ECAI 2024
U. Endriss et al. (Eds.)

doi:10.3233/FAIA240624

A Novel View of Analogical Proportion Between Formulas

Andreas Herzig[a], Emiliano Lorini[a] and Henri Prade[a]

[a]IRIT, CNRS, Toulouse University, France

Abstract. Analogical proportions are statements of the form "α is to β as γ is to δ", noted $\alpha : \beta :: \gamma : \delta$, and can be understood as "α differs from β as γ differs from δ" and conversely "β differs from α as δ differs from γ". In this paper, $\alpha, \beta, \gamma, \delta$ are supposed to be propositional logic formulas, which are appropriate for representing concepts. There exists one approach, developed over the last 15 years, where "α differs from β" is understood in terms of the negation of the material implication $\alpha \to \beta$. The paper investigates another view where "α differs from β" is interpreted in terms of transformations where some variables become false, some variables become true, and some variables become irrelevant. Both approaches satisfy the three basic postulates of analogical proportions (reflexivity, symmetry, and stability under central permutation), as well as other interesting properties such as transitivity and unicity of δ such that $\alpha : \beta :: \gamma : \delta$. However, the two approaches depart from each other since they do not validate the same analogical proportions. In particular, when p, q, r are atoms the proportion $p : (p \wedge r) :: q : (q \wedge r)$ holds in the new approach, while it fails to do so for the other. The new approach exhibits also a good behaviour with respect to integrity constraints. It is advocated that this makes it appropriate for handling analogy between concepts, while the other approach has proved to be fruitful for Boolean features-based representations. The paper provides a thorough analysis of the differences between the two approaches.

1 Introduction

For a long time, analogy and logic belonged to separate worlds. The main modeling of analogical reasoning, the Structure Mapping Theory [7], has little to do with logic, even if some have proposed to integrate it into a second-order logic framework; see [8] and their HDTP (heuristic-driven theory projection) approach. However, following the intuition of the great mathematician Polya [18] about the solving of mathematical problems, the use of analogy in theorem proving has been widely studied [5]. Let us also mention the proposal of an encoding of analogical inference in first-order logic by [24], but it results in a framework that is too narrow and difficult to apply in practice.

However, a form of analogical inference can be encoded in terms of analogical proportions; see, e.g., [4]. Analogical proportions are statements of the form "α is to β as γ is to δ", denoted $\alpha{:}\beta :: \gamma{:}\delta$. Recently a logical characterization of a class of analogical proportions has been proposed [13]. Analogical proportions have attracted growing interest in various fields of artificial intelligence research [21]. They can be described as a quaternary connective in propositional logic as first shown in [16]. Yet, analogical proportions *between logical formulas* have been little considered until now. This is the topic of this paper.

So in this study, analogical proportions are 4-ary relations $\alpha{:}\beta :: \gamma{:}\delta$, where $\alpha, \beta, \gamma, \delta$ are propositions. Up to now their logical analysis was mainly restricted to the case where $\alpha, \beta, \gamma, \delta$ are restricted to *maximal terms*: maximal consistent conjunctions of literals on some finite language [3]. We here explore how such an analysis can be generalised to arbitrary formulas.

Our approach is based on what we call transformations, which are functions mapping α to β and γ to δ. For maximal terms such transformations are simple: the truth values of the variables get flipped in the same sense. This means that the transformation turns some truths into falsities, turns some falsities into truths, and leaves the rest unchanged. It will be convenient to reformulate this as: (1) the `tf`-difference between α and β equals the `tf` between γ and δ and (2) the `tf`-difference between β and α equals the `tf` between δ and γ, where the `tf`-difference between two terms α and β is the set of variables that occur positively in α and negatively in β.

While this idea can be transferred from maximal terms to arbitrary propositional formulas, it however does not account for several manifest cases of analogy, such as the proportion $p : (p \wedge r) :: q : (q \wedge r)$. The sentence "a man is to a king as a woman is to a queen", due to [23], is a well-known example of such analogical proportion. Indeed, the concept of king can be seen as the conjunction of the two more primitive concepts of man (M) and royal (R), while the concept of queen can been as the conjunction of the two more primitive concepts of (W) and royal (R). Therefore, we should have $\mathsf{M} : (\mathsf{M} \wedge \mathsf{R}) :: \mathsf{W} : (\mathsf{W} \wedge \mathsf{R})$.

We therefore generalise transformations as follows: the truth values of some variables get flipped in the same sense, and the truth values of some others are maximally varied. The latter means that they become irrelevant. We show that this restriction makes that $\alpha{:}\beta :: \gamma{:}\delta$ holds if and only if

- $\alpha, \beta, \gamma, \delta$ are all terms;
- The `tf`-difference between α and β equals the `tf`-difference between γ and δ, and the `tf`-difference between β and α equals the `tf`-difference between δ and γ;
- The `ti`-difference between α and β equals the `ti`-difference between γ and δ and the `ti`-difference between β and α equals the `ti`-difference between δ and γ;
- The `fi`-difference between α and β equals the `fi`-difference between γ and δ and the `fi`-difference between β and α equals the `fi`-difference between δ and γ;

where the `ti`-difference between α and β is the set of variables that occur positively in α and do not occur in β; and the `fi`-difference between α and β is the set of variables that occur negatively in α and do not occur in β. Observe that while for maximal terms we only have to consider two differences, viz. the `tf`-difference between

α and β and the tf-difference between β and α, the extension to arbitrary terms requires 6 difference sets.

The paper is organised as follows. In Section 2 we recall the main principles for analogical proportions that have been studied in the literature. In Section 3, we offer an overview of how the notion of analogical proportion has been formalized in the literature. We show that the few attempts to formalize analogical proportion between propositional formulas proposed in the literature have drawbacks: they are not able to satisfy at the same time the three basic postulates of analogical proportions, namely reflexivity, transitivity and central permutation, and the property of uniformity that is required to validate the intuitive analogical proportion $\mathsf{M} : (\mathsf{M} \wedge \mathsf{R}) :: \mathsf{W} : (\mathsf{W} \wedge \mathsf{R})$ (i.e., "a man is to a king as a woman is to a queen"). In Section 4, we present our novel approach to analogical proportion between propositional formulas based on the notion of transformation between valuations we sketched above. The king/queen example by Rumelhart & Abrahamson does not full exploit the expressiveness of propositional logic since the terms of comparison in the analogical proportion do not include negation or disjunction. To fill this gap, in Section 5 we present a richer example in which the analogical proportion is between complex propositional formulas with negation and disjunction. Section 6 we conclude.

2 Postulates for Analogical Proportions

Probably inspired by the works of mathematicians such as Archytas of Taras on numerical proportions, Aristotle was the first (at least in the Western World) to consider analogical proportions between words [1]. On this basis, it is natural to assume the following basic postulates for analogical proportions: reflexivity, symmetry, and stability under central permutation. Note that they hold for arithmetic proportions ($\alpha - \beta = \gamma - \delta$) and geometric proportions ($\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$) where $\alpha, \beta, \gamma, \delta$ are numbers. Mind that in this background section $\alpha, \beta, \gamma, \delta$ are not necessarily propositions but denote any elements that can participate in an analogical proportion. The three postulates write

(refl) $\alpha{:}\beta :: \alpha{:}\beta$;
(sym) If $\alpha{:}\beta :: \gamma{:}\delta$ then $\gamma{:}\delta :: \alpha{:}\beta$;
(cperm) If $\alpha{:}\beta :: \gamma{:}\delta$ then $\alpha{:}\gamma :: \beta{:}\delta$.

These traditional postulates are currently supposed to hold for any four items (numbers, words, sentences, drawings, pictures, etc.) realizing an analogical proportion. Note however that *(cperm)* is a strong requirement that may appear debatable. Indeed while the analogical proportion "a man is to a king what a woman is to a queen" leads to another meaningful proportion by central permutation: "a man is to a woman what a king is to a queen", it is much less clear that *(cperm)* applied to "wine is to the French as beer is to the English" yields a statement that makes sense.

Moreover *(cperm)* has a strong consequence on the classical view of analogy as a parallel between two situations where the entities of the first situation can be mapped onto the entities of the second one on the basis of the properties and relations that hold for them [7][25]. Indeed, as noticed in Hesse [9], if α and β are two entities of the first situation that can be mapped onto the entities γ and δ of the second situation, one can say that "α is to β as γ is to δ". But permuting β and γ presupposes that the entities belong to the same universe where the two situations can be embedded. As it can be noticed in the last example where central permutation does not make sense, there exist two distinct universes: the one of the beverages and the one of the people.

Besides, there exists a "functional" view of analogical proportion where $\alpha{:}\beta :: \gamma{:}\delta$ is understood as: there is an (invertible) function f such that $\beta = f(\alpha)$ (and $f^{-1}(\beta) = \alpha$) together with $\delta = f(\gamma)$ (and $f^{-1}(\delta) = \gamma$) see, e.g., [10]. Thus $\alpha{:}\beta :: \gamma{:}\delta$ reads
$\alpha : f(\alpha) :: \gamma : f(\gamma)$.
Applying central permutation, we obtain
$\alpha : \gamma :: f(\alpha) : f(\gamma)$.
Thus it is also expected that there is a function g such that $\gamma = g(\alpha)$, which leads to
$\alpha : f(\alpha) :: g(\alpha) : f(g(\alpha))$,
but because of central permutation, we should also have
$\alpha : g(\alpha) :: f(\alpha) : g(f(\alpha))$.
since f and g play symmetrical roles. So we need to have $f(g(\alpha)) = g(f(\alpha))$ (thus assuming unicity of the fourth term, given the first three). As a conclusion, f and g need to be invertible and to commute [3], but all these requirements are not always assumed, leading to a violation of the central permutation postulate.

Accepting the three postulates entails that the following properties hold:

- $\alpha : \alpha :: \beta : \beta$ (*identity*);
- $\alpha : \beta :: \gamma : \delta \Rightarrow \delta : \beta :: \gamma : \alpha$ (*extreme permutation*);
- $\alpha : \beta :: \gamma : \delta \Rightarrow \beta : \alpha :: \delta : \gamma$ (*internal reversal*);
- $\alpha : \beta :: \gamma : \delta \Rightarrow \delta : \gamma :: \beta : \alpha$ (*complete reversal*).

However, it is important to note that other potential properties of interest such as unicity and transitivity are not consequences of the postulates (and are not always accepted):

(unic) If $\alpha{:}\beta :: \gamma{:}\delta$ and $\alpha{:}\beta :: \gamma{:}\epsilon$ then $\delta = \epsilon$;
(trans) If $\alpha{:}\beta :: \gamma{:}\delta$ and $\gamma{:}\delta :: \epsilon{:}\zeta$ then $\alpha{:}\beta :: \epsilon{:}\zeta$.

3 Analogy between Formulas: A First Approach

About fifteen years ago, a first logical modeling of analogical proportions was proposed [15, 16], coming after a number of previous proposals outside logic [11, 14, 17, 26]. We first recall this modeling and exemplify it. We then explain how it can be applied to the definition of analogical proportions between logical formulas, and we conclude the section by pointing out that this approach misses some desirable properties.

3.1 A Logical Modeling between Atomic Variables

We now assume that $\alpha, \beta, \gamma, \delta$ are Boolean variables with value in $\mathbb{B} = \{0, 1\}$. Applying the three postulates *(refl)*, *(sym)*, *(cperm)*, and viewing $\alpha : \beta :: \gamma : \delta$ as a quaternary connective, it is easy to see that $\alpha : \beta :: \gamma : \delta$ should be true for the 6 valuations given in Table 1

α	β	γ	δ
0	0	0	0
1	1	1	1
0	1	0	1
1	0	1	0
0	0	1	1
1	1	0	0

Table 1. The 6 valuations making true an analogical proportion

The first four lines are forced by reflexivity, the last two are obtained by applying *(cperm)*. This is the minimal Boolean model [20] validating reflexivity and stability under central permutation, and it is also symmetrical. This can be logically expressed by the following formula, first proposed in [15] (where $\wedge$, $\neg$, and $\equiv$ respectively denote conjunction, negation and equivalence in propositional calculus):

$$\alpha : \beta :: \gamma : \delta = ((\alpha \wedge \neg\beta) \equiv (\gamma \wedge \neg\delta)) \wedge ((\neg\alpha \wedge \beta) \equiv (\neg\gamma \wedge \delta)) \quad (1)$$

This formula is true only for the 6 valuations given in Table 1 and is false for the $2^4 - 6 = 10$ remaining valuation patterns. As can be also seen, 0 and 1 play exchangeable roles, which indicates that the truth value of $\alpha : \beta :: \gamma : \delta$ does not depend on the encoding of truth and falsity. The above formula readily states that "α differs from β as γ differs from δ and β differs from α as δ differs from γ" (indeed $\alpha \wedge \neg\beta$ expresses a logical difference).

Although it is not immediately obvious, the expression (1) of an analogical proportion may receive a functional reading in the sense of Section 2 [3]. Indeed when $\alpha : \beta :: \gamma : \delta$ holds true,

let $\lambda = \alpha \wedge \neg\beta \equiv \gamma \wedge \neg\delta$, and $\mu = \neg\alpha \wedge \beta \equiv \neg\gamma \wedge \delta$.
Then the function $f(\chi) = (\chi \wedge \neg\lambda) \vee \mu$ is such that $\beta = f(\alpha)$ as can be checked.

Similarly let $g(\chi) = (\chi \wedge \neg\nu) \vee \xi$
with $\nu = \alpha \wedge \neg\gamma \equiv \beta \wedge \neg\delta$ and $\xi = \neg\alpha \wedge \gamma \equiv \neg\beta \wedge \delta$.
It can be checked that the above expression of $\alpha : \beta :: \gamma : \delta$ agrees with a functional conception of analogical proportion, namely we have

$\alpha : \beta :: \gamma : \delta = \alpha : f(\alpha) :: g(\alpha) : f(g(\alpha))$
with $f(g(\alpha)) = g(f(\alpha))$.

Besides, expression (1) (and Table 1) not only satisfy the 3 postulates *(refl)*, *(sym)*, *(cperm)*, but also the unicity (of δ given α, β, γ) and transitivity properties, as can be checked [16].

3.2 Example

Analogical proportions can be extended componentwise to items represented by vectors of Boolean variables such as $\alpha = (\alpha_1, ..., \alpha_n)$ defined on the same set of Boolean variables, namely:

$$\alpha : \beta :: \gamma : \delta \quad \text{iff} \quad \forall i \in [1, n],\ \alpha_i : \beta_i :: \gamma_i : \delta_i$$

We illustrate this view of analogical proportion on Boolean vector representations with an example inspired from [23] who very early proposed a parallelogram view of analogical proportions between words modeled by an arithmetic proportion $\beta - \alpha = \delta - \gamma$ for items represented by real-valued vectors; we can check that all the valuations in Table 1 satisfy these equalities.

Here the four considered items, namely `man` (M), `king` (K), `woman` (W) and `queen` (Q) are represented in terms of 6 Boolean variables (describing the sex, the position and the nature of the items) as given in Table 2. In this representation it is clear that the analogical proportion "a man is to a king as a woman is to a queen" holds true, since for each variable an analogical proportion holds.

	sexM	*sexF*	*power pos.*	*ordinary pos.*	*human*	*god*
M	1	0	0	1	1	0
K	1	0	1	0	1	0
W	0	1	0	1	1	0
Q	0	1	1	0	1	0

Table 2. The `man`/`king`/`woman`/`queen` example

3.3 Application to Analogical Proportions between Formulas

The above logical modeling of analogical proportions has been successfully used in machine learning classification [4], or in solving IQ tests [6]. However, such applications do not require the definition of analogical proportions between logical formulas of any kind. This later topic was only briefly touched in [19].

Any propositional formula with n variables can be described by a bitstring of size 2^n. For instance, if we consider two Boolean variables p and q, we have have the interpretations pq, $p\neg q$, $\neg pq$, $\neg p\neg q$. Taking in this order, p is encoded by 1100, $p \wedge q$ by 1000, and so on.

Thus given 4 logical formulas $\alpha, \beta, \gamma, \delta$, involving together n variables, and thus representable by bitstrings of size 2^n, we say that $\alpha : \beta :: \gamma : \delta$ holds if $(\alpha \wedge \neg\beta) \equiv (\gamma \wedge \neg\delta)$ and $(\neg\alpha \wedge \beta) \equiv (\neg\gamma \wedge \delta)$.

It can be shown that this is equivalent to having an analogical proportion $\alpha_i : \beta_i :: \gamma_i : \delta_i$ on each component i of the bitstrings that represent the formulas, i.e., for each possible interpretation [22] (note that i refers here to an interpretation, not to a Boolean variable as in the example of Table 2). Let us take an example that appears in [19] and that also holds in any lattice structure [2]:

$$\alpha \vee \beta : \alpha :: \beta : \alpha \wedge \beta$$

It can be easily checked that $(\alpha \vee \beta) \wedge \neg\alpha \equiv \beta \wedge \neg(\alpha \wedge \beta) \equiv \neg\alpha \wedge \beta$ and $\alpha \wedge \neg(\alpha \vee \beta) \equiv (\alpha \wedge \beta) \wedge \neg\beta \equiv \bot$. Thus the above analogical proportion does hold, and their respective bitstrings are indeed in analogical proportion as well: indeed $1110 : 1100 :: 1010 : 1000$ holds componentwise for the interpretations $\alpha\beta, \alpha\neg\beta, \neg\alpha\beta, \neg\alpha\neg\beta$ (since for each interpretation we recognize one of the 6 valuations of Table 1: 1:1 :: 1:1, 1:1 :: 0:0, 1:0 :: 1:0, 0:0 :: 0:0, respectively).

Another remarkable analogical proportion between logical formulas describes the behavior of negation:

$$\alpha : \beta :: \neg\beta : \neg\alpha$$

This analogical proportion holds as it can be easily checked (since $\alpha \wedge \neg\beta = \neg\beta \wedge \neg(\neg\alpha)$). This can be directly related to the contraposition of material implication. One may have rather expected that we have $\alpha : \beta :: \neg\alpha : \neg\beta$ (which does not hold with the definition proposed above). Still if we consider that it holds at the level of the syntactic presentation of formulas, we can justify it. Indeed Table 3 provides a justification using a representation based on the symbol(s) present or not in each formula.

	presence of α	*presence of* β	*presence of* $\neg$
α	1	0	0
β	0	1	0
$\neg\alpha$	1	0	1
$\neg\beta$	0	1	1

Table 3. Justifying $\alpha : \beta :: \neg\alpha : \neg\beta$ by the syntactic presentation

Thus, in the view of analogical proportion between logical formulas presented in this subsection, we focus on the semantic equality of logical differences between formulas, and this is compatible with a vector-based representation in terms of bitstrings.

3.4 Failure of a Desirable Property

We furthermore study the following property of uniformity:

(unif) If α and β are terms and p is a variable neither occurring in α nor in β then $\alpha{:}(\alpha \wedge p) :: \beta{:}(\beta \wedge p)$ and $\alpha{:}(\alpha \wedge \neg p) :: \beta{:}(\beta \wedge \neg p)$.

This property accounts for classical cases of analogical proportions such as "a man is to a king what a woman is to a queen" if we formalise 'king' as the conjunction of 'man' and 'royal' and 'queen' as the conjunction of 'woman' and 'royal'.

Unfortunately, the logical difference $\alpha \wedge \neg(\alpha \wedge p)$ is not equal in general to the logical difference $\beta \wedge \neg(\beta \wedge p)$. So the above desirable property fails to hold in the above approach.

This should not come as surprise, since it has been observed [19] that if we have two analogical proportions between Boolean variables $\alpha : \beta :: \gamma : \delta$ and $\alpha' : \beta' :: \gamma' : \delta'$, then neither $\alpha \wedge \alpha' : \beta \wedge \beta' :: \gamma \wedge \gamma' : \delta \wedge \delta'$ nor $\alpha \vee \alpha' : \beta \vee \beta' :: \gamma \vee \gamma' : \delta \vee \delta'$ hold in the general case.

Thus if we consider the example of Table 2, and we create a new column for the conjunction of two variables, say $sex\ M$ and $power\ pos.$, we get 0100 (since the conjunction is true only for `king`), which is not a valuation validating an analogical proportion according to Table 1. The only way to justify it would be to use a representation based on the symbol(s) present or not in each formula as in the case of Table 3.

Yet, given an analogical proportion $\alpha : \beta :: \gamma : \delta$ which holds between formulas $\alpha, \beta, \gamma, \delta$, for any formula ϵ, the proportion $\alpha \wedge \epsilon : \beta \wedge \epsilon :: \gamma \wedge \epsilon : \delta \wedge \epsilon$ still holds. This enables us to take into account an integrity constraint ϵ in the evaluation of the analogical proportion, as done in subsection 4.7.

Due to the failure of *(unif)*, we are in need of a novel view of analogical proportions between logical formulas if we want to preserve the above desirable property. This is the topic of the next sections.

4 Analogy between Formulas: A New Approach

We now propose an alternative definition that has not been investigated before. We start by formulating it for the special case of maximal terms and then generalise it to arbitrary formulas. We show that the generalisation restricts the four formulas making up the arguments of analogical proportions to be terms. We also show that it satisfies the principles *(refl)*, *(sym)*, *(cperm)*, *(unic)*, *(trans)*, and *(unif)*. But before dwelving into the new proposal some preliminary notation and notions of propositional logic are introduced in the next section.

4.1 Background

We suppose given a finite set of propositional variables $\mathbb{P} = \{p, q, r, \ldots\}$. We use $a, b, c, d, \ldots$ to denote subsets of $\mathbb{P}$. We can think of these subsets as valuations: for every propositional variable $p \in \mathbb{P}$, when $p \in a$ then p is true at the valuation a; and when $p \notin a$ then p is false at a.

A *literal* is a propositional variable or the negation of a propositional variable. A *term* is a consistent conjunction of literals. A *maximal term* is a maximal consistent conjunction of literals on $\mathbb{P}$. We can identify any maximal term with a valuation on $\mathbb{P}$; that is, we identify a valuation a on $\mathbb{P}$ with $(\bigwedge_{p \in a} p) \wedge (\bigwedge_{p \in \mathbb{P} \setminus a} \neg p)$. Valuations can therefore be viewed as particular formulas.

We use $\alpha, \beta, \gamma, \delta, \ldots$ to denote arbitrary propositional formulas on $\mathbb{P}$. Formulas can be identified with sets of valuations, that is, with subsets of $2^{\mathbb{P}}$. In particular, a maximal term can be identified with a singleton valuation; and an arbitrary term $p_1 \wedge \ldots \wedge p_n \wedge \neg q_1 \wedge \ldots \wedge \neg q_m$ can be identified with the set of valuations

$$\{\{p_1, \ldots, p_n\} \cup Q \ : \ Q \subseteq \mathbb{P} \setminus \{q_1 \ldots q_m\}\}.$$

The other way round, a set of valuations α can be identified with a term if and only if

$$\alpha = \left\{a \subseteq \mathbb{P} \ : \ \left(\bigcap \alpha\right) \subseteq a \subseteq \mathbb{P} \setminus \left(\bigcup \alpha\right)\right\}.$$

As usual, α *entails* β, noted $\alpha \models \beta$, when all models of α are also models of β; and α is *logically equivalent* to β, noted $\alpha \equiv \beta$, when the models of α equal the models of β.

We say that a variable p is *irrelevant* for α if $\alpha[p/\top] \equiv \alpha[p/\bot]$. Semantically, p is irrelevant for α if for every $a \in \alpha$, $a \cup \{p\} \in \alpha$ and $a \setminus \{p\} \in \alpha$. When α is a term then p is irrelevant for α if and only if p does not occur in $\bigcap \alpha$.

4.2 Analogy between Maximal Terms

Let a, b, c, d be maximal terms, alias valuations. The analogical proportion $a{:}b :: c{:}d$ is the case if and only if $b \backslash a = d \backslash c$ and $a \backslash b = c \backslash d$. Intuitively, the first condition says that what becomes true from a to b equals what becomes true from c to d; and the second condition says that what becomes false from a to b equals what becomes false from c to d.

This seems to appropriately capture our intuitions about analogicaly proportions between maximal terms. Moreover, the definition satisfies all the postulates. In particular, symmetry is the case because what becomes true from a to b is what becomes false from b to a.

Example 1 *Let $\mathbb{P}_{\mathsf{M,W,R}} = \{\mathsf{M}, \mathsf{W}, \mathsf{R}\}$ where M stands for "man", W stands for "woman", R stands for "royal". Then the analogies*

$$\{\mathsf{M}\}{:}\{\mathsf{M},\mathsf{R}\} :: \{\mathsf{W}\}{:}\{\mathsf{W},\mathsf{R}\},$$
$$\{\mathsf{M}\}{:}\{\mathsf{W}\} :: \{\mathsf{M},\mathsf{R}\}{:}\{\mathsf{W},\mathsf{R}\}$$

are the case. Written as maximal consistent conjunctions of literals on $\mathbb{P}_{\mathsf{M,W,R}}$:

$$(\mathsf{M} \wedge \neg\mathsf{W} \wedge \neg\mathsf{R}){:}(\mathsf{M} \wedge \neg\mathsf{W} \wedge \mathsf{R}) :: (\mathsf{W} \wedge \neg\mathsf{M} \wedge \neg\mathsf{R}){:}(\mathsf{W} \wedge \neg\mathsf{M} \wedge \mathsf{R});$$
$$(\mathsf{M} \wedge \neg\mathsf{W} \wedge \neg\mathsf{R}){:}(\mathsf{W} \wedge \neg\mathsf{M} \wedge \neg\mathsf{R}) :: (\mathsf{M} \wedge \neg\mathsf{W} \wedge \mathsf{R}){:}(\mathsf{W} \wedge \neg\mathsf{M} \wedge \mathsf{R}).$$

It remains to generalise our definition to formulas on the alphabet $\mathbb{P}$; or, alternatively, to sets of $\mathbb{P}$-valuations. We work towards such a definition in the rest of the paper. To warm up we reformulate our definition in terms of transformations: partial functions on valuations $\xrightarrow{\mathtt{f},\mathtt{t}}$ that are parametrised by two disjoint sets of variables $\mathtt{f}, \mathtt{t} \subseteq \mathbb{P}$. The variables in $\mathtt{t}$ are made true by the transformation and those in $\mathtt{f}$ are made false. Therefore the application of the transformation $\xrightarrow{\mathtt{f},\mathtt{t}}$ to a valuation a yields the valuation $(a \setminus \mathtt{f}) \cup \mathtt{t}$.

Definition 2 *Let $\mathtt{f}, \mathtt{t} \subseteq \mathbb{P}$ be disjoint. For two valuations $a, b \subseteq \mathbb{P}$, $a \xrightarrow{\mathtt{f},\mathtt{t}} b$ is the case if and only if $b = (a \setminus \mathtt{f}) \cup \mathtt{t}$.*

For example, we have both $\emptyset \xrightarrow{\emptyset,\{p\}} \{p\}$ and $\{p\} \xrightarrow{\emptyset,\{p\}} \{p\}$. Such transformations have the property that for any two valuations a and b there is at least one transformation $\xrightarrow{\mathtt{f},\mathtt{t}}$ such that $a \xrightarrow{\mathtt{f},\mathtt{t}} b$, namely $\mathtt{t} = b \setminus a$ and $\mathtt{f} = a \setminus b$. The operation of composition of transformations (defined as function composition) fails to be commutative and $\xrightarrow{\mathtt{t},\mathtt{f}}$ is not necessarily the inverse of $\xrightarrow{\mathtt{f},\mathtt{t}}$.[1] We therefore define analogy in terms of two transformations, a 'forth' transformation $\xrightarrow{\mathtt{f}_1,\mathtt{t}_1}$ and a 'back' transformation $\xrightarrow{\mathtt{f}_2,\mathtt{t}_2}$.

Definition 3 *For $a, b, c, d \subseteq \mathbb{P}$, the analogical proportion $a{:}b :: c{:}d$ is the case if and only if there are two transformations $\xrightarrow{\mathtt{f}_1,\mathtt{t}_1}$ and $\xrightarrow{\mathtt{f}_2,\mathtt{t}_2}$ such that:*

[1] As an aside, we can think of $\xrightarrow{\mathtt{f},\mathtt{t}}$ as a STRIPS action with positive effects $\mathtt{t}$ and negative effects $\mathtt{f}$.

- $a \xrightarrow{\mathtt{f}_1,\mathtt{t}_1} b$ *and* $c \xrightarrow{\mathtt{f}_1,\mathtt{t}_1} d$;
- $b \xrightarrow{\mathtt{f}_2,\mathtt{t}_2} a$ *and* $d \xrightarrow{\mathtt{f}_2,\mathtt{t}_2} c$.

Proposition 4 *Let $a, b, c, d \subseteq \mathbb{P}$. Then $a{:}b :: c{:}d$ holds if and only if $b \setminus a = d \setminus c$ and $a \setminus b = c \setminus d$.*

PROOF. From the right to the left, suppose $b \setminus a = d \setminus c$ and $a \setminus b = c \setminus d$. Let $\mathtt{f}_1 = \mathtt{t}_2 = b \setminus a$ and $\mathtt{t}_1 = \mathtt{f}_1 = a \setminus b$. Then we have both $a \xrightarrow{\mathtt{f}_1,\mathtt{t}_1} b$ and $c \xrightarrow{\mathtt{f}_1,\mathtt{t}_1} d$ and $b \xrightarrow{\mathtt{f}_2,\mathtt{t}_2} a$ and $d \xrightarrow{\mathtt{f}_2,\mathtt{t}_2} c$. ∎

It follows from Proposition 4 that reflexivity, symmetry, central permutation, unicity, and transitivity hold [3].

4.3 Analogy between Sets of Valuations: First Attempt

We start by a straightforward generalisation of Definition 3 and discuss its shortcomings; we then refine it to our official definition of analogical proportion.

An obvious way of lifting the definition of transformation from valuations to sets of valuations is the following: for $\alpha, \beta \subseteq 2^{\mathbb{P}}$, $\alpha \xrightarrow{\mathtt{f},\mathtt{t}} \beta$ if and only if β entails $\neg(\bigvee \mathtt{f}) \wedge (\bigwedge \mathtt{t})$ and the status of all variables in $\mathbb{P} \setminus (\mathtt{f} \cup \mathtt{t})$ is the same for α and β; that is, for every $p \in \mathbb{P} \setminus (\mathtt{f} \cup \mathtt{t})$, α entails p iff β entails p and α entails $\neg p$ iff β entails $\neg p$. However, the desirable principle $\alpha{:}(\alpha \wedge p) :: \beta{:}\beta \wedge p$ fails for that definition. For example,

$$\{\{\mathsf{M}\}, \{\mathsf{M}, \mathsf{R}\}\}{:}\{\{\mathsf{M}, \mathsf{R}\}\} :: \{\{\mathsf{W}\}, \{\mathsf{W}, \mathsf{R}\}\}{:}\{\{\mathsf{W}, \mathsf{R}\}\}$$

does not hold, alias its syntactic counterpart

$$(\mathsf{M} \wedge \neg\mathsf{W}){:}(\mathsf{M} \wedge \neg\mathsf{W} \wedge \mathsf{R}) :: (\mathsf{W} \wedge \neg\mathsf{M}){:}(\mathsf{W} \wedge \neg\mathsf{M} \wedge \mathsf{R}).$$

For our example the reason is that there is no 'back' transformation $\xrightarrow{\mathtt{f}_2,\mathtt{t}_2}$ such that $\{\{\mathsf{M}, \mathsf{R}\}\} \xrightarrow{\mathtt{f}_2,\mathtt{t}_2} \{\{\mathsf{M}\}, \{\mathsf{M}, \mathsf{R}\}\}$.

4.4 Generalising the Transformation

The shortcomings of our tentative definition motivate a generalisation of the transformation function. The idea is to go beyond making some variables true and some others false by moreover varying the truth values of a third set of variables. Intuitively, for disjoint $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$, we say that $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$ is the case if β is obtained from α by (1) making all variables of $\mathtt{t}$ true; (2) making all variables of $\mathtt{f}$ false; and (3) varying the truth value of the variables of $\mathtt{v}$ in all possible ways.[2] The following generalisation of Definition 2 formalises this.

Definition 5 *Let $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$ be disjoint. For two valuations $a, b \subseteq \mathbb{P}$, $a \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} b$ is the case if and only if $b = (a \setminus \mathtt{f} \cup \mathtt{v}) \cup \mathtt{t} \cup \mathtt{v}'$ for some $\mathtt{v}' \subseteq \mathtt{v}$.*

The following notion of reachable set is useful to extend transformations $\xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}$ from valuations to sets of valuations.

Definition 6 *Let $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$ be disjoint. For $a \subseteq \mathbb{P}$ and $\beta \subseteq 2^{\mathbb{P}}$, we say that β is the reachable set from a through the transformation $\xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}$ if and only if $\beta = \{b \subseteq \mathbb{P} \,:\, a \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} b\}$.*

[2] In terms of actions this amounts to the move from STRIPS actions to non-deterministic actions. In terms of belief change operations this corresponds to a 'forgetting' operation [12].

As a tranformation $\xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}$ maps every valuation $a \in \alpha$ to β we say that it is *uniform*.

Definition 7 *Let $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$ be disjoint. For $\alpha, \beta \subseteq 2^{\mathbb{P}}$, $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$ is the case if and only if, for every valuation $a \in \alpha$, β is the reachable set from a through $\xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}$.*

Contrarily to the transformation of single valuations (alias complete terms), there are α and β such that no $\mathtt{f}, \mathtt{t}, \mathtt{v}$ with $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$ exist. For example, there are no $\mathtt{f}, \mathtt{t}, \mathtt{v}$ such that $\{\emptyset\} \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \{\{p\}, \{q\}\}$. However:

Proposition 8 *For every α and β, there exist sets $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$ such that $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$ if and only if β is equivalent to a term.*

PROOF. Suppose $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$. For every $a \in \alpha$, the set $\{b \subseteq \mathbb{P} \,:\, a \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} b\}$ is a term. Therefore β is a term.

For the other sense suppose β is a term. Then $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$ holds if we set $\mathtt{f} = \mathbb{P} \setminus (\bigcup \beta)$, $\mathtt{t} = \bigcap \beta$, and $\mathtt{v} = \mathbb{P} \setminus (\mathtt{f} \cup \mathtt{t})$. ∎

Observe that $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \beta$ does not guarantee that a 'backwards' transformation exists such that $\beta \xrightarrow{\mathtt{f}',\mathtt{t}',\mathtt{v}'} \alpha$. To witness, we have $\{\{p\}, \{q\}\} \xrightarrow{\{p,q\},\emptyset,\emptyset} \{\emptyset\}$. while we have seen above that no uniform transformation $\{\emptyset\} \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} \{\{p\}, \{q\}\}$ exists. This observation makes us conclude that the definition of analogy has to include the converse direction of the transformation.

4.5 Definition of Analogy

Definition 9 *For $\alpha, \beta, \gamma, \delta \subseteq 2^{\mathbb{P}}$, the analogical proportion $\alpha{:}\beta :: \gamma{:}\delta$ is the case if and only if there are two uniform transformations $\xrightarrow{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}$ and $\xrightarrow{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}$ such that:*

1. $\alpha \xrightarrow{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1} \beta$ *and* $\beta \xrightarrow{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2} \alpha$;
2. $\gamma \xrightarrow{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1} \delta$ *and* $\delta \xrightarrow{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2} \gamma$.

When the terms $\alpha, \beta, \gamma, \delta$ are maximal (that is, when the valuations $\alpha, \beta, \gamma, \delta$ are singletons) then the sets $\mathtt{v}_1$ and $\mathtt{v}_2$ are empty and our definition boils down to the above definition of analogy between individual valuations: Definition 9 coincides with Definition 3.

Example 10 *The analogy "men are to kings what women are to queens" holds. Formally,*

$$\{\{\mathsf{M}\}, \{\mathsf{M}, \mathsf{R}\}\}{:}\{\{\mathsf{M}, \mathsf{R}\}\} :: \{\{\mathsf{W}\}, \{\mathsf{W}, \mathsf{R}\}\}{:}\{\{\mathsf{W}, \mathsf{R}\}\},$$

holds thanks to the 'forth' transformation $\xrightarrow{\emptyset,\{\mathsf{R}\},\emptyset}$ and the 'back' transformation $\xrightarrow{\emptyset,\emptyset,\{\mathsf{R}\}}$. Its syntactic counterpart is:

$$(\mathsf{M} \wedge \neg\mathsf{W}){:}(\mathsf{M} \wedge \neg\mathsf{W} \wedge \mathsf{R}) :: (\mathsf{W} \wedge \neg\mathsf{M}){:}(\mathsf{W} \wedge \neg\mathsf{M} \wedge \mathsf{R})$$

Furthermore, $\mathsf{M}{:}(\mathsf{M} \wedge \mathsf{R}) :: \mathsf{W}{:}(\mathsf{W} \wedge \mathsf{R})$ also holds thanks to the same transformation.

Example 11 *Let us enrich the previous example by adding the new variable P which stands for "he/she has the legal right to receive somebody's property". The following analogy holds: "a heir is to a prince what a heiress is to a princess", or, formulated more explicitly, "A man who has the legal right to receive somebody's property (be it royal or not) is to a prince what a woman who has the legal right to* *receive somebody's property (royal or not) is to a princess". Namely, we have:*

$$\{\{\mathsf{M},\mathsf{P}\},\{\mathsf{M},\mathsf{P},\mathsf{R}\}\}{:}\{\{\mathsf{M},\mathsf{R}\},\{\mathsf{M},\mathsf{P},\mathsf{R}\}\} :: \{\{\mathsf{W},\mathsf{P}\},\{\mathsf{W},\mathsf{P},\mathsf{R}\}\}{:}\{\{\mathsf{W},\mathsf{R}\},\{\mathsf{W},\mathsf{P},\mathsf{R}\}\}$$

thanks to the 'forth' transformation $\overset{\emptyset,\{\mathsf{R}\},\{\mathsf{P}\}}{\longrightarrow}$ *and the 'back' transformation* $\overset{\emptyset,\{\mathsf{P}\},\{\mathsf{R}\}}{\longrightarrow}$. *Syntactically:*

$$\mathsf{M}\wedge\neg\mathsf{W}\wedge\mathsf{P}{:}\mathsf{M}\wedge\neg\mathsf{W}\wedge\mathsf{R} :: \mathsf{W}\wedge\neg\mathsf{M}\wedge\mathsf{P}{:}\mathsf{W}\wedge\neg\mathsf{M}\wedge\mathsf{R}.$$

Moreover:

$$\mathsf{M}\wedge\mathsf{P}{:}\mathsf{M}\wedge\mathsf{R} :: \mathsf{W}\wedge\mathsf{P}{:}\mathsf{W}\wedge\mathsf{R}.$$

4.6 Simplifying the Definition

Proposition 8 tells us that if we base our definition of analogy on transformations then analogical proportions can only hold between terms. This result allows us to reformulate our official definition of analogy in a way that is more in line with our definition of analogy between maximal terms of Section 4.2.

First, the tf-*difference* between two formulas α and β is the set of variables $p \in \mathbb{P}$ such that α entails p and β entails $\neg p$:

$$\alpha -_{\mathtt{tf}} \beta = \{p \in \mathbb{P} \ : \ \alpha \models p \text{ and } \beta \models \neg p\}.$$

For maximal terms we have $a{:}b :: c{:}d$ if and only if $a -_{\mathtt{tf}} b = c -_{\mathtt{tf}} d$ and $b -_{\mathtt{tf}} a = d -_{\mathtt{tf}} c$. For arbitrary terms α and β, $\alpha -_{\mathtt{tf}} \beta$ is the set of variables that occur positively in α and negatively in β. For example, $p \wedge \neg q -_{\mathtt{tf}} \neg p \wedge q = \{p\}$ and $p \wedge \neg q -_{\mathtt{tf}} q \wedge \neg r = \emptyset$.

Second, the ti-*difference* between α and β is the set of variables $p \in \mathbb{P}$ such that α entails p and p is irrelevant for β:

$$\alpha -_{\mathtt{ti}} \beta = \{p \in \mathbb{P} \ : \ \alpha \models p \text{ and } p \text{ is irrelevant for } \beta\};$$

and the fi-*difference* between α and β is the set of variables $p \in \mathbb{P}$ such that α entails $\neg p$ and p is irrelevant for β:

$$\alpha -_{\mathtt{fi}} \beta = \{p \in \mathbb{P} \ : \ \alpha \models \neg p \text{ and } p \text{ is irrelevant for } \beta\}.$$

For maximal terms $\alpha -_{\mathtt{ti}} \beta = \alpha -_{\mathtt{fi}} \beta = \emptyset$. When α and β are arbitrary terms then $\alpha -_{\mathtt{ti}} \beta$ is the set of variables that occur in α but not in β; and vice versa for $\alpha -_{\mathtt{fi}} \beta$. For example, $p\wedge q -_{\mathtt{ti}} \neg p \wedge q = \emptyset$ and $p \wedge q -_{\mathtt{ti}} \neg p \wedge r = \{q\}$.

For the sake of readability we also define $\alpha -_{\mathtt{ft}} \beta = \beta -_{\mathtt{tf}} \alpha$, $\alpha -_{\mathtt{it}} \beta = \beta -_{\mathtt{ti}} \alpha$, and $\alpha -_{\mathtt{if}} \beta = \beta -_{\mathtt{fi}} \alpha$.

Proposition 12 *Given formulas $\alpha, \beta, \gamma, \delta$, an analogical proportion $\alpha{:}\beta :: \gamma{:}\delta$ is the case if and only if*

1. *$\alpha, \beta, \gamma, \delta$ are respectively equivalent to terms $\alpha', \beta', \gamma', \delta'$;*
2. *$\alpha' -_{\mathtt{tf}} \beta' = \gamma' -_{\mathtt{tf}} \delta'$ and $\alpha' -_{\mathtt{ft}} \beta' = \gamma' -_{\mathtt{ft}} \delta'$;*
3. *$\alpha' -_{\mathtt{ti}} \beta' = \gamma' -_{\mathtt{ti}} \delta'$ and $\alpha' -_{\mathtt{it}} \beta' = \gamma' -_{\mathtt{it}} \delta'$;*
4. *$\alpha' -_{\mathtt{fi}} \beta' = \gamma' -_{\mathtt{fi}} \delta'$ and $\alpha' -_{\mathtt{if}} \beta' = \gamma' -_{\mathtt{if}} \delta'$.*

PROOF. For the left-to-right direction, suppose $\alpha{:}\beta :: \gamma{:}\delta$, that is, we have $\alpha \overset{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}{\longrightarrow} \beta$, $\beta \overset{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}{\longrightarrow} \alpha$, $\gamma \overset{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}{\longrightarrow} \delta$, and $\delta \overset{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}{\longrightarrow} \gamma$. The first item follows from Proposition 8. In order to establish the remaining three items it suffices to observe the following:

$$\alpha' -_{\mathtt{tf}} \beta' = \mathtt{f}_1 \cap \mathtt{t}_2 = \gamma' -_{\mathtt{tf}} \delta' \text{ and } \alpha' -_{\mathtt{ft}} \beta' = \mathtt{t}_1 \cap \mathtt{f}_2 = \delta' -_{\mathtt{ft}} \gamma';$$
$$\alpha' -_{\mathtt{ti}} \beta' = \mathtt{v}_1 \cap \mathtt{t}_2 = \gamma' -_{\mathtt{ti}} \delta' \text{ and } \alpha' -_{\mathtt{it}} \beta' = \mathtt{t}_1 \cap \mathtt{v}_2 = \gamma' -_{\mathtt{it}} \delta';$$
$$\alpha' -_{\mathtt{fi}} \beta' = \mathtt{v}_1 \cap \mathtt{f}_2 = \gamma' -_{\mathtt{fi}} \delta' \text{ and } \alpha' -_{\mathtt{if}} \beta' = \mathtt{f}_1 \cap \mathtt{v}_2 = \gamma' -_{\mathtt{if}} \delta'.$$

For the right-to-left direction, suppose $\alpha', \beta', \gamma', \delta'$ are terms such that $\alpha' -_{\mathtt{tf}} \beta' = \gamma' -_{\mathtt{tf}} \delta'$ and $\alpha' -_{\mathtt{ft}} \beta' = \gamma' -_{\mathtt{ft}} \delta'$; $\alpha' -_{\mathtt{ti}} \beta' = \gamma' -_{\mathtt{ti}} \delta'$ and $\alpha' -_{\mathtt{it}} \beta' = \gamma' -_{\mathtt{it}} \delta'$; and $\alpha' -_{\mathtt{fi}} \beta' = \gamma' -_{\mathtt{fi}} \delta'$ and $\alpha' -_{\mathtt{if}} \beta' = \gamma' -_{\mathtt{if}} \delta'$. Let

$$\mathtt{f}_1 = \alpha' -_{\mathtt{tf}} \beta' \cup \alpha' -_{\mathtt{if}} \beta', \quad \mathtt{f}_2 = \alpha' -_{\mathtt{ft}} \beta' \cup \alpha' -_{\mathtt{fi}} \beta',$$
$$\mathtt{t}_1 = \alpha' -_{\mathtt{ft}} \beta' \cup \alpha' -_{\mathtt{it}} \beta', \quad \mathtt{t}_2 = \alpha' -_{\mathtt{tf}} \beta' \cup \alpha' -_{\mathtt{ti}} \beta',$$
$$\mathtt{v}_1 = \alpha' -_{\mathtt{fi}} \beta' \cup \alpha' -_{\mathtt{ti}} \beta', \quad \mathtt{v}_2 = \alpha' -_{\mathtt{if}} \beta' \cup \alpha' -_{\mathtt{it}} \beta'.$$

It can be shown that $\alpha' \overset{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}{\longrightarrow} \beta'$ and $\beta' \overset{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}{\longrightarrow} \alpha'$ on the one hand, and $\gamma' \overset{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}{\longrightarrow} \delta'$ and $\delta' \overset{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}{\longrightarrow} \gamma'$ on the other. ■

Proposition 13 *Definition 9 satisfies the postulates of symmetry, transitivity, unicity, and central permutation. Moreover, it satisfies reflexivity when α and β are equivalent to terms.*

PROOF. Symmetry and transitivity are obvious from the definition. Reflexivity with α, β equivalent to terms follows from Proposition 8.

For unicity, suppose $\alpha_1{:}\beta :: \gamma{:}\delta$ and $\alpha_2{:}\beta :: \gamma{:}\delta$. By Proposition 8 there are terms $\alpha'_1, \alpha'_2, \beta', \gamma', \delta'$ that are respectively equivalent to $\alpha_1, \alpha_2, \beta, \gamma, \delta$ such that

$$\alpha'_1 -_{\mathtt{tf}} \beta' = \gamma' -_{\mathtt{tf}} \delta' \text{ and } \alpha'_1 -_{\mathtt{ft}} \beta' = \gamma' -_{\mathtt{ft}} \delta',$$
$$\alpha'_2 -_{\mathtt{tf}} \beta' = \gamma' -_{\mathtt{tf}} \delta' \text{ and } \alpha'_2 -_{\mathtt{ft}} \beta' = \gamma' -_{\mathtt{ft}} \delta',$$
$$\alpha'_1 -_{\mathtt{ti}} \beta' = \gamma' -_{\mathtt{ti}} \delta' \text{ and } \alpha'_1 -_{\mathtt{it}} \beta' = \gamma' -_{\mathtt{it}} \delta'.$$
$$\alpha'_2 -_{\mathtt{ti}} \beta' = \gamma' -_{\mathtt{ti}} \delta' \text{ and } \alpha'_2 -_{\mathtt{it}} \beta' = \gamma' -_{\mathtt{it}} \delta',$$
$$\alpha'_1 -_{\mathtt{fi}} \beta' = \gamma' -_{\mathtt{fi}} \delta' \text{ and } \alpha'_1 -_{\mathtt{if}} \beta' = \gamma' -_{\mathtt{if}} \delta',$$
$$\alpha'_2 -_{\mathtt{fi}} \beta' = \gamma' -_{\mathtt{fi}} \delta' \text{ and } \alpha'_2 -_{\mathtt{if}} \beta' = \gamma' -_{\mathtt{if}} \delta'.$$

Hence α'_1 and α'_2 must be terms containing the same literals; they are therefore logically equivalent. Central permutation is the most delicate part of the proof. Suppose $\alpha{:}\beta :: \gamma{:}\delta$ holds. Thanks to Proposition 8 $\alpha, \beta, \gamma, \delta$ can all be supposed to be terms. Consider the sets

$$\alpha - \beta = (\alpha -_{\mathtt{tf}} \beta) \cup (\alpha -_{\mathtt{ti}} \beta) \cup (\alpha -_{\mathtt{ft}} \beta) \cup (\alpha -_{\mathtt{fi}} \beta) \cup (\alpha -_{\mathtt{it}} \beta) \cup (\alpha -_{\mathtt{if}} \beta),$$
$$\alpha - \gamma = (\alpha -_{\mathtt{tf}} \gamma) \cup (\alpha -_{\mathtt{ti}} \gamma) \cup (\alpha -_{\mathtt{ft}} \gamma) \cup (\alpha -_{\mathtt{fi}} \gamma) \cup (\alpha -_{\mathtt{it}} \gamma) \cup (\alpha -_{\mathtt{if}} \gamma).$$

Let us prove that these two sets are disjoint. Suppose to the contrary that there is a p such that $p \in \alpha - \beta$ and $p \in \alpha - \gamma$. We analyse the possible cases. When $p \in \alpha -_{\mathtt{tf}} \beta$ then we can only have either $p \in \alpha -_{\mathtt{tf}} \gamma$ or $p \in \alpha -_{\mathtt{ti}} \gamma$. In the former subcase we have $\gamma \models \neg p$, which cannot be the case because our hypothesis that $p \in \alpha -_{\mathtt{tf}} \beta$ implies that $p \in \gamma -_{\mathtt{tf}} \delta$ and therefore $\gamma \models p$. In the latter subcase p does not occur in γ, which again cannot be the case because our hypothesis $p \in \alpha -_{\mathtt{tf}} \beta$ implies $p \in \gamma -_{\mathtt{tf}} \delta$ and therefore $\gamma \models p$. The 5 other cases are proved in a similar way. Therefore $\alpha - \beta$ and $\alpha - \gamma$ are disjoint, which means that the corresponding transformations between α and β and between α and γ are independent and can be permuted. Then unicity allows us to conclude that there is a unique δ that can be reached from β via the same transformation as the one going from α to γ. ■

Proposition 14 *Definition 9 satisfies uniformity (unif): If α and β are terms and p is a variable neither occurring in α nor in β then $\alpha{:}(\alpha \wedge p) :: \beta{:}(\beta \wedge p)$ and $\alpha{:}(\alpha \wedge \neg p) :: \beta{:}(\beta \wedge \neg p)$.*

PROOF. For the 'forth' transformation we set $\mathtt{f}_1 = \mathtt{v}_1 = \emptyset$ and $\mathtt{t}_1 = \{p\}$; then we have $\alpha \overset{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}{\longrightarrow} \alpha \wedge p$. For the 'back' transformation we set $\mathtt{f}_2 = \mathtt{t}_2 = \emptyset$ and $\mathtt{v}_2 = \{p\}$; then we have $\alpha \wedge p \overset{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}{\longrightarrow} \alpha$. The proof for $\alpha{:}(\alpha \wedge \neg p) :: \beta{:}(\beta \wedge \neg p)$ is analogous. ■

4.7 Integrity Constraints

In this last subsection, we briefly show how our novel approach to analogical proportions between formulas can be generalized to integrity constraints (ICs). The idea is simple and is organized in three steps. First of all, we need to generalise Definition 6 of reachable set.

Definition 15 *Let* $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$ *be disjoint. For* $a \subseteq \mathbb{P}$ *and* $\beta, \chi \subseteq 2^{\mathbb{P}}$*, we say that* β *is the* χ*-reachable set from* a *through the transformation* $\xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}$ *if and only if* $\beta = \{b \in \chi \ : \ a \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}} b\}$.

Then, we need to generalise Definition 7 of uniform transformation.

Definition 16 *Let* $\mathtt{f}, \mathtt{t}, \mathtt{v} \subseteq \mathbb{P}$ *be disjoint. For* $\alpha, \beta, \chi \subseteq 2^{\mathbb{P}}$ *with* χ, $\alpha \xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}_{\chi} \beta$ *is the case if and only if, for every valuation* $a \in (\alpha \cap \chi)$, β *is the* χ*-reachable set from* a *through* $\xrightarrow{\mathtt{f},\mathtt{t},\mathtt{v}}$.

The third step consists in generalising Definition 9.

Definition 17 *For* $\alpha, \beta, \gamma, \delta, \chi \subseteq 2^{\mathbb{P}}$*, the IC-based analogical proportion* $\alpha{:}\beta \ ::_{\chi} \ \gamma{:}\delta$ *("*α *is to* β *as* γ *is to* δ*, under the integrity constraint* χ*") is the case if and only if there are two uniform transformations* $\xrightarrow{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}_{\chi}$ *and* $\xrightarrow{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}_{\chi}$ *such that:*

1. $\alpha \xrightarrow{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}_{\chi} \beta$ *and* $\beta \xrightarrow{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}_{\chi} \alpha$*;*
2. $\gamma \xrightarrow{\mathtt{f}_1,\mathtt{t}_1,\mathtt{v}_1}_{\chi} \delta$ *and* $\delta \xrightarrow{\mathtt{f}_2,\mathtt{t}_2,\mathtt{v}_2}_{\chi} \gamma$*.*

Interestingly, this notion of IC-based analogical proportion can be reduced to simple analogical proportion. Indeed, the following holds:

$$\alpha{:}\beta ::_{\chi} \gamma{:}\delta \text{ iff } (\alpha \wedge \chi){:}(\beta \wedge \chi) :: (\gamma \wedge \chi){:}(\delta \wedge \chi).$$

In the light of the previous property, it is straightforward to show that the postulates of symmetry and central permutation are preserved under the integrity constraints. Indeed, the following hold:

$$\text{If } \alpha{:}\beta ::_{\chi} \gamma{:}\delta \text{ then } \gamma{:}\delta ::_{\chi} \alpha{:}\beta;$$
$$\text{If } \alpha{:}\beta ::_{\chi} \gamma{:}\delta \text{ then } \alpha{:}\gamma ::_{\chi} \beta{:}\delta.$$

5 Example

The running example "a man is to a king as a woman is to a queen" we used in the previous sections does not fully exploit the expressiveness of propositional logic since the terms of comparison of the analogical proportion, i.e. $\mathsf{M}{:}(\mathsf{M} \wedge \mathsf{R}) :: \mathsf{W}{:}(\mathsf{W} \wedge \mathsf{R})$, do not involve negation or disjunction. In this section, we present a richer example in which formulas in the analogical proportions include negation and disjunction. This is to show that generalising analogy between maximal terms to analogy between arbitrary propositional formulas is not a mere mathematical exercise. It is worth doing it from the modeling point of view, since analogies are often between concepts. The latter are not necessarily represented as maximal terms, the full expressiveness of propositional logic may be needed to represent them.

According to zoological taxonomy, mammals (MA) can be organized into two groups: theria (TH) and prototheria (PR). Prototheria lay eggs and therefore are not viviparous ($\neg\mathsf{VIV}$), are not marsupial ($\neg\mathsf{MAR}$), are homeothermic (HOM) and their young are fed milk produced by the mother's mammary glands (MIL). This category include platipus and echidnas. Theria can be divided into two subgroups: metatheria (ME) and eutheria (EU). Metatheria are viviparous, marsupial, homeothermic and their young are fed milk. This category includes kangaroos, koalas and opossums. Finally, eutheria are viviparous, homeothermic, not marsupial and their young are fed milk. This category includes the majority of mammals, e.g., humans, cats, dogs, elephants, whales. That is,

$$\begin{aligned}
&\mathsf{MA} =_{def} \mathsf{TH} \vee \mathsf{PR},\\
&\mathsf{TH} =_{def} \mathsf{ME} \vee \mathsf{EU},\\
&\mathsf{ME} =_{def} \mathsf{VIV} \wedge \mathsf{MAR} \wedge \mathsf{HOM} \wedge \mathsf{MIL},\\
&\mathsf{EU} =_{def} \mathsf{VIV} \wedge \neg\mathsf{MAR} \wedge \mathsf{HOM} \wedge \mathsf{MIL},\\
&\mathsf{PR} =_{def} \neg\mathsf{VIV} \wedge \neg\mathsf{MAR} \wedge \mathsf{HOM} \wedge \mathsf{MIL}.
\end{aligned}$$

The following analogical proportions are validated by the formal semantics we provided in Section 4 (Definition 9). They are reasonable since their left sides relate a more general concept to a less general one in the same way as their right sides do:

$$\mathsf{TH}{:}\mathsf{EU} :: (\neg\mathsf{VIV} \wedge \mathsf{HOM} \wedge \mathsf{MIL}){:}\mathsf{PR}. \tag{2}$$

Notice that the previous analogical proportion could be simplified by introducing a fictitious subgroup of mammals, called extratheria (EX), that are marsupial, homeothermic, their young are fed milk but are not viviparous and by calling pseudotheria (PS) the group of mammals including existing metatheria and the fictitious extratheria:

$$\begin{aligned}
&\mathsf{PS} =_{def} \mathsf{PR} \vee \mathsf{EX},\\
&\mathsf{EX} =_{def} \neg\mathsf{VIV} \wedge \mathsf{MAR} \wedge \mathsf{HOM} \wedge \mathsf{MIL}.
\end{aligned}$$

Using these abbreviations, we get the following simplified version of the analogical proportion (2):

$$\mathsf{TH}{:}\mathsf{EU} :: \mathsf{PS}{:}\mathsf{PR}. \tag{3}$$

Now consider the following integrity constraint specifying that metatheria and prototheria can only be native to Oceania:

$$\mathsf{IC} =_{def} \neg\mathsf{OC} \rightarrow (\neg\mathsf{ME} \wedge \neg\mathsf{PR}).$$

Due to the co-extensionality of $(\mathsf{MA} \wedge \neg\mathsf{OC})$ and $(\mathsf{TH} \wedge \neg\mathsf{OC})$ under the integrity constraint IC, the following analogy holds:

$$(\mathsf{TH} \wedge \mathsf{OC}){:}(\mathsf{MA} \wedge \neg\mathsf{OC}) ::_{\mathsf{IC}} (\mathsf{TH} \wedge \mathsf{OC}){:}(\mathsf{TH} \wedge \neg\mathsf{OC}). \tag{4}$$

6 Conclusion

We have proposed a new definition of analogical proportion between logical formulas. This definition obeys the usual postulates expected for analogical proportions, as well as properties such as transitivity and unicity. As suggested by the final example, this definition offers an approach for modeling analogical proportions between concepts. The comparison with another approach in the setting of formal concept analysis [2] is a topic for further research. In that respect the failure of $\alpha \vee \beta : \alpha :: \beta : \alpha \wedge \beta$ in the proposed approach indicates a clear difference between the two approaches.

Future work will also be devoted to studying in more detail the axiomatic properties of our notion of analogical proportion between formulas. We plan to come up with a representation theorem providing a sound and complete axiomatic characterization of our notion. Another research avenue we intend to explore is the generalization of our approach to more expressive languages that go beyond the propositional language including the modal language of the epistemic logic S5. We believe it is worth to investigate analogical proportions between epistemic formulas such as "*knowing that* he is a man is to *knowing that* he is a king as *knowing that* he is a woman is to *knowing that* he is a queen".

Acknowledgements

This research has been supported by the French ANR project "Analogies: from Theory to Tools and Applications" (AT2TA), ANR-22-CE23-0023 as well as by the project LIANDA (BBVA).

References

[1] Aristotle. *Nicomachean Ethics*. Univ. of Chicago Press, 2011. Trans. by R. C. Bartlett and S. D. Collins.
[2] N. Barbot, L. Miclet, and H. Prade. Analogy between concepts. *Artif. Intell.*, 275:487–539, 2019.
[3] N. Barbot, L. Miclet, H. Prade, and G. Richard. A new perspective on analogical proportions. In G. Kern-Isberner and Z. Ognjanovic, editors, *Proc. 15th Europ. Conf. on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, (ECSQARU'19), Belgrade, Sept. 18-20*, volume 11726 of *LNCS*, pages 163–174. Springer, 2019.
[4] M. Bounhas, H. Prade, and G. Richard. Analogy-based classifiers for nominal or numerical data. *Int. J. Approx. Reason.*, 91:36–55, 2017.
[5] T. Boy de la Tour and N. Peltier. Analogy in automated deduction: A survey. In H. Prade and G. Richard, editors, *Computational Approaches to Analogical Reasoning: Current Trends*, volume 548 of *Studies in Computational Intelligence*, pages 103–130. Springer, 2014.
[6] W. Correa Beltran, H. Prade, and G. Richard. Constructive solving of Raven's IQ tests with analogical proportions. *Int. J. Intell. Syst.*, 31(11): 1072–1103, 2016.
[7] D. Gentner. Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, 7(2):155–170, 1983.
[8] H. Gust, K. Kühnberger, and U. Schmid. Metaphors and heuristic-driven theory projection (HDTP). *Theoretical Computer Science*, 354 (1):98 – 117, 2006.
[9] M. Hesse. On defining analogy. *Proceedings of the Aristotelian Society*, 60:79–100, 1959.
[10] D. Hofstadter and The Fluid Analogies Research Group, editors. *Fluid Concepts and Creative Analogies: Computer Models of the Fundamental Mechanisms of Thought*. Basic Books, Inc., New York, NY, 1995.
[11] S. Klein. Culture, mysticism & social structure and the calculation of behavior. In *Proc. 5th Europ. Conf. in Artificial Intelligence (ECAI'82), Orsay, France*, pages 141–146, 1982.
[12] J. Lang, P. Liberatore, and P. Marquis. Propositional independence: Formula-variable independence and forgetting. *J. Artif. Intell. Res.*, 18: 391–443, 2003. doi: 10.1613/JAIR.1113. URL https://doi.org/10.1613/jair.1113.
[13] M. Leemhuis, D. Wolter, and Ö. L. Özçep. Decomposing analogy: A logic characterization. In A. Meier and M. Ortiz, editors, *Proc. 13th Int. Symp. on Foundations of Information and Knowledge Systems (FoIKS'24), Sheffield, April 8-11*, volume 14589 of *LNCS*, pages 256–274. Springer, 2024.
[14] Y. Lepage. De l'analogie rendant compte de la commutation en linguistique. *Habilit. à Diriger des Recher., Univ. J. Fourier, Grenoble*, 2003.
[15] L. Miclet and H. Prade. Logical definition of analogical proportion and its fuzzy extensions. In *Proc. Annual Meeting of the North Amer. Fuzzy Information Proc. Soc. (NAFIPS), New-York, May, 19-22*, pages 1–6. IEEE, 2008.
[16] L. Miclet and H. Prade. Handling analogical proportions in classical logic and fuzzy logics settings. In *Proc. 10th Eur. Conf. on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09),Verona*, pages 638–650. Springer, LNCS 5590, 2009.
[17] V. Pirrelli and F. Yvon. Analogy in the lexicon: a probe into analogy-based machine learning of language. In *Proc. 6th Int. Symp. on Human Communication*, 1999. Santiago de Cuba, 6 p.
[18] G. Polya. *How to Solve It*. Princeton University Press, 2nd ed. 1957, 1945.
[19] H. Prade and G. Richard. From analogical proportion to logical proportions. *Logica Universalis*, 7(4):441–505, 2013.
[20] H. Prade and G. Richard. Analogical proportions: From equality to inequality. *Int. J. of Approximate Reasoning*, 101:234 – 254, 2018.
[21] H. Prade and G. Richard. Analogical proportions: Why they are useful in AI. In Z. Zhou, editor, *Proc. of the 30th Int. Joint Conf. on Artificial Intelligence (IJCAI'21), Virtual Event / Montreal, Aug. 19-27*, pages 4568–4576. ijcai.org, 2021.
[22] H. Prade and G. Richard. Diagrammatic analogical reasoning. In *Proc. of the 14th Int. Conf. on the Theory and Application of Diagrams (DIAGRAMS'24), Münster, Sept. 27 - Oct. 1*. Springer, 2024.
[23] D. E. Rumelhart and A. A. Abrahamson. A model for analogical reasoning. *Cognitive Psychol.*, 5:1–28, 1973.
[24] S. J. Russell. *The Use of Knowledge in Analogy and Induction*. Pitman, UK, 1989.
[25] P. H. Winston. Learning and reasoning by analogy. *Commun. ACM*, 23 (12):689–703, 1980.
[26] F. Yvon and N. Stroppa. Formal models of analogical proportions. Technical report, Ecole Nationale Supérieure des Télécommunications, no D008, 2006.