facility at $1/8m$ with an optimal minimum utility of $\frac{8m-1}{8m}$. To meet the approximation ratio, the mechanism must locate the facility so that the minimum utility is greater than $\frac{8m-1}{8m}\frac{8m-2}{8m-1}$ (which is $\frac{4m-1}{4m}$). This puts the facility at some location to the left of $1/4m$, which itself is to the left of b. Thus if agents are at 0, b and 1 then it pays for the agent located at b to mis-report their location as $1/m$. □

6 Randomized mechanisms

Randomization is often a simple and attractive device to achieve better approximation ratios in expectation.

6.1 Single facility

In the classic setting of minimizing the maximum distance, the randomized ENDORAV mechanism stands out. This is strategy proof and $3/2$-approximates the optimal maximum distance in expectation (Theorem 3.3 in [20]). This beats the 2-approximation lower bound for deterministic mechanisms. Indeed, it is optimal as no randomized and strategy proof can do better than an approximation ratio for the maximum distance of $3/2$ in expectation (Theorem 3.4 in [20]).

In terms of optimizing the minimum utility, the ENDORAV mechanism performs less well, only 2-approximating the optimal minimum utility ex ante. This is not even as good as some deterministic mechanisms ex post. In particular, the deterministic MIDORNEAREST mechanism $3/2$-approximates the optimal minimum utility ex post, and this is the best possible approximation ratio for deterministic mechanisms. With respect to minimum utility, the ex post performance of MIDORNEAREST therefore beats the ex ante performance of the ENDORAV mechanism significantly. The ex ante performance of the ENDORAV mechanism is also, as we show next, far from the best that randomized and strategy proof mechanisms can achieve.

Theorem 7. *The* ENDORAV *mechanism 2-approximates the optimal minimum utility in expectation.*

Proof. Without loss of generality, we shift agents to left so $x_1 = 0$ and $x_n = z$. The optimal minimum utility is $1 - z/2$. The expected minimum utility of the solution returned by the ENDORAV mechanism is $1/4(1-z) + 1/2(1 - z/2) + 1/4(1-z)$. That is $1 - 3z/4$. The approximation ratio $\alpha = (1 - z/2)/(1 - 3z/4) = \frac{4-2z}{4-3z}$. For $z \in [0,1]$, this takes the maximum value of 2 at $z = 1$. □

We next show that there do exist randomized mechanisms that perform better than any deterministic mechanism at maximizing the minimum utility. By considering carefully how the ENDORAV mechanism can perform poorly, we design a new randomized mechanism that is optimal with respect to approximating the minimum utility. The essential problem with the ENDORAV mechanism is that it can place the facility close to 0 which gives low utility when an agent is close to 1 (and vice versa). We therefore modify the ENDORAV mechanism to avoid this. The ENDORAVTRUNC mechanism works as follows. If x_1 is the leftmost agent and x_n is the rightmost agent, then let $y = \max(1/3, \min(x_1, 2/3))$ and $z = \max(1/3, \min(2/3, x_n))$. If $y = z = 1/3$ then we locate the facility using RIGHTMOST (i.e at x_n). If $y = z = 2/3$ then we locate the facility using LEFTMOST (i.e. at x_1). Otherwise we locate the facility using ENDORAV applied to y and z (i.e. at y with probability $1/4$, at $(y+z)/2$ with probability $1/2$ and z otherwise). We now prove this new three part randomized mechanism is strategy proof and achieves the best possible approximation ratio of the optimal minimum utility.

Theorem 8. *The* ENDORAVTRUNC *mechanism is strategy proof, $4/3$-approximates the optimal minimum utility and 2-approximates the optimal maximum distance in expectation.*

Proof. Strategy proofness of the ENDORAVTRUNC mechanism follows from strategy proofness of LEFTMOST when all agents are to left of $1/3$, RIGHTMOST when all agents are to right of $2/3$, and ENDORAV otherwise, with the additional observations that truncating reduces the ability of agents to misreport (e.g. reporting less than $1/3$ is the same as reporting $1/3$), and that misreporting which moves between one of these three regimes is never advantageous.

To determine the approximation ratio of the minimum utility, there are seven cases to consider. In the first case $y = z = 1/3$. The worst subcase has $x_1 = 0$, $x_n = 1/3$ and an approximation ratio of $5/4$. In the second case $y = z = 2/3$. This case is dual to the first case. In the third case, $y = 1/3$, $z = 2/3$ and $x_1 \leq 1 - x_n$. The expected minimum utility of the returned solution is $1/2 + x_1$. This compares to an optimal of $1 - 1/2(x_n - x_1)$. The approximation ratio is a maximum when $x_1 = 0$ and $x_n = 2/3$. In this situation, the optimal minimum utility is $2/3$ while the expected minimum utility of the solution returned by the mechanism is $1/2$. This corresponds to an approximation ratio of $4/3$. In the fourth case, $y = 1/3$, $z = 2/3$ and $x_1 > 1 - x_n$. This is dual to the third case. In the fifth case, $y \geq 1/3$ and $z = 2/3$. The expected minimum utility of the returned solution is $4/3 - x_n + y/2$. This compares to an optimal of $1 - (x_n - x_1)/2$. The approximation ratio is a maximum when $x_1 = 1/3$ and $x_n = 1$. In this situation, the optimal minimum utility is $2/3$ while the expected minimum utility of the solution returned by the mechanism is $1/2$. This corresponds to an approximation ratio of $4/3$. In the sixth case, $y = 1/3$ and $z \leq 2/3$. This is dual to the fifth case. In the seventh and final case, $y > 1/3$ and $z < 2/3$. In this situation, the optimal minimum utility is $1 - (z - y)/2$ while the expected minimum utility of the solution returned by the mechanism is $1 - 3(z - y)/4$. The worst case is when $y \rightsquigarrow 1/3$, $z \rightsquigarrow 2/3$ and the approximation ratio approaches $10/9$ from below. Over the seven cases, the worst approximation ratio achieved by the mechanism is $4/3$.

To determine the approximation ratio of the maximum distance, there are again seven cases to consider. In the first case $y = z = 1/3$. The worst subcase has $x_1 = 0$, $x_n = 1/3$ and an approximation ratio of 2. In the second case $y = z = 2/3$. This case is dual to the first case. In the third case, $y = 1/3$, $z = 2/3$ and $x_1 \leq 1 - x_n$. The expected maximum distance of the returned solution is $1/2 - x_1$. This compares to an optimal of $1/2(x_n - x_1)$. The approximation ratio is a maximum when $x_1 = 0$ and $x_n = 2/3$. In this situation, the optimal maximum distance is $1/3$ while the expected minimum utility of the solution returned by the mechanism is $1/2$. This corresponds to an approximation ratio of $3/2$. In the fourth case, $y = 1/3$, $z = 2/3$ and $x_1 > 1 - x_n$. This is dual to the third case. In the fifth case, $y \geq 1/3$ and $z = 2/3$. The expected maximum distance of the returned solution is $x_n - y/2 - 1/3$. This compares to an optimal of $(x_n - x_1)/2$. The approximation ratio is a maximum when $x_1 = 1/3$ and $x_n = 1$. In this situation, the optimal maximum distance is $1/3$ while the expected maximum distance of the solution returned by the mechanism is $1/2$. This corresponds to an approximation ratio of $3/2$. In the sixth case, $y = 1/3$ and $z \leq 2/3$. This is dual to the fifth case. In the seventh and final case, $y > 1/3$ and $z < 2/3$. In this situation, the optimal maximum distance is $(z - y)/2$ while the expected maximum distance of the solution returned by the mechanism is $3(z - y)/4$. This corresponds to an approximation ratio of $3/2$. Over the seven cases, the worst approximation ratio achieved by the mechanism is 2. □

We next show that the approximation ratio of the minimum utility achieved by EndOrAvTrunc is optimal, being the best possible for a randomized and strategy proof mechanism.

Theorem 9. *No randomized and strategy proof mechanism for a single facility can do better than $4/3$-approximate the minimum utility in expectation.*

Proof. To determine the lower bound on the approximation ratio of the optimal minimum utility, we consider just the leftmost and rightmost agents as these alone determine the minimum utility of an agent. Suppose the leftmost agent is at $1/3$ and the rightmost is at $2/3$. By Lemma 3.6 in [20], one of the agents is at least an expected distance of $1/6$ from the facility. Suppose this is the agent at $2/3$. There is a dual argument if it is the agent at $1/3$. Suppose we now shift the agent at $2/3$ to 1. By strategy proofness, the expected distance of the facility from $2/3$ must remain at least $1/6$. By Lemma 3.5 in [20], as the agents are at a distance of $2/3$ apart, the expected maximum distance of an agent from a facility is therefore at least $1/6 + (2/3)/2$, which is $1/2$. The expected minimum utility then is $1 - 1/2$, which is $1/2$. This compares to an optimal minimum utility of $2/3$, giving an approximation ratio of $(2/3)/(1/2)$ or $4/3$. □

6.2 Two facilities

The best performing randomized and strategy proof mechanism for locating two facilities currently known is the EndsOrAv mechanism (named "Mechanism 2" in [20]. This $5/3$-approximates the optimal maximum distance in expectation. This is only slightly greater than the $3/2$ lower bound known for any randomized and strategy proof mechanism (Corollary 4.6 in [20]). This mechanism also performs well at optimizing the minimum utility.

Theorem 10. *When locating two facilities, the* EndsOrAv *mechanism $9/7$-approximates the minimum utility in expectation.*

Proof. EndsOrAv works as follows. Let $mid = (x_1 + x_n)/2$, $l = \max_i\{i \mid x_i \leq mid\}$, $r = \min_i\{i \mid x_i \geq mid\}$, $\Delta = \max(x_l - x_1, x_n - x_r)$. Then the mechanism constructs the following probability distribution of facility locations: x_1 and x_n with probability $1/2$, $x_1 + \Delta$ and $x_n - \Delta$ with probability $1/6$, and $x_1 + \Delta/2$ and $x_n - \Delta/2$ otherwise (i.e. with probability $1/3$). By a pigeonhole argument, $\Delta \in [0, 1/2]$. The optimal minimum utility is $1 - \Delta/2$. With probability $1/3$, the mechanism outputs a solution with minimum utility $1 - \Delta/2$. With probability $2/3$, the mechanism outputs a solution with minimum utility $1 - \Delta$. The expected minimum utility therefore is $1 - 5\Delta/6$. The approximation ratio is $(1 - \Delta/2)/(1 - 5\Delta/6)$. This is $\frac{6-3\Delta}{6-5\Delta}$. For $\Delta \in [0, 1/2]$, this has a maximum value of $9/7$ when $\Delta = 1/2$. □

Finally, we provide a lower bound on the best possible approximation ratio achievable in expectation.

Theorem 11. *No randomized and strategy proof mechanism for two facilities can do better than $10/9$-approximate the minimum utility in expectation.*

Proof. Suppose we have agents at $1/6$, $1/3$, and 1. To meet the approximation bound, one facility is likely close to agents at $1/6$ and $1/3$, while the other is likely close to the agent at 1. By Lemma 3.6 in [20], one of the two leftmost agents is at least an expected distance of $1/12$ from the leftmost facility. Suppose this is the agent at $1/6$. There is a dual argument if it is the agent at $1/3$. Suppose we now shift the agent at $1/6$ to 0. By strategy proofness, the expected distance of the leftmost facility from $1/6$ must remain at least $1/12$. By Lemma 3.5 in [20], as the agents are at a distance of $1/3$ apart, the expected maximum distance of an agent from this facility is at least $1/12 + (1/3)/2$, which is $1/4$. The expected minimum utility then is $3/4$. This compares to an optimal minimum utility of $5/6$, giving an approximation ratio of $(5/6)/(3/4)$ or $10/9$. □

6.3 Three or more facilities

When locating three or more facilities, the randomized EqualCost mechanism stands out as it is strategy proof and achieves an approximation ratio of 2 for the optimal maximum distance [9]. Deterministic mechanisms, by comparison, do not have a bounded approximation ratio of the optimal maximum distance for three or more facilities. Recall that EQUALCOST work as follows: it computes an optimal covering of all agent locations with m disjoint intervals $[\alpha_i, \alpha_i + p]$ that minimize p and that fit within $[0, 1]$, then generates a random bit $z \in \{0, 1\}$ uniformly, and locates the ith facility at $\alpha_i + zp$ for i odd and $\alpha_i + (1 - z)p$ for i even.

Theorem 12. *With m facilities ($m \geq 2$), the* EQUALCOST *mechanism $\frac{2m-1}{2m-2}$-approximates the minimum utility. With a single facility, the ratio is unbounded.*

Proof. For $m \geq 2$, EQUALCOST computes an optimal covering of all agent locations with m disjoint intervals of length l. By a pigeonhole argument, $l \in [0, {}^1\!/_m]$. The worst performance of EQUALCOST is when agents are at the interval endpoints. The optimal minimum utility is $1 - {}^l\!/_2$ compared to the expected minimum utility of the solution returned by EQUALCOST of $1 - l$. This gives an approximation ratio of $\frac{2-l}{2-2l}$. For $l \in [0, {}^1\!/_m]$, this takes a maximum value of $\frac{2m-1}{2m-2}$. For $m = 1$, the worst case has $x_1 = 0$ and $x_n = 1$. The optimal solution puts the facility at ${}^1\!/_2$, with a minimum utility of ${}^1\!/_2$. However, the EQUALCOST returns a lottery over solutions, each of which has a minimum utility of zero units. The approximation ratio is therefore unbounded. □

Similar to ENDORAVTRUNC, we could modify EQUALCOST for a single facility so that the facility is never placed in $[0, {}^1\!/_3)$ or $({}^2\!/_3, 1]$ but truncated to lie in $[{}^1\!/_3, {}^2\!/_3]$. This would improve the approximation ratio of the optimal minimum utility to ${}^4\!/_3$.

Our results for different randomized mechanisms are summarized in Table 3. For a single facility, ENDORAV and ENDORAVTRUNC are incomparable, both dominating EQUALCOST. For two facilities, ENDSORAV dominates EQUALCOST. And for three or more facilities, EQUALCOST performs well on both objectives.

mechanism \ measure	maximum distance	minimum utility
$m = 1$		
lower bound	$3/2$	$\mathbf{4/3}$
ENDORAV	$3/2$	$\mathbf{2}$
ENDORAVTRUNC	$\mathbf{2}$	$\mathbf{4/3}$
EQUALCOST	2	$\boldsymbol{\infty}$
$m = 2$		
lower bound	$3/2$	$\mathbf{10/9}$
ENDSORAV	$5/3$	$\mathbf{9/7}$
EQUALCOST	2	$\mathbf{3/2}$
$m > 2$		
EQUALCOST	2	$\mathbf{\frac{2m-1}{2m-2}}$

Table 3: Summary of approximation ratios achieved by different randomized mechanisms for the m facility location problem. **Bold** for results proved here

7 Happiness objective

Closely related (but different) to this work is [16, 17]. They define the "happiness" of agent i as $h_i = 1 - {}^{d_i}\!/_{d^i_{max}}$ where d^i_{max} is the maximum possible distance agent i can travel. Compare this to utilities where $u_i = 1 - d_i$. With agents and facilities constrained to $[0, 1]$ (both here and in their work), $d^i_{max} = \max(x_i, 1 - x_i)$.

Mei *et al.* [17] consider strategy proof mechanisms maximizing the minimum happiness, limiting their analysis to deterministic mechanisms locating a single facility. Their normalization of distances changes the approximation ratios that can be achieved. They prove that the mechanism locating the facility at the midpoint of the interval 2-approximates the optimal minimum happiness, and that no deterministic and strategy proof mechanism has an approximation ratio better than ${}^4\!/_3$. They observe,

> *"Note that there is a big gap between the upper bound 2 and the lower bound 4/3. It is worth figuring out a tighter bound."* [17]

We now close this gap by proving that the MIDORNEAREST mechanism achieves this lower bound.

Theorem 13. *The* MIDORNEAREST *mechanism is a ${}^4\!/_3$-approximation of the optimal minimum happiness.*

Proof. There are three cases. In the first, $x_1 \leq {}^1\!/_2 \leq x_n$ and the mechanism locates the facility at ${}^1\!/_2$. Algebraic reasoning shows that the approximation ratio takes a maximum value of ${}^4\!/_3$ when $x_1 = {}^1\!/_2$ and $x_n = 1$, or $x_1 = 0$ and $x_n = {}^1\!/_2$. In both subcases, the minimum happiness is ${}^1\!/_2$ unit compared to an optimal of ${}^2\!/_3$ unit. In the second case, $x_n \leq {}^1\!/_2$. Algebraic reasoning again shows that the approximation ratio takes a maximum value of ${}^4\!/_3$ for $x_1 = 0$ and $x_n = {}^1\!/_2$. The third case, with $x_1 > {}^1\!/_2$ is symmetric to the second. □

Recall that the MIDORNEAREST mechanism ${}^3\!/_2$-approximates the optimal minimum utility. This demonstrates that approximation ratios for happiness and utility are different. Surprisingly then the MIDORNEAREST mechanism is optimal with respect to approximating maximum distance, minimum utility and minimum happiness.

8 Conclusions

We have explored the landscape of strategy proof mechanisms for egalitarian facility location. Our major novelty is to consider whether mechanisms approximate well *both* the optimal maximum distance *and* the optimal minimum utility. We identified several strategy proof mechanisms with optimal approximation ratios with respect to the two objectives. Somewhat surprisingly, with respect to the worst off agent, neither the deterministic MEDIAN nor the randomized ENDORAV mechanisms always perform well, despite performing well with respect to just the maximum distance.

For deterministic mechanisms, we proved that the MIDORNEAREST mechanism matches the optimal approximation ratio of the MEDIAN mechanism for the objective of maximum distance, but offers a better and optimal approximation ratio for the objective of minimum utility. We also closed an open problem from Mei *et al.* [17] by proving that the MIDORNEAREST mechanism has an optimal approximation ratio for maximizing the minimum happiness, a closely related objective to minimum utility.

For randomized mechanisms, we proved that the ENDORAV mechanism, which has previously been shown to be optimal with respect to minimizing the maximum distance, is not optimal with respect to maximizing minimum utility. By considering carefully how it can perform poorly, we proposed a new three part randomized mechanism that is optimal with respect to approximating the minimum utility. Our results for one facility are optimal, providing upper and lower bounds on approximation ratios that match. For multiple facilities, our results are asymptotically tight, providing bounds that are within a constant factor. Our result uncover a richer landscape of strategy proof mechanisms with good approximation ratios than was uncovered in the "classic" setting when considering just distances.

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