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Control by Adding Players to Change or Maintain the Shapley–Shubik or the Penrose–Banzhaf Power Index in Weighted Voting Games Is Complete for $\mathrm{NP}^{\mathrm{PP}}$

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Abstract. Weighted voting games are a well-known and useful class of succinctly representable simple games that have many real-world applications, e.g., to model collective decision-making in legislative bodies or shareholder voting. Among the structural control types being analyzing, one is control by adding players to weighted voting games, so as to either change or to maintain a player's power in the sense of the (probabilistic) Penrose–Banzhaf power index or the Shapley–Shubik power index. For the problems related to this control, the best known lower bound is PP-hardness, where PP is "probabilistic polynomial time," and the best known upper bound is the class $\mathrm{NP}^{\mathrm{PP}}$, i.e., the class NP with a PP oracle. We optimally raise this lower bound by showing $\mathrm{NP}^{\mathrm{PP}}$-hardness of all these problems for the Penrose–Banzhaf and the Shapley–Shubik indices, thus establishing completeness for them in that class. Our proof technique may turn out to be useful for solving other open problems related to weighted voting games with such a complexity gap as well.

1 Introduction

Weighted voting games (WVGs) are a central, very popular class of simple coalitional games with many real-world applications. They can be used to model and analyze collective decision-making in legislative bodies and in parliamentary voting [28], such as the European Union or the International Monetary Fund [12], in joint stock companies, etc. For more information, we refer to the books by Chalkiadakis *et al.* [5], Taylor and Zwicker [30], and Peleg and Sudhölter [22] and the book chapters by Chalkiadakis and Wooldridge [4] and Bullinger *et al.* [3]. Especially important is the analysis of how significant players are in WVGs, i.e., what they contribute to forming winning coalitions. Their influence can be measured by so-called power indices among which some well-known examples are: the *Shapley–Shubik index* due to Shapley and Shubik [29], the *probabilistic Penrose–Banzhaf index* due to Dubey and Shapley [9], and also the *normalized Penrose–Banzhaf index* due to Penrose [23] and Banzhaf [2]. We are concerned with the former two.

Much work has been done on how one can tamper with a given player's power in a WVG. For example, the effect of merging or splitting players (the latter a.k.a. "false-name manipulation") was studied by Aziz *et al.* [1] and later on by Rey and Rothe [25]. Zuckerman *et al.* [35] studied the impact of manipulating the quota in WVGs on the power of players. Another way of tampering with the players' power was introduced by Rey and Rothe [26] who studied control problems by adding players to or by deleting players from a WVG; their results have recently been improved by Kaczmarek and Rothe [16].

Control attempts in voting (e.g., by adding or deleting either voters or candidates) have been studied in depth [11]. Surprisingly, however, much less work has been done on control attempts in cooperative game theory, such as for WVGs (e.g., by adding or deleting players). Control by adding players to WVGs is inspired by the analogous notion of control by adding either candidates or voters to elections in voting. There are many real-world scenarios where WVGs and power indices are used to analyze the power of agents and where there is an incentive to change the power in the situation to somebody's advantage (e.g., in politics or to measure control in corporate structures). Concretely, WVGs are the typical way to model decision-making in the EU, as countries can be assigned a weight (essentially related to their population size). The EU is constantly expanding: New members join in (or, rarely, they leave), which is exactly control by adding players, raising the question of if and how the power of old EU members is changed by adding new ones to the EU—just one clear-cut case of motivation among various others. If new members join, an old one may insist on having the same power afterwards (motivating the goal of "maintaining one's power"), or at least not lose power ("nondecreasing one's power"), or Poland may insist that Germany's power does not increase when Ukraine joins ("nonincreasing one's power"). We continue the work on the computational complexity of structural control by adding players to a given WVG, which was started by Rey and Rothe [26]. They showed PP-hardness for the related problems and an upper bound of $\mathrm{NP}^{\mathrm{PP}}$. Extending our initial results (for the Penrose–Banzhaf index only, presented at AAMAS 2024 [15]), we optimally improve their results by showing $\mathrm{NP}^{\mathrm{PP}}$-completeness of these problems for both the Penrose–Banzhaf and the Shapley–Shubik index.

Many of the problems related to WVGs are computationally hard. For instance, under suitable functional reducibilities, computing the Shapley–Shubik power index [8] and the Penrose–Banzhaf power indices [24] is #P-complete, where #P is the *counting version of the class* NP [33]. This is employed by Faliszewski and Hemaspaandra [10] in their result that comparing a given player's probabilistic Penrose–Banzhaf index or a given player's Shapley–Shubik index in two given WVGs is PP-complete. PP is *probabilistic polynomial*

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time [14], a class that presumably is larger than the class NP.

Adding players is just one possibility to change the outcome of a WVG; as mentioned above, Aziz *et al.* [1] proposed merging or splitting players so as to change their power. The problems related to merging players in WVGs were later proven to be PP-complete [25]. However, interestingly, the same complexity gap we are concerned with here—PP-hardness versus membership in $\mathrm{NP}^{\mathrm{PP}}$—is also persistent for false-name manipulation, i.e., for the problems related to splitting players [25]. The novel proof techniques developed in the current paper may thus turn out to be useful for closing this huge complexity gap as well, which provides another strong motivation of our work. Our novel approach might be useful for many interesting open problems in the literature on WVGs (e.g., for control by adding or deleting edges in graph-restricted WVGs, again with a complexity gap between PP-hardness and membership in $\mathrm{NP}^{\mathrm{PP}}$ [18]).

We start with providing the needed notions from cooperative game theory and computational complexity in Section 2, and introduce a new $\mathrm{NP}^{\mathrm{PP}}$-complete problem that will be used in some of our reductions. In Section 3, we prepare some tools and show their properties that are needed in our proofs. Finally, we present our results in Section 4. Due to space limitations, some of our proofs are omitted here; they can be found in the full version of this paper [17].

2 Preliminaries

Let $N = \{1, \dots, n\}$ be a set of players. For $v : 2^N \to \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers, $v(\emptyset) = 0$, a *coalitional game* is a pair (N, v) and each subset of N is called a *coalition*. (N, v) is a *simple* coalitional game if it is *monotonic* (i.e., $v(T) \leq v(T')$ for any T, T' with $T \subseteq T' \subseteq N$), and $v(S) \in \{0, 1\}$ for each coalition $S \subseteq N$. We focus on the following type of simple coalitional games.

Definition 1. *A* weighted voting game $\mathcal{G} = (w_1, \dots, w_n; q)$ *is a simple coalitional game with player set N that consists of a natural number q called the* quota *and nonnegative integer weights, where w_i is the* weight of player $i \in N$. *For each coalition $S \subseteq N$, let $w_S = \sum_{i \in S} w_i$ and define the* characteristic function $v : 2^N \to \{0, 1\}$ of $\mathcal{G}$ *as $v(S) = 1$ if $w_S \geq q$, and $v(S) = 0$ otherwise. We say that S is a* winning coalition *if $v(S) = 1$, and it is a* losing coalition *if $v(S) = 0$. Moreover, we call a player i* pivotal for coalition $S \subseteq N \setminus \{i\}$ *if $v(S \cup \{i\}) - v(S) = 1$.*

How significant are players in a given game? We usually measure this by so-called *power indices*. The main information used in determining the power index of a player i is the number of coalitions i is pivotal for. We study two of the most popular and well-known power indices. One of them is the *probabilistic Penrose–Banzhaf power index*, which was introduced by Dubey and Shapley [9] as an alternative to the original *normalized Penrose–Banzhaf index* [23, 2].

Definition 2. *Let $\mathcal{G}$ be a WVG. The* probabilistic Penrose–Banzhaf power index of a player i in $\mathcal{G}$ *is defined by*

$$\beta(\mathcal{G}, i) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)).$$

The other index we will study is the *Shapley–Shubik power index*, introduced by Shapley and Shubik [29] as follows:

Definition 3. *Let $\mathcal{G}$ be a WVG. The* Shapley–Shubik power index of a player i in $\mathcal{G}$ *is defined by*

$$\varphi(\mathcal{G}, i) = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} |S|!(n - 1 - |S|)!(v(S \cup \{i\}) - v(S)).$$

We assume familiarity with the basic concepts of computational complexity theory, such as the well-known complexity classes P (*deterministic polynomial time*), NP (*nondeterministic polynomial time*), and PP (*probabilistic polynomial time* [14]). $\mathrm{NP}^{\mathrm{PP}}$ is the class of problems that can be solved by an NP oracle Turing machine accessing a PP oracle. It is a very large complexity class containing even the entire polynomial hierarchy by Toda's result [31].

We will use the notions of completeness and hardness for a complexity class based on the polynomial-time many-one reducibility: A problem X *(polynomial-time many-one) reduces to a problem* Y ($X \leq_{\mathrm{m}}^{\mathrm{p}} Y$) if there is a polynomial-time computable function ρ such that for each input x, $x \in X \iff \rho(x) \in Y$; Y is hard for a complexity class $\mathcal{C}$ if $C \leq_{\mathrm{m}}^{\mathrm{p}} Y$ for each $C \in \mathcal{C}$; and Y is complete for $\mathcal{C}$ if Y is $\mathcal{C}$-hard and $Y \in \mathcal{C}$. For more background on complexity theory, we refer to some of the common text books [13, 20, 27].

Valiant [33] introduced #P as the class of functions that give the number of solutions of NP problems. #P is a.k.a. the *"counting version of* NP*"*: For every NP problem X, $\#X$ denotes the function that maps each instance of X to the number of its solutions. For example, for the problem SAT $= \{\phi \mid \phi$ is a boolean formula satisfied by at least one truth assignment$\}$, which is NP-complete [6], #SAT maps each boolean formula to the number of its satisfying assignments. Clearly, any NP problem X is closely related to its counting version $\#X$ because if we can efficiently count the number of solutions of an instance x, we can immediately tell whether x is a yes- or a no-instance of X: $x \in X$ exactly if the number of solutions of x is positive.

Deng and Papadimitriou [8] showed that computing the Shapley–Shubik index of a player in a given WVG is complete for #P via *functional* many-one reductions. Prasad and Kelly [24] proved that computing the probabilistic Penrose–Banzhaf index is parsimoniously complete for #P. #P and PP, even though the former is a class of functions and the latter a class of decision problems, are closely related by the well-known result that $\mathrm{P}^{\mathrm{PP}} = \mathrm{P}^{\#\mathrm{P}}$. For more complexity-theoretic background on the *counting (polynomial-time) hierarchy*, which contains $\mathrm{NP}^{\mathrm{PP}}$, we refer to [34, 21, 32, 31, 27]. Using the standard problem complete for PP due to Gill [14], i.e., MAJSAT $= \{\phi \mid \phi$ is a boolean formula satisfied by a majority of truth assignments$\}$, Littman *et al.* [19] introduced and studied the following problem, which they proved to be $\mathrm{NP}^{\mathrm{PP}}$-complete:

EXIST-MAJORITY-SAT (E-MAJSAT)	
Given:	A boolean formula ϕ with n variables $x_1, \dots, x_n$ and an integer k, $1 \leq k \leq n$.
Question:	Is there an assignment to $x_1, \dots, x_k$, the first k variables, such that a majority of assignments to the remaining $n - k$ variables $x_{k+1}, \dots, x_n$ satisfies ϕ?

Another closely related $\mathrm{NP}^{\mathrm{PP}}$-complete decision problem was introduced by de Campos et al. [7]:

EXIST-MINORITY-SAT (E-MINSAT)	
Given:	A boolean formula ϕ with n variables $x_1, \dots, x_n$ and an integer k, $1 \leq k \leq n$.
Question:	Is there an assignment to $x_1, \dots, x_k$, the first k variables, such that at most half of the assignments to the remaining $n - k$ variables $x_{k+1}, \dots, x_n$ satisfies ϕ?

Note that if $k = 0$, E-MAJSAT is equivalent to the PP-complete problem MAJSAT, and E-MINSAT is equivalent to the complement of MAJSAT, which is also PP-complete since the class PP is closed under complement [14]. If $k = n$, E-MAJSAT is equivalent to the NP-complete problem SAT, and E-MINSAT is equivalent to the complement of SAT, i.e., it is coNP-complete. Therefore, we can omit these cases ($k = 0$ and $k = n$) when proving $\mathrm{NP}^{\mathrm{PP}}$-hardness of our problems. Moreover, we can also assume that a given formula in CNF does not contain any variable x in both forms, x and $\neg x$, in any of its clause (which can be checked in polynomial time) because then the clause would be true for any possible truth assignment. Also, we will assume that our inputs for these problems contain only those variables that actually occur in the given boolean formula.

Rey and Rothe [26] defined problems capturing control by adding players to a given WVG so as to change a given player's power in the modified game. To increase this power for an index PI, the control problem is defined as follows:

CONTROL-BY-ADDING-PLAYERS-TO-INCREASE-PI	
Given:	A WVG $\mathcal{G}$ with a set N of players, a set M of players (given by their weights) that can be added to $\mathcal{G}$, a distinguished player $p \in N$, and a positive integer $k \leq \lvert M \rvert$.
Question:	Can at most k players $M' \subseteq M$ be added to $\mathcal{G}$ such that for the new game $\mathcal{G}_{\cup M'}$, it holds that $\mathrm{PI}(\mathcal{G}_{\cup M'}, p) > \mathrm{PI}(\mathcal{G}, p)$?

The corresponding control problems for decreasing, nonincreasing, nondecreasing, and maintaining PI are defined analogously, by changing the relation sign in the question to "$<$," "$\leq$," "$\geq$," and "$=$," respectively. Additionally, we assume that we add at least one new player in case of nondecreasing, nonincreasing, or maintaining PI (otherwise, the control problems would be trivial).

For both the Penrose–Banzhaf and the Shapley–Shubik power index, Rey and Rothe [26] showed that these five control problems are PP-hard, and they observed that $\mathrm{NP}^{\mathrm{PP}}$ is the best known upper bound for them. Our goal in this paper is to raise the PP-hardness lower bound of these problems to $\mathrm{NP}^{\mathrm{PP}}$-hardness, thus establishing their completeness for this class. We now introduce another problem that will be used in some of our proofs, and we state its $\mathrm{NP}^{\mathrm{PP}}$-completeness (the proof can be found in the full version [17]):

EXIST-EXACT-SAT (E-EXASAT)	
Given:	A boolean formula ϕ with n variables $x_1, \ldots, x_n$, an integer k, $1 \leq k \leq n$, and an integer ℓ.
Question:	Is there an assignment to the first k variables $x_1, \ldots, x_k$ such that *exactly* ℓ assignments to the remaining $n - k$ variables $x_{k+1}, \ldots, x_n$ satisfy ϕ?

Lemma 1. E-EXASAT *is* $\mathrm{NP}^{\mathrm{PP}}$*-complete.*

3 Transforming Truth Assignments of Boolean Formulas to Weight Vectors

First, let us define a transformation from a truth assignment for a given boolean formula to vectors of weights to be used for some players in our reductions later on.

Definition 4. *Let ϕ be given boolean formula in CNF with variables $x_1, \ldots, x_n$ and m clauses. Let $k \in \mathbb{N}$ with $k \leq n$ and $r = \lceil \log_2 n \rceil - 1$. Let us define the following two sets of weight vectors which are going to be assigned as weights to players divided either into three sets—M, A, and C—or into four sets—M, A, C, and C'—in our proofs later on:*

Set 1: *For some $t \in \mathbb{N} \setminus \{0\}$ such that $10^t > 2^{\lceil \log_2 n \rceil + 1}$, and for $i \in \{1, \ldots, n\}$, define*

$$a_i = 10^{t(m+1)+i} + \sum_{\substack{j\,:\,\text{clause } j\\ \text{contains } x_i}} 10^{tj} \text{ and}$$

$$b_i = 10^{t(m+1)+i} + \sum_{\substack{j\,:\,\text{clause } j\\ \text{contains } \neg x_i}} 10^{tj},$$

and for $j \in \{1, \ldots, m\}$ and $s \in \{0, \ldots, r\}$, define

$$c_{j,s} = 2^s \cdot 10^{tj}.$$

Define the following three weight vectors:

$$W_M = (a_1, \ldots, a_k, b_1, \ldots, b_k),$$
$$W_A = (a_{k+1}, \ldots, a_n, b_{k+1}, \ldots, b_n),$$
$$W_C = (c_{1,0}, \ldots, c_{m,r}).$$

Set 2: *For some $t, t' \in \mathbb{N} \setminus \{0\}$ such that $10^{t'} > 2^{\lceil \log_2 n \rceil + 1}$ and $10^t > 10^{t'} + 2^{\lceil \log_2 n \rceil + 1} \sum_{l=1}^{m} 10^{lt'}$, and for $i \in \{1, \ldots, n\}$, define a_i and b_i as in Set 1, and for $j \in \{1, \ldots, m\}$ and $s \in \{0, \ldots, r\}$, let*

$$c'_{j,s} = 2^s \cdot 10^{t'j} \text{ and } c_{j,s} = 2^s \cdot 10^{tj} + c'_{j,s}.$$

In addition to W_M and W_A defined as in Set 1, define the following two weight vectors:

$$W_{C'} = (c'_{1,0}, \ldots, c'_{m,r}) \text{ and } W_C = (c_{1,0}, \ldots, c_{m,r}).$$

Additionally, let

$$q_1 = \sum_{i=1}^{n} 10^{t(m+1)+i} + 2^{\lceil \log_2 n \rceil} \sum_{j=1}^{m} 10^{tj} \text{ and}$$

$$q_2 = \sum_{i=1}^{n} 10^{t(m+1)+i} + 2^{\lceil \log_2 n \rceil} \sum_{j=1}^{m} 10^{tj} + \left(2^{\lceil \log_2 n \rceil} - 1\right) \sum_{j=1}^{m} 10^{t'j}.$$

Lemma 2. *Let $i \in \{1, 2\}$. There exists a bijective transformation from the set of truth assignments satisfying a boolean formula ϕ to the family of subsets of players with weights defined in Set i of Definition 4 whose total weight equals q_i.*

Proof Sketch. It can be shown that for each set S of weight q_i, for $i \in \{1, 2\}$, S has to contain exactly n players from $M \cup A$ (namely, n players, each with exactly one weight from $\{a_j, b_j\}$, $j \in \{1, \ldots, n\}$), and for each $S \cap (M \cup A)$, there exists exactly one set of weight q_1 with players from C for Set 1 and q_2 from $C \cup C'$ for Set 2 (but there can exist subsets of $M \cup A$ of the mentioned form that are not contained in any set of weight q_i).

Let us prove that there exists a bijection between the sets of weight q_i and the set of truth assignments to the variables $x_1, \ldots, x_n$ satisfying the given formula ϕ.

For each truth assignment to the variables $x_1, \ldots, x_n$, let 1 represent `true` and 0 `false`, and let

$$d_l = \begin{cases} a_l & \text{if } x_l = 1, \\ b_l & \text{if } x_l = 0. \end{cases} \quad (1)$$

The resulting weight vector $\vec{d} = (d_1, \dots, d_n)$ is unique for each assignment to $x_1, \dots, x_n$ (from the previously mentioned assumption that no clause contains both a variable and its negation, so $a_l \neq b_l$ for any $l \in \{1, \dots, n\}$). Also, if this vector $\vec{d}$ corresponds to a satisfying assignment of ϕ, the total weight of the players' subset in both cases of Set 1 and Set 2 equals

$$\sum_{l=1}^{n} d_l = \sum_{l=1}^{n} 10^{t(m+1)+l} + \sum_{j=1}^{m} p_j 10^{tj},$$

where p_j, $1 \leq p_j \leq n$, is at least 1 since each clause is satisfied by our fixed assignment: For each clause j, there exists some x_l making it true (i.e., either $x_l = 1$ and the clause j contains x_l, or $x_l = 0$ and j contains $\neg x_l$), which implies that the corresponding d_l has 10^{tj} as one of its summands (i.e., either $d_l = a_l$ if x_l is contained in clause j, or $d_l = b_l$ if $\neg x_l$ is contained in j). From the fact that $p_j \neq 0$ for all $j \in \{1, \dots, m\}$ and the previous analysis, there exists exactly one subset of C when $i = 1$ or exactly one subset of $C \cup C'$ when $i = 2$ such that the players with the corresponding weights together with the players whose weights correspond to $\vec{d}$ form a coalition of weight q_i. Therefore, for each truth assignment satisfying ϕ, there exists a unique set of players from $A \cup M \cup C$ (respectively, $A \cup M \cup C \cup C'$) with total weight q_i.

Conversely, let $S \subseteq M \cup A \cup C$ for $i = 1$, and $S \subseteq M \cup A \cup C \cup C'$ for $i = 2$, be a coalition of players whose total weight is q_i. From the previous analysis, S can contain exactly one player with weight from $\{a_j, b_j\}$ for $j \in \{1, \dots, n\}$, and for $S \cap (M \cup A)$, there exists exactly one subset of C for $i = 1$, and exactly one subset of $C \cup C'$ for $i = 2$, which creates with the former a coalition of players with total weight q_i, i.e., there exist no two different sets S and S' both with $w_S = w_{S'} = q_i$ such that $S \cap (M \cup A) = S' \cap (M \cup A)$.

For the set $S \cap (M \cup A)$ with the weight vector $(d_1, \dots, d_n)$, set

$$x_\ell = \begin{cases} 1 & \text{if } d_\ell = a_\ell \\ 0 & \text{if } d_\ell = b_\ell \end{cases} \quad (2)$$

for $\ell \in \{1, \dots, n\}$. For each clause $j \in \{1, \dots, m\}$, there exists some d_ℓ corresponding to the player whose weight's part is equal to 10^{tj}; and if the weight is a_ℓ, clause j contains x_ℓ, so assigning `true` to x_ℓ makes clause j true; otherwise, the player's weight is b_ℓ and the clause j contains $\neg x_\ell$, so assigning `false` to x_ℓ makes j true. Hence, this is a unique truth assignment to the variables $x_1, \dots, x_n$ that satisfies ϕ and is obtained by the described transformation from the set S. □

The full proof of Lemma 2 can be found in the full version [17].

4 NP$^{\text{PP}}$-Hardness of Control by Adding Players to a Weighted Voting Game

In this section, we show our results, i.e., we prove NP$^{\text{PP}}$-hardness of the control problems by adding players to a given WVG. Specifically, we will present full proofs of NP$^{\text{PP}}$-hardness for three of the problems. The remaining proofs (see Theorem 5) can be found in the full version of this paper [17].

Theorem 3. Control-by-Adding-Players-to-Increase-β *is* NP$^{\text{PP}}$*-complete.*

Proof. We will prove NP$^{\text{PP}}$-hardness by using a reduction from E-MajSAT. Let (ϕ, k) be a given instance of E-MajSAT, where ϕ is a boolean formula in CNF with variables $x_1, \dots, x_n$ and m clauses, and $1 \leq k < n$. Before we construct an instance of our control problem from (ϕ, k), we need to choose some numbers and introduce some notation.

Let $t \in \mathbb{N}$ be such that

$$10^t > \max\left\{2^{\lceil \log_2 n \rceil + 1}, k + (n-k-1)(k+1)\right\}, \quad (3)$$

and for that t, given ϕ and k, we define q_1 and W_A, W_C, and W_M as in Set 1 of Definition 4 for player sets A, C, and M.

Now, we construct an instance of Control-by-Adding-Players-to-Increase-β: Let k be the limit for the number of players that can be added, and let M be the set of $2k$ players that can be added with the list of weights W_M. Further, we define the quota of the WVG $\mathcal{G}$ by

$$q = 2 \cdot (w_A + w_M + w_C + (n-k)(k+1)) + 1, \quad (4)$$

and we let N be the set of $4n - 2k + m(r+1)$ players in $\mathcal{G}$, subdivided into the following seven groups:

- player p with weight 1 will be our distinguished player,
- group A contains $2(n-k)$ players with weight list W_A,
- group C contains $m(r+1)$ players with weight list W_C,
- group W contains k players with weight list
$$(q - q_1 - 2, q - q_1 - 3, \dots, q - q_1 - (k+1)),$$
- group X contains k players with weight 1 each,
- group Y contains $n - k$ players with weight list
$$(q-1, q-1-(k+1), \dots, q-1-(n-k-1)(k+1)), \quad \text{and}$$
- group Z contains $n - k - 1$ players with weight $k + 1$ each.

This concludes the description of how to construct the instance $(\mathcal{G}, M, p, k)$ of our control problem from the given instance (ϕ, k) of E-MajSAT. Obviously, this can be done in polynomial time.

Let us first discuss which coalitions player p can be pivotal for in any of the games $\mathcal{G}_{\cup M'}$ for some $M' \subseteq M$.[1] Player p is pivotal for those coalitions of players in $(N \setminus \{p\}) \cup M'$ whose total weight is $q - 1$. First, note that any two players from $W \cup Y$ together have a weight larger than q. Therefore, at most one player from $W \cup Y$ can be in any coalition player p is pivotal for. Moreover, by (4), all players from $A \cup C \cup M \cup X \cup Z$ together have a total weight smaller than $q - 1$. This means that any coalition $S \subseteq (N \setminus \{p\}) \cup M'$ with a total weight of $q - 1$ has to contain *exactly* one of the players in $W \cup Y$. Now, whether this player is in W or Y has consequences as to which other players will also be in such a weight-$(q-1)$ coalition S:

Case 1: If S contains a player from W with weight, say, $q - q_1 - \ell - 1$ for some ℓ, $1 \leq \ell \leq k$, S also has to contain those players from $A \cup C \cup M$ whose weights sum up to q_1 and j players from X. Indeed, $w_{X \cup Z} < 10^t$, so players from $A \cup C \cup M$ are needed to achieve $q_1 + \ell$. Moreover, they are able to achieve only the value q_1 because any subset of $A \cup C \cup M$ is divisible by 10^t. At the same time, each player in Z has weight $k + 1 > \ell$, so no coalition with them achieves $q_1 + \ell$. Also, recall that q_1 can be achieved only by a set of players whose weights take exactly one of the values from $\{a_i, b_i\}$ for each $i \in \{1, \dots, n\}$, so S must contain exactly $n - k$ players from A that already are in $\mathcal{G}$ (either a_i or b_i, for $k + 1 \leq i \leq n$) and exactly k players from M (either a_i or b_i, for $1 \leq i \leq k$); these k players must have been added to the game, i.e., $|M'| = k$.

[1] This also includes the case of the unchanged game $\mathcal{G}$ itself, namely for $M' = \emptyset$.

Case 2: If S contains a player from Y with weight, say, $q-1-\ell(k+1)$ for some ℓ, $0 \leq \ell \leq n-k-1$, then either S already achieves weight $q-1$ for $\ell = 0$, or S has to contain $\ell > 0$ players from Z. The players from X are not heavy enough and since each player from $A \cup C \cup M$ has a weight larger than $w_{X \cup Z}$ (which, together with any player from S, gives a total weight exceeding the quota).

Since there are no players with weights a_i or b_i for $i \in \{1, \ldots, k\}$ in game $\mathcal{G}$, player p can be pivotal only for the coalitions described in the second case above, and therefore,

$$\beta(\mathcal{G}, p) = \frac{\sum_{j=0}^{n-k-1} \binom{n-k-1}{j}}{2^{|N|-1}} = \frac{2^{n-k-1}}{2^{|N|-1}}.$$

We now show the correctness of our reduction: (ϕ, k) is a yes-instance of E-MAJSAT if and only if $(\mathcal{G}, M, p, k)$ as defined above is a yes-instance of CONTROL-BY-ADDING-PLAYERS-TO-INCREASE-β.

Only if: Suppose that (ϕ, k) is a yes-instance of E-MAJSAT, i.e., there exists an assignment to $x_1, \ldots, x_k$ such that a majority of assignments to the remaining $n-k$ variables yields a satisfying assignment for the boolean formula ϕ. Let us fix one of these satisfying assignments to $x_1, \ldots, x_n$. From this fixed assignment, we get the vector $(d_1, \ldots, d_n)$ as defined in the proof of Lemma 2, where the first k positions correspond to the players $M' \subseteq M$, $|M'| = k$, which we add to the game $\mathcal{G}$.

Since there are more than 2^{n-k-1} assignments to $x_{n-k}, \ldots, x_n$ which—together with the fixed assignments to $x_1, \ldots, x_k$—satisfy ϕ, by Lemma 2 there are more than 2^{n-k-1} subsets of $A \cup C \cup M'$ such that the players' weights in each subset sum up to q_1. Each of these subsets with total weight q_1 can form coalitions of weight $q-1$ with each player from W having weight $q-q_1-(\ell+1)$, $\ell \in \{1, \ldots, k\}$, and ℓ weight-1 players from X—and there are $\binom{k}{\ell}$ such coalitions. Therefore, recalling from Case 2 above that $Y \cup Z$ already contains 2^{n-k-1} coalitions of weight $q-1$, we have

$$\begin{aligned}\beta(\mathcal{G}_{\cup M'}, p) &> \frac{2^{n-k-1} + 2^{n-k-1} \sum_{\ell=1}^{k} \binom{k}{\ell}}{2^{|N|+k-1}} \\ &= \frac{2^{n-k-1} + (2^k - 1) \cdot 2^{n-k-1}}{2^{|N|+k-1}} \\ &= \frac{2^k \cdot 2^{n-k-1}}{2^{|N|+k-1}} = \frac{2^{n-k-1}}{2^{|N|-1}} = \beta(\mathcal{G}, p),\end{aligned}$$

so player p's Penrose–Banzhaf index is strictly larger in the new game $\mathcal{G}_{\cup M'}$ than in the old game $\mathcal{G}$, i.e., we have constructed a yes-instance of our control problem.

If: Assume now that (ϕ, k) is a no-instance of E-MAJSAT, i.e., there does not exist any assignment to the variables $x_1, \ldots, x_k$ such that a majority of assignments to the remaining $n-k$ variables satisfies the boolean formula ϕ. In other words, for each assignment to $x_1, \ldots, x_k$, there exist at most 2^{n-k-1} assignments to $x_{k+1}, \ldots, x_n$ that yield a satisfying assignment for ϕ. Again, we consider subsets $M' \subseteq M$ of players that uniquely correspond to the assignments of $x_1, \ldots, x_k$ according to Lemma 2. Note that any other possible subset will not allow to form new coalitions for which player p could be pivotal in the new game, i.e., p's Penrose–Banzhaf index will not increase unless we add any player with weight either a_i or b_i for each $i \in \{1, \ldots, k\}$.

By Lemma 2 and our assumption, there are at most 2^{n-k-1} subsets of $A \cup C \cup M'$ such that the players' weights in each subset sum up to q_1. As in the proof of the "Only if" direction, for each $\ell \in \{1, \ldots, k\}$, each of these subsets of $A \cup C \cup M'$ forms $\binom{k}{\ell}$ coalitions of weight $q-1$ with a player in W having weight $q-q_1-(\ell+1)$ and ℓ players in X. Again recalling from Case 2 above that $Y \cup Z$ already contains 2^{n-k-1} coalitions of weight $q-1$, we have

$$\begin{aligned}\beta(\mathcal{G}_{\cup M'}, p) &\leq \frac{2^{n-k-1} + (2^k - 1) \cdot 2^{n-k-1}}{2^{|N|+k-1}} \\ &= \frac{2^k \cdot 2^{n-k-1}}{2^{|N|+k-1}} = \frac{2^{n-k-1}}{2^{|N|-1}} = \beta(\mathcal{G}, p).\end{aligned}$$

Thus player p's Penrose–Banzhaf index cannot increase by adding up to k players from M to the game $\mathcal{G}$, and we have a no-instance of our control problem. □

Theorem 4. CONTROL-BY-ADDING-PLAYERS-TO-INCREASE-φ *and* CONTROL-BY-ADDING-PLAYERS-TO-NONDECREASE-φ *are* $\mathrm{NP}^{\mathrm{PP}}$*-complete.*

Proof. We prove $\mathrm{NP}^{\mathrm{PP}}$-hardness of both control problems using one and the same reduction from E-MAJSAT (and argue slightly differently for them). Let (ϕ, k) be a given instance of E-MAJSAT, where ϕ is a boolean formula in CNF with variables $x_1, \ldots, x_n$ and m clauses, and let $k < n$.

Before we construct an instance of our control problems from (ϕ, k), we need to choose some numbers and introduce some notation. Let

$$P = 6n^2 m + 26n^2 + 8k^2 + 8nm + 18n + 4k - 2m - 3$$

be the number of players in our game (note that P is an odd number). The numbers

$$\begin{aligned}\delta &= 3n^2 m + 13n^2 + 4k^2 + 3nm + 5n + 4k - 2m - 5, \\ x &= \delta + nm + 4n - 2k + m + 3 = \frac{P-1}{2}, \quad \text{and} \\ k' &= \left(1 + \frac{x+1}{P-x}\right) \cdot \ldots \cdot \left(1 + \frac{x+1}{P-x+k-1}\right) \leq 2^k\end{aligned}$$

with $k' \geq 2$, will be used in our calculations later in the proof. Finally, let

$$z = \lceil 2^{n-k+1}(k'-1) \rceil - 1 < 2^{n+1}$$

and choose $y_1, \ldots, y_u$ with $y_1 > \cdots > y_u$ such that

$$z = 2^{y_1} + \cdots + 2^{y_u}$$

is satisfied. Note that $y_1 \leq n$ and $u \leq n$.

To make the calculations in our proof simpler, we want all coalitions counted for computing the Shapley–Shubik indices to be equally large (to be more specific, we want these coalitions to have size x). Therefore, we define the following values. For $i \in \{0, 1, \ldots, 2n-2k\}$, let

$$\alpha_i = nm + 4n - 2k + m + 2 - i,$$

and for $i \in \{0, \ldots, y_1\}$, let

$$\beta_i = (n-r)m + 3n - 2k + 2 - i.$$

Finally, let $t' \in \mathbb{N}$ be such that

$$10^{t'} > \max\left\{2^{\lceil \log_2 n \rceil + 1}, (2n - 2k + 1)w'\right\}$$

for $w' = (\alpha_{2n-2k} + 1)w^*_{2n-2k}$ as defined in Table 1. For ϕ, k, and t', let t, q_2, M, A, C, and C' with weight lists W_M, W_A, W_C, and $W_{C'}$ be defined as in Set 2 of Definition 4.

Table 1: Groups of players in the proof of Theorem 4, with their categories, numbers, and weights (note that, e.g., the sum $\sum_{j=0}^{i-1} \beta_j v_j$ in the second (size) row has value 0 for $i = 0$)

Category	**Group**	**Number of Players**	**Weights**
	distinguished player p	1	1
(ms)	A	$2n - 2k$	W_A
(ms)	C	$m(r + 1)$	W_C
(ms)	C'	$m(r + 1)$	$W_{C'}$
(size)	D	δ	1
(def)	S	$\sum_{i=1}^{u}(y_i + 1)$	$q - q_2 - \beta_{j_i} v_{j_i} - j_i v_i' - \delta - 1$ for $i \in \{1, \ldots, u\}$ and $j_i \in \{0, \ldots, y_i\}$
(size)	V_i for $i \in \{0, \ldots, y_1\}$	β_i	$v_i = 1 + \delta + \sum_{j=0}^{i-1} \beta_j v_j$
(num)	V_i' for $i \in \{1, \ldots, u\}$	y_i	$v_i' = (\beta_{y_1} + 1)v_{y_1} + \sum_{i'=1}^{i-1} y_{i'} v_{i'}'$
(def)	T	$2n - 2k + 1$	$q - \alpha_i w_i^* - iw' - \delta - 1$ for $i \in \{0, \ldots, 2n - 2k\}$
(size)	W_i^* for $i \in \{0, \ldots, 2n - 2k\}$	α_i	$w_i^* = (y_u + 1)v_u' + \sum_{i'=0}^{i-1} \alpha_{i'} w_{i'}^*$
(num)	W'	$2n - 2k$	$w' = (\alpha_{2n-2k} + 1)w_{2n-2k}^*$
	Z	remaining players	q

Now, we are ready to construct the instance of our two control problems by adding players to increase or to nondecrease a given player's Shapley–Shubik power index as follows: Let k be the limit for the number of players that can be added, let M be the set of $2k$ players that can be added and let W_M be the list of their weights, let

$$q = 2 \cdot \left(w_A + w_M + w_C + w_{C'} + 10^{t'} + 1\right)$$

be the quota of $\mathcal{G}$, and let N be the set of P players in game $\mathcal{G}$, subdivided into groups as presented in Table 1.

Note that each group of players in Table 1 (except the distinguished player p and group Z whose players are not part of any coalition for which p is pivotal) belongs to some category: We categorize players by their function, i.e., there are groups of players who are responsible for defining coalitions that are counted when computing the Shapley–Shubik indices; other groups of players are responsible for the size of the coalition they are in (again, when counted in these indices); and there are players who are responsible for the number of coalitions. Some of these players are defined by setting their weights to the quota minus some values that have to be satisfied by other players (for a sufficiently large quota, so as to make it impossible for the distinguished player to be pivotal for any coalition containing more than one of these players). For the remaining players, we define their weights in such a way that they are not interchangeable.

In more detail, the players with category (def) "define" which other players are needed to create a coalition of weight $q - 1$, among the players with category (ms) and the players in M, we will focus on those coalitions whose total weight is q_2. The main purpose of the players from the groups marked (num) is to specify the number of coalitions for which player p can be pivotal. The players from groups with category (size) are used to make all these coalitions of equal size (among these players, the players with the same weight are together part of the same coalitions). Now, we will discuss the coalitions counted in our proof in detail.

Let us analyze for what coalitions player p can be pivotal in $\mathcal{G}$ or any new game resulting from $\mathcal{G}$ by adding players from M. Player p is pivotal for coalitions of weight $q - 1$. First, note that any two players from $S \cup T$ together have a total weight larger than q. Next, the total weight of $N \setminus (\{p\} \cup S \cup T \cup Z)$ is smaller than $q - 1$. Therefore, a coalition with a total weight of $q - 1$ has to contain exactly one of the players in $S \cup T$ and whether this player is in S or T has consequences as to which other players have to be in such a coalition:

Case 1: If the coalition contains a player from S, it also has to contain the players from $M \cup A \cup C \cup C'$ whose weights sum up to q_2, some players from $V_i \cup V_i'$ (for i defined as in Table 1), and all players from D—the players from

$$\bigcup_{i=0}^{y_1} V_i \cup \bigcup_{i=1}^{u} V_i' \cup \bigcup_{i=0}^{2n-2k} W_i^* \cup W' \cup D$$

have total weight smaller than $10^{t'}$. Therefore, q_2 can be achieved only by the players from $M \cup A \cup C \cup C'$. Recalling that q_2 can be achieved by a set consisting of those players whose weights take exactly one value in $\{a_i, b_i\}$ for each $i \in \{1, \ldots, n\}$, we have to add a set $M' \subseteq M$ with $|M'| = k$ to $\mathcal{G}$. But weights of players from $M \cup A \cup C \cup C'$ can sum up only to values which are divisible by $10^{t'}$ therefore they can achieve only the q_2-part. Each player from $\bigcup_{i=0}^{2n-2k} W_i^* \cup W'$ also is too heavy to achieve the required value.

Case 2: If the coalition contains a player from T, the coalition also has to contain some of the players from $W_i^* \cup W'$ and all players from D. Also here, we do not find any other combination of players which could form a weight-$(q - 1)$ coalition with a player in T—all players in

$$\bigcup_{i=0}^{y_1} V_i \cup \bigcup_{i=1}^{u} V_i' \cup D$$

have a total weight too small to be able to replace even one player from $\bigcup_{i=0}^{2n-2k} W_i^* \cup W'$ and (as mentioned in Case 1) any player in $M \cup A \cup C \cup C'$ together with any player from T has total weight larger than $q-1$.

In both cases, each coalition has the same size of

$$1 + \delta + n + m(r+1) + \beta_j + j = 1 + \delta + \alpha_i + i = x$$

for any $i \in \{0, \ldots, 2n-2k\}$ and $j \in \{0, \ldots, y_1\}$.

Since there are no players with weights a_i or b_i for $i \in \{1, \ldots, k\}$ in game $\mathcal{G}$, player p can be pivotal only for the coalitions described in the second case above and therefore,

$$\varphi(\mathcal{G}, p) = 2^{2n-2k} \frac{x!(P-x-1)!}{P!}.$$

To prove the correctness of the reduction, we show that the following three statements are pairwise equivalent:

- (ϕ, k) is a yes-instance of E-MAJSAT;
- $(\mathcal{G}, M, p, k)$ is a yes-instance of CONTROL-BY-ADDING-PLAYERS-TO-INCREASE-φ;
- $(\mathcal{G}, M, p, k)$ is a yes-instance of CONTROL-BY-ADDING-PLAYERS-TO-NONDECREASE-φ.

Suppose (ϕ, k) is a yes-instance of E-MAJSAT, i.e., there exists an assignment to $x_1, \ldots, x_k$ such that a majority of assignments of the remaining $n-k$ variables satisfies the boolean formula ϕ. Let us fix one of these satisfying assignments. From this fixed assignment, we get the vector $\vec{d} = (d_1, \ldots, d_n)$ as defined in the proof of Lemma 2, where the first k positions correspond to the players in $M' \subseteq M$, $|M'| = k$, which we add to the game $\mathcal{G}$.

Since there are at least $2^{n-k-1}+1$ assignments for $x_{n-k}, \ldots, x_n$ which—together with the fixed assignments for $x_1, \ldots, x_k$—satisfy ϕ, by Lemma 2 there are more than 2^{n-k-1} subsets of $M' \cup A \cup C \cup C'$ such that the players' weights in each subset sum up to q_2. Now, each of these subsets can form $2^{y_1} + \cdots + 2^{y_u} = z$ coalitions with the players from

$$S \cup \bigcup_{i=0}^{y_1} V_i \cup \bigcup_{i=1}^{u} V_i' \cup D$$

for which player p is pivotal in the new game $\mathcal{G}_{\cup M'}$. Therefore,

$$\begin{aligned}
&\varphi(\mathcal{G}_{\cup M'}, p)\\
&\geq \left(2^{2n-2k} + z \cdot (2^{n-k-1}+1)\right) \frac{x!(P+k-1-x)!}{(P+k)!}\\
&= \left(2^{2n-2k} + \left(\lceil 2^{n-k+1}(k'-1)\rceil - 1\right) \cdot \left(2^{n-k-1}+1\right)\right)\\
&\quad \cdot \frac{x!(P-1-x)!}{P!} \cdot \frac{(P-x)\cdots(P+k-1-x)}{(P+1)\cdots(P+k)}\\
&\geq \left(2^{2n-2k} + \left(2^{n-k+1}(k'-1) - 1\right) \cdot \left(2^{n-k-1}+1\right)\right)\\
&\quad \cdot \frac{1}{k'} \frac{x!(P-1-x)!}{P!}\\
&= \left(2^{2n-2k}k' - 2^{n-k-1} + 2^{n-k+1}(k'-1) - 1\right)\\
&\quad \cdot \frac{1}{k'} \cdot \frac{x!(P-1-x)!}{P!}\\
&> \varphi(\mathcal{G}, p),
\end{aligned}$$

so player p's Shapley–Shubik power index is strictly larger in the new game $\mathcal{G}_{\cup M'}$ than in the old game $\mathcal{G}$, i.e., we have constructed a yes-instance of both our control problems.

Conversely, suppose now that (ϕ, k) is a no-instance of E-MAJSAT, i.e., for each assignment to $x_1, \ldots, x_k$, there exist at most 2^{n-k-1} assignments of $x_{k+1}, \ldots, x_n$ which satisfy ϕ. It is enough to consider subsets $M' \subseteq M$ of players that uniquely correspond to the assignments of $x_1, \ldots, x_k$ according to Lemma 2, because any other possible subset will not allow to form new coalitions for which player p could be pivotal in the new game, i.e., p's Shapley–Shubik index will only decrease if we do not add any player with weight either a_i or b_i for each $i \in \{1, \ldots, k\}$.

Now let $M' \subseteq M$ be any subset of players that corresponds to some assignment to $x_1, \ldots, x_k$. By Lemma 2 and our assumption, there are at most 2^{n-k-1} subsets of $M' \cup A \cup C \cup C'$ such that the players' weights in each subset sum up to q_2. For each of these sets, there are exactly z new coalitions described in Case 1 for which p is pivotal after adding the new players from M'. Therefore,

$$\begin{aligned}
&\varphi(\mathcal{G}_{\cup M'}, p)\\
&\leq \left(2^{2n-2k} + \left(\lceil 2^{n-k+1}(k'-1)\rceil - 1\right) \cdot 2^{n-k-1}\right)\\
&\quad \cdot \frac{x!(P-1-x)!}{P!} \cdot \frac{(P-x)\cdots(P+k-1-x)}{(P+1)\cdots(P+k)}\\
&< \left(2^{2n-2k} + 2^{n-k+1}(k'-1) \cdot 2^{n-k-1}\right)\\
&\quad \cdot \frac{1}{k'} \cdot \frac{x!(P-1-x)!}{P!}\\
&= \frac{2^{2n-2k}k'}{k'} \cdot \frac{x!(P-1-x)!}{P!} \;=\; \varphi(\mathcal{G}, p),
\end{aligned}$$

which means that the Shapley–Shubik index of player p decreases. Thus the Shapley–Shubik index of player p can neither increase nor nondecrease by adding up to k players from M to the game $\mathcal{G}$, and we have a no-instance of both our control problems. □

Theorem 5. *The following seven problems are* $\mathrm{NP}^{\mathrm{PP}}$*-complete:*

(a) CONTROL-BY-ADDING-PLAYERS-TO-NONDECREASE-β; *and for* $\gamma \in \{\beta, \varphi\}$*,*

(b) CONTROL-BY-ADDING-PLAYERS-TO-DECREASE-γ*,*

(c) CONTROL-BY-ADDING-PLAYERS-TO-NONINCREASE-γ*, and*

(d) CONTROL-BY-ADDING-PLAYERS-TO-MAINTAIN-γ*.*

5 Conclusions

We have shown that control by adding players to WVGs so as to change or maintain a given player's Shapley–Shubik or Penrose–Banzhaf index is $\mathrm{NP}^{\mathrm{PP}}$-complete, thus settling the complexity of these problems by raising their lower bounds so as to match their upper bound. Compared with the eminently rich body of results on control attacks in voting [11], these results fill a glaring gap in the literature on WVGs which—perhaps due to the immense hardness of these problems that is proven here—fairly much has neglected issues of control attacks to date.

For future work, we propose to study the corresponding problems for deleting players from WVGs. Further, it would be interesting to study these problems in the model proposed by Kaczmarek and Rothe [16] in which the quota is indirectly changed when players are added or deleted. Our techniques may also turn out to be useful for closing the complexity gap for other problems in $\mathrm{NP}^{\mathrm{PP}}$ only known to be PP-hard, such as false-name manipulation [1, 25] and control by adding or deleting edges in graph-restricted WVGs [18].

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