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From Causation (and Parthood) to Time: The Case of EMMO

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Abstract. We investigate the construction of time in EMMO, a foundational ontology developed to improve the strictness in the representation of applied sciences' knowledge. We show how temporal individuals and temporal relations can be defined from the primitives of *causation* and *parthood*, at the core of EMMO; we then prove that our construction satisfies van Benthem's requirements for temporal structures. Our analysis contributes to clarifying the overall landscape of causal relational theories of time, and to the ongoing effort of aligning foundational ontologies. We conclude by sketching how our results can be generalised, employing a strategy to simulate relations' transitive closure in FOL. This generalisation makes the described construction of time exploitable in ontology engineering with minimal preconditions and sets up the groundwork for a systematic analysis of the connections between (discrete) causal and temporal structures.

Keywords. Foundational Issues, Causation, Time, EMMO, Alignment

1. Introduction

The relationship between causation and time has been investigated at length both in the context of philosophy and in the natural sciences. The default position attributes precedence to time over causation; however, there is a notable, if somewhat obscure, tradition of *causal relational theories of time* dating back at least to Leibniz [1]. This research program re-emerged contemporaneously with Einstein's *special relativity*, and its association with *Minkowski spacetime*. In 1914, Robb produced the first axiomatisation of Minkowski spacetime employing qualitative geometrical notions, a development which can be compared to Euclid's geometry [2]. Robb's theory is capable of expressing the equivalence of temporal intervals and supports a coordinate system by relying only on the primitive $\ll(x,y)$, which reads "x is (causally) after y" and should be interpreted as *the possibility* of y to influence x. This approach enjoyed popularity throughout the 20^{th} Century, even among physicists (e.g., Roger Penrose).[2]

Robb's work greatly contributed to *causal relational theories of time*'s popularity around the '70s, yet not all the authors took direct inspiration from it. For instance, Re-

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[2]Most developments focused on simplifying the axiomatisation, via different primitives or by making use of more expressive formal systems. See for instance [3], [4], [5] and [6]. Compare also with [7].

ichenbach [8], Grünbaum [9] and van Fraassen [10] all produced theoretical frameworks with significant elements of originality. That said, most approaches have some common points: (*i*) the adoption of a geometrical approach; (*ii.a*) the focus on a very specific notion of causation with a naturalistic interpretation and on (*ii.b*) (causal) connectibility rather than factual connection; (*iii*) a general preference for continuity over discreteness, in line with (*i*); (*iv*) the aim of fully recovering Minkowski spacetime, and, thus, special relativity; (*v*) the choice of events or spacetime points as relata of the causation relation.

Causal relational theories of time were heavily criticised by Earman [11] for their incompatibility with *general relativity* and the *questionable nature of the connectibility relation*, understood by the author in strictly spatiotemporal (rather than causal) terms. As an indirect consequence of that, and through the work of Malament [12] (systematising results of Stephen Hawking [13] among others) discrete approaches started gaining traction. Among them, causal set theory [7,14] is particularly noteworthy, though the focus shifted from the construction of axiomatic theories to scientific research finalised to the unification of relativity and quantum mechanics.

Recently a new foundational ontology has been developed under the flag of the European Materials Modelling Council (EMMC):[3] the **E**lementary **M**ultiperspective **M**aterial **O**ntology (EMMO).[4] Its aim is to provide a general framework tailored to improve the rigorousness in the representation of scientific and industrial knowledge. EMMO's core revolves around two primitive binary relations: *causation* and *parthood*. To roughly situate EMMO in the context of the discussed approaches, we note that, taking inspiration from (special) relativity and Feynman diagrams [15], and sharing similarities with causal set theory, EMMO opts for (*ii.b*) a productive and factual relation of causation; (*iii*) discreteness; (*iv*) providing a simple, relativity-friendly framework which can be used in practical scenarios over recovering Minkowski spacetime, with all its mathematical properties; (*v*) *real particles* (as they are described by the standard model) in-between interactions as the relata of causation; see Sect. 2. Given the significant dissimilarities with orthodox positions, most of the results contained in the literature cannot be co-opted, and a new approach to the construction of time is made necessary.

In this paper we analyse how time can be reconstructed in EMMO on the basis of its mereocausal commitments grounded in applied sciences. Our analysis provides both theoretical and applicative contributions: it furthers the clarification of the overall landscape of causal relational theories of time; it offers technical insights on the interaction between causation and parthood; and it can also play a role in ontology engineering/alignment where the notion of time is usually central. More specifically, our work was undertaken in the context of OntoCommons,[5] an ongoing H2020 CSA project dedicated to the standardisation of data documentation across all domains related to materials and manufacturing. The project adopts a pluralistic approach, hosting and aligning different ontologies, capturing heterogeneous worldviews to obtain an integrated modelling framework able to cover a multitude of scenarios. The construction put forward in this paper was deemed a necessary step to link EMMO with other foundational ontologies part of the network which endorse more traditional stances on time.[6]

[3] https://emmc.eu/.
[4] https://github.com/emmo-repo/EMMO.
[5] https://ontocommons.eu/.
[6] See [16] for an in-depth discussion of the alignment methodology resting on partial mappings.

The discussion will proceed as follows: Sect. 2 offers a general introduction to EMMO and its axiomatisation in FOL. Sect. 3 describes all the technical steps necessary to define, starting from EMMO's mereocausality, temporal individuals and the temporal relations among them. Finally, Sect. 4 offers the outline of a more general framework and makes the previous results exploitable in ontology engineering, leaving some room for tweaks depending on the intended application, the desired characteristics of the emergent time and divergences in theoretical commitments. A (potentially reusable) strategy to simulate the transitive closure of a relation in FOL is sketched in the process.

2. EMMO

As anticipated, EMMO has been developed with the aim of grounding the representation of knowledge and data from applied sciences, with a focus on materials modelling. EMMO's uppermost module is built upon formal ontology and naturalistic constraints; it establishes clear and objective numerical identity conditions, sets up the preconditions for a multi-scale approach and provides the means to deal with spatio-temporal reasoning. As such, it provides all the elements for our construction.

For what concerns us here, EMMO endorses a form of *ontological naturalism* and a pragmatic *reductionistic* stance, electing the standard model of particle physics as the theory of choice: given the preferred interpretation, the world is seen as made up entirely of fundamental (non-virtual) elementary particles (e.g., electrons, photons), and sums of the latter, understood as portions of the world individuated via a linguistic label. Specifically, EMMO's focus is on non-virtual elementary particles in-between interactions, given the endorsement of a form of perdurantism [17] to be understood in terms of persistence across change (rather than time), and the further assumption that change is intrinsically related to causal interactions mimicking Feynman diagrams at the micro level. As such, EMMO supports *nominalism* across the board (e.g., abstract entities and sets have no place in EMMO's domain), and it favours a "materialistic", "physicalistic", worldview.

Formally, the uppermost module of EMMO is based on two primitives: $\mathtt{P}(x,y)$, which reads "x is *part of* y" and $\mathtt{C}(x,y)$, which reads "x *causes* y". EMMO's definitions and axioms relevant for the construction of time are reported below.

d1 $\mathtt{O}(x,y) := \exists z(\mathtt{P}(z,x) \wedge \mathtt{P}(z,y))$ (*Overlap*)
d2 $\sigma x\langle\phi(x)\rangle := \iota z(\forall y(\mathtt{O}(y,z) \leftrightarrow \exists x(\phi(x) \wedge \mathtt{O}(x,y))))$ (*Fusion*)
d3 $\mathtt{SUM}(x,y,z) := x = \sigma w\langle\mathtt{P}(w,y) \vee \mathtt{P}(w,z)\rangle$ (*Binary Sum*)
d4 $\mathtt{PRD}(x,y,z) := x = \sigma w\langle\mathtt{P}(w,y) \wedge \mathtt{P}(w,z)\rangle$ (*Binary Product*)
d5 $\mathtt{DIF}(x,y,z) := x = \sigma w\langle\mathtt{P}(w,y) \wedge \neg\mathtt{O}(w,z)\rangle$ (*Binary Difference*)
d6 $\mathtt{u} := \sigma x\langle\mathtt{P}(x,x)\rangle$ (*Universe*)
d7 $\mathtt{Q}(x) := \neg\exists y(\mathtt{PP}(y,x))$ (*Quantum [Mereological Atom]*)
d8 $\mathtt{qP}(x,y) := \mathtt{P}(x,y) \wedge \mathtt{Q}(x)$ (*Quantum Part*)
d9 $\mathtt{dC}(x,y) := \mathtt{C}(x,y) \wedge \neg\exists z(\mathtt{C}(x,z) \wedge \mathtt{C}(z,y))$ (*Direct Causation*)
d10 $\mathtt{MDC}(x,y) := \neg\mathtt{O}(x,y) \wedge \exists wz(\mathtt{qP}(w,x) \wedge \mathtt{qP}(z,y) \wedge \mathtt{dC}(w,z))$ (*Macro Direct Causat.*)
d11 $\mathtt{ITEM}(x) := \forall yz(\mathtt{SUM}(x,y,z) \wedge \neg\mathtt{O}(y,z) \rightarrow (\mathtt{MDC}(y,z) \vee \mathtt{MDC}(z,y)))$ (*Item*)
d12 $\mathtt{CSTR}(x) := \mathtt{ITEM}(x) \wedge \neg\mathtt{Q}(x)$ (*Causal Structure*)
d13 $\mathtt{qSNK}(x,y) := \mathtt{qP}(x,y) \wedge \neg\exists z(\mathtt{qP}(z,y) \wedge \mathtt{C}(x,z))$ (*Quantum Sink*)
d14 $\mathtt{qSRC}(x,y) := \mathtt{qP}(x,y) \wedge \neg\exists z(\mathtt{qP}(z,y) \wedge \mathtt{C}(z,x))$ (*Quantum Source*)

a1 $\mathtt{P}(x,x)$ (*Parthood: Reflexivity*)
a2 $\mathtt{P}(x,y) \wedge \mathtt{P}(y,x) \rightarrow x = y$ (*Parthood: Antisymmetry*)
a3 $\mathtt{P}(x,y) \wedge \mathtt{P}(y,z) \rightarrow \mathtt{P}(x,z)$ (*Parthood: Transitivity*)
a4 $\neg\mathtt{P}(y,x) \rightarrow \exists z(\mathtt{P}(z,y) \wedge \neg\mathtt{O}(z,x))$ (*Strong Supplementation*)
a5 $\exists x(\phi(x)) \rightarrow \exists y(y = \sigma x\langle\phi(x)\rangle)$ (*Unrestricted Composition*)
a6 $\exists y(\mathtt{qP}(y,x))$ (*Atomicity*)
a7 $\neg\mathtt{C}(x,x)$ (*Causation: Irreflexivity*)
a8 $\mathtt{C}(x,y) \wedge \mathtt{C}(y,z) \rightarrow \mathtt{C}(x,z)$ (*Causation: Transitivity*)
a9 $\mathtt{C}(x,y) \rightarrow \mathtt{dC}(x,y) \vee \exists zw(\mathtt{C}(x,z) \wedge \mathtt{dC}(z,y) \wedge \mathtt{dC}(x,w) \wedge \mathtt{C}(w,y))$
(*Causation: Discreteness / Direct Causation's Necessity*)
a10 $\mathtt{C}(x,y) \rightarrow \mathtt{Q}(x) \wedge \mathtt{Q}(y)$ (*Quantum Causation*)
a11 $\mathtt{ITEM}(\mathtt{u})$ (*Self-Connected Universe*)
a12 $\mathtt{dC}(x,y) \rightarrow \exists z((\mathtt{dC}(x,z) \vee \mathtt{dC}(z,y)) \wedge y \neq z \wedge x \neq z)$ (*Minimal Causal Structure*)
a13 $\mathtt{dC}(x,y) \wedge \mathtt{dC}(x,z) \wedge \mathtt{dC}(w,y) \rightarrow \mathtt{dC}(w,z)$ (*Locality*)

Definitions (d1) to (d7) and axioms (a1) to (a6) are a standard formalization of Atomistic General Extensional Mereology (AGEM),[7] where *quanta* correspond to *mereological atoms*.[8] (d8) introduces a specification of parthood restricting the first relata to quanta – a definition particularly useful given EMMO's theoretical assumptions. (t1)-(t3), which are listed below, are standard theorems of AGEM. Notably, (t1) grounds the unicity of mereological sums, products and differences among other things. Hence, to simplify the notation, sometimes we note with $x+y$, $x-y$, and $x\times y$ the unique z such that, respectively, $\mathtt{SUM}(z,x,y)$, $\mathtt{DIF}(z,x,y)$, and $\mathtt{PRD}(z,x,y)$.

t1 $\mathtt{P}(x,y) \leftrightarrow \forall z(\mathtt{qP}(z,x) \rightarrow \mathtt{qP}(z,y))$ (*Quantum Extensionality*)
t2 $\mathtt{SUM}(s,x,y) \leftrightarrow \forall z(\mathtt{qP}(z,s) \leftrightarrow (\mathtt{qP}(z,x) \vee \mathtt{qP}(z,y)))$ (*Binary Sum's Atoms*)
t3 $\mathtt{SUM}(s,x,y) \rightarrow \mathtt{P}(x,s) \wedge \mathtt{P}(y,s)$ (*Binary Sum's Parts*)

(a7) and (a8) characterise *causation* as a strict partial order, i.e., an irreflexive, transitive, and thus asymmetric, relation. The characterisation is made *prima facie* more plausible considering the relation's intended interpretation, specifically with respect to its domain of application: in fact, (a10) states that causation can only hold between quanta. Strictness is micro-physics-friendly, while partiality is in line with relativity.

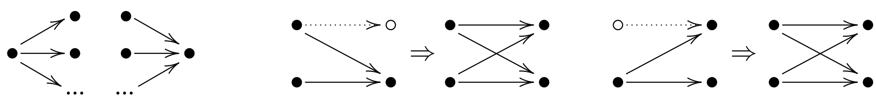
(d9) simply defines *direct causation* as the transitive reduction of causation. The centrality of dC in EMMO is reinforced by (a9), a standard expression of (causation's) *discreteness*. (d10) provides a possible generalisation of direct causation for the *macro level* (from whence the label *macro direct causation*): an entity is a macro direct cause of another if and only if they do not overlap and there is a quantum part of the first which is a direct cause of a quantum part of the second. As it is defined, MDC is not irreflexive, nor asymmetric or intransitive, trading strictness for a wider applicability.

(d11) provides a way to identify (causally) self-connected entities, called *items*: something is an item if and only if there is a macro direct causation relation among any pair of non-overlapping, complementary parts of the entity. (a11) states that the *universe*, i.e., the *fusion* of all the entities in the domain, is self-connected: a pragmatic choice also supported by epistemological and cosmological considerations. (d12) simply restricts the

[7] As such, the reader can refer directly to [18]. (t1)-(t3) are established consequences; see [19].
[8] The standard label is not employed in EMMO to avoid confusions related to EMMO's domain of application.

definition of item to macro (non quantum) entities. (d13) and (d14) intuitively serve to identify the "firsts" and "lasts" quanta of a given entity (with respect to causation).

(a12) and (a13) require a more in-depth discussion, as they are expression of naturalistic commitments of EMMO which are especially relevant for the proposed construction. (a12) states that at least three entities have to participate in any given interaction, as it is shown in Fig. 1(a). (a13) states that, given four generic entities x, y, z, w, if x (directly) causes y and z, and w (directly) causes y, then w also (directly) causes z, as exemplified in Fig. 1(b). Intuitively, (direct) causal connections spread within interactions: no entity can be the (direct) cause of only a subset of the effects, and no entity can be the (direct) effect of only a subset of the causes in a given interaction; likewise, there cannot be distinct interactions in which a given entity plays the same role (cause/effect). Both axioms are extrapolated from Feynman diagrams via an analysis of the common constraints, in line with what has been anticipated above. (a12) can be seen as identifying the *minimal causal structure*, which takes the form of the annihilation or production of real particles. (a13) can be understood as enforcing interactions' locality and would be systematically violated if virtual particles were also considered in EMMO (in line with expectations based on quantum mechanics).



(*a*) Minimal Causal Structure (*b*) Locality

Figure 1. EMMO's Core Naturalistic Commitments (the arrows stand for dC relations)

3. From Mereocausality to Time in EMMO

Our construction of time rests on the idea that, in EMMO, causation (direct or not) between quanta intuitively implies complete temporal precedence, i.e., the effect-quantum starts to be present after the cause-quantum ceases to be present. This assumption is supported by EMMO's perdurantist stance and by the intransitivity and lack of loops of dC, ruling out causal connections among causes (effects) of a given interaction. Indeed, given the productive nature of causation in EMMO and the lack of entities beside fusions of quanta in the domain, it seems appropriate to assume that, in a given interaction, the direct causes cease to be present when effects appear, as the former transform into the latter without intermediary causal interactions. Since in EMMO time emerges from causality, it makes no sense to speak of interactions' temporal duration, i.e., time is marked only by quanta going in and out of existence.

To build time, we proceed by strengthening the notion of macro causation introduced in (d10) to assure that the cause completely precedes the effect, where causes and effects are now generic sums of quanta. The new notion, called *macro causal precedence* (d15), assumes that all the quantum parts of the cause are linked by a causation relation to (and hence, intuitively, temporally precede) all the quantum parts of the effect.

Times, our temporal individuals, are built (by simulating *equivalence classes*) as maximal mereological sums of sums of quanta that are *indistinguishable* with respect to macro causal precedence, see Sect. 3.1, i.e., following the reductionist stance of EMMO, times are reduced to specific sums of quanta. Notice that (i) causal chains of several quanta can be part of times and (ii) times can be proper parts ($\mathtt{PP}$) of other times. Intuitively, (i) and (ii) indicate that the ontological nature of times is close to the one of periods or intervals (as opposed to points), i.e., times can have a "duration".

A reconstruction of time cannot stop at temporal individuals, as it should cover the individuals' temporal arrangement. We have just seen that there are several options concerning the nature of temporal individuals (e.g., points vs. periods); the same holds for the nature of the structuring relations of times (e.g, discrete vs. continuous orders). Hence, we introduce a precedence relation and a parthood relation on times (see respectively (d21) and (d22)) and we study their properties. In particular, we show that they satisfy the formal requirements assumed by van Benthem in [20] for a theory of time based on periods. This formally validates our construction of time from mereocausality.

3.1. The Construction of Times

In line with what has been said above, first we define the notion of *macro causal precedence*, (d15), where $\mathtt{pC}(x,y)$ stands for "x macro causally precedes y". Intuitively this relation assures that y starts to be present after x ceases to be present, even when macro-objects are concerned. Theorems (t4)-(t10) show that $\mathtt{pC}$ is a discrete partial order which is monotone with respect to $\mathtt{P}$, possibly left/right bounded, and possibly non-linear.

d15 $\mathtt{pC}(x,y) \coloneqq \forall uv(\mathtt{qP}(u,x) \wedge \mathtt{qP}(v,y) \rightarrow \mathtt{C}(u,v))$ ((*Macro*) *Causal Precedence*)

t4 $\neg\mathtt{pC}(x,x)$ ($\mathtt{pC}$*: Irreflexivity*)
Proof. By contradiction, assume $\mathtt{pC}(x,x)$. From (d15) it follows that $\forall q(\mathtt{qP}(q,x) \rightarrow \mathtt{C}(q,q))$; given (a6), $\exists q(\mathtt{qP}(q,x) \wedge \mathtt{C}(q,q))$ that contradicts (a7). □

t5 $\mathtt{pC}(x,y) \wedge \mathtt{pC}(y,z) \rightarrow \mathtt{pC}(x,z)$ ($\mathtt{pC}$*: Transitivity*)
Proof. It follows trivially from (a8) and (d15). □

t6 $(\mathtt{pC}(x,y) \rightarrow \exists z(\mathtt{pC}(x,z) \wedge \neg\exists u(\mathtt{pC}(x,u) \wedge \mathtt{pC}(u,z)))) \wedge$
$(\mathtt{pC}(x,y) \rightarrow \exists z(\mathtt{pC}(z,y) \wedge \neg\exists u(\mathtt{pC}(z,u) \wedge \mathtt{pC}(u,y))))$ ($\mathtt{pC}$*: Discreteness*)
Proof. Consider the first conjunct. By (d15) we have that $\mathtt{pC}(x,y) \rightarrow \forall uv(\mathtt{qP}(u,x) \wedge \mathtt{qP}(v,y) \rightarrow \mathtt{C}(u,v))$. Thus, by (a6), $\mathtt{pC}(x,y) \rightarrow \exists v(\mathtt{qP}(v,y) \wedge \forall u(\mathtt{qP}(u,x) \rightarrow \mathtt{C}(u,v)))$, and then $\mathtt{pC}(x,y) \rightarrow \exists v(\forall u(\mathtt{qP}(u,x) \rightarrow \mathtt{C}(u,v)))$. By (a5), $\mathtt{pC}(x,y) \rightarrow \exists z(z = \sigma v\langle\forall u(\mathtt{qP}(u,x) \rightarrow \mathtt{C}(u,v))\rangle)$. From the hypotheses, by (a6), (a7), (d15), and the construction of z, we have that $\mathtt{pC}(x,z) \wedge \forall u(\mathtt{pC}(x,u) \rightarrow \mathtt{P}(u,z))$ and then $\mathtt{pC}(x,z) \wedge \forall u(\mathtt{pC}(x,u) \rightarrow \neg\mathtt{pC}(u,z))$. Similarly for the second conjunct. □

t7 $\mathtt{pC}(x,y) \wedge \mathtt{P}(u,x) \wedge \mathtt{P}(v,y) \rightarrow \mathtt{pC}(u,v)$ ($\mathtt{pC}$*: Monotonicity*)
Proof. Directly from (d15) and (a3) by observing that all the quanta of u are also quanta of x and all the quanta of v are also quanta of y. □

t8 $(\mathtt{pC}(x,z) \wedge \mathtt{pC}(y,z) \wedge \mathtt{SUM}(s,x,y) \rightarrow \mathtt{pC}(s,z)) \wedge$
$(\mathtt{pC}(z,x) \wedge \mathtt{pC}(z,y) \wedge \mathtt{SUM}(s,x,y) \rightarrow \mathtt{pC}(z,s))$ ($\mathtt{pC}$ *preservation w.r.t Sum*)
Proof. Consider the first conjunct. By (d15) we have that $\mathtt{pC}(x,z) \rightarrow \forall uv(\mathtt{qP}(u,x) \wedge \mathtt{qP}(v,z) \rightarrow \mathtt{C}(u,v))$, and analogously for $\mathtt{pC}(y,z)$. By (d2), (d3), and extensionality (t1) together with (a2), $\mathtt{SUM}(s,x,y) \rightarrow \forall q((\mathtt{qP}(q,x) \vee \mathtt{qP}(q,y)) \leftrightarrow \mathtt{qP}(q,s))$, thus $\mathtt{pC}(x,z) \wedge \mathtt{pC}(y,z) \wedge \mathtt{SUM}(s,x,y) \rightarrow \forall uv(\mathtt{qP}(u,s) \wedge \mathtt{qP}(v,z) \rightarrow \mathtt{C}(s,z))$. The conclusion follows from (d15). Similarly for the second conjunct. □

t9 $\nvdash \exists y(\mathtt{pC}(y,x)) \vee \exists y(\mathtt{pC}(x,y))$ (pC*: Existence of the Predecessor/Successor*)
Proof. Consider $\mathtt{dC}(1,2)$, $\mathtt{dC}(1,3)$, $\mathtt{dC}(2,4)$, $\mathtt{dC}(3,4)$ where $1,2,3,4$ are all different. $\neg\exists y(\mathtt{pC}(y,1))$. Consider $\mathtt{dC}(1,2)$, $\mathtt{dC}(1,3)$, $\mathtt{dC}(2,4)$, $\mathtt{dC}(3,4)$ where $1,2,3,4$ are all different. $\neg\exists y(\mathtt{pC}(4,y))$. □

t10 $\nvdash \mathtt{pC}(x,y) \vee \mathtt{pC}(y,x) \vee \mathtt{O}(x,y)$ (pC*: Linearity*)
Proof. Consider the situation where $\mathtt{dC}(0,1)$, $\mathtt{dC}(0,2)$, $\mathtt{dC}(1,3)$, $\mathtt{dC}(1,4)$, $\mathtt{dC}(2,5)$, and $\mathtt{dC}(2,6)$. We have $\neg\mathtt{pC}(1,2)$, $\neg\mathtt{pC}(2,1)$, and $\neg\mathtt{O}(1,2)$. □

The second step to build the temporal individuals consists in the introduction of a relation of *temporal equivalence* (noted EQ), intuitively holding between temporally co-extensional sums of quanta. Our idea is to reduce EQ to indistinguishability with respect to pC.[9] Technically, we introduce a notion of *temporal inclusion* on items (i.e., self-connected macro-entities made up of quanta; see (d11)) in (d16) which is successively extended to general, possibly non self-connected, sums of quanta in (d17). Temporal equivalence is then characterised as mutual temporal inclusion (d18). Two things are worth mentioning on this constructive step. First, note that (d16) does not work for generic sums of quanta; for instance, an item with a single source *s* and a single sink *e* would be temporally included in any entity overlapping *s* and *e*, "gappy" collections included. Second, (d16) and (d17) also apply to sums of quanta including qSRCs or qSNKs of the universe (u), i.e., quanta intuitively at the beginning/end of time. Other solutions might have been equally valid. Geometrical approaches closer to Robb's orthodoxy do not have to take a stance, as the relevant models are unbounded; thus, in this respect, EMMO offers a more neutral framework. Theorems (t11)-(t25) prove some important properties of iIN, IN, and EQ. In particular they show that EQ is an equivalence relation that does not collapse on numerical identity and that IN is a partial order on sums of quanta that is more general than parthood, preserves pC, and is preserved by P and SUM.

d16 $\mathtt{iIN}(x,y) := \mathtt{ITEM}(x) \wedge \mathtt{ITEM}(y) \wedge \forall z(\mathtt{pC}(z,y) \rightarrow \mathtt{pC}(z,x)) \wedge \forall z(\mathtt{pC}(y,z) \rightarrow \mathtt{pC}(x,z))$ (*Temporal Inclusion among Items*)

d17 $\mathtt{IN}(x,y) := \forall u(\mathtt{ITEM}(u) \wedge \mathtt{P}(u,x) \rightarrow \exists v(\mathtt{P}(v,y) \wedge \mathtt{iIN}(u,v)))$ (*Temporal Inclusion*)

d18 $\mathtt{EQ}(x,y) := \mathtt{IN}(x,y) \wedge \mathtt{IN}(y,x)$ (*Temporal Equivalence*)

t11 $\mathtt{iIN}(x,x)$ (iIN*: Reflexivity*)
Proof. Directly from (d16). □

t12 $\mathtt{iIN}(x,y) \wedge \mathtt{iIN}(y,z) \rightarrow \mathtt{iIN}(x,z)$ (iIN*: Transitivity*)
Proof. From the hypothesis, by (d16), $\mathtt{ITEM}(x) \wedge \mathtt{ITEM}(z)$. Consider a such that $\mathtt{pC}(a,z)$. From $\mathtt{iIN}(y,z)$, by (d16), $\mathtt{pC}(a,y)$ that from $\mathtt{iIN}(x,y)$, by (d16), implies $\mathtt{pC}(a,x)$. Similarly for a such that $\mathtt{pC}(z,a)$. □

t13 $\mathtt{iIN}(x,y) \wedge \mathtt{P}(y,z) \wedge \mathtt{ITEM}(z) \rightarrow \mathtt{iIN}(x,z)$ (iIN *preservation w.r.t.* P)
Proof. By contradiction assume $\neg\mathtt{iIN}(x,z)$. If $\neg\mathtt{ITEM}(x)$ or $\neg\mathtt{ITEM}(z)$ we have a contradiction. Assume then that $\mathtt{ITEM}(x)$ and $\mathtt{ITEM}(z)$. From $\neg\mathtt{iIN}(x,z)$, by (d16), there exists u s.t. $\mathtt{pC}(u,z) \wedge \neg\mathtt{pC}(u,x)$ or $\mathtt{pC}(z,u) \wedge \neg\mathtt{pC}(x,u)$. Then, assuming $\mathtt{P}(y,z)$, by (t7) and the reflexivity of P, $\mathtt{pC}(u,y) \wedge \neg\mathtt{pC}(u,x)$ or $\mathtt{pC}(y,u) \wedge \neg\mathtt{pC}(x,u)$, that contradicts $\mathtt{iIN}(x,y)$. □

t14 $\mathtt{pC}(x,y) \wedge \mathtt{iIN}(u,x) \wedge \mathtt{iIN}(v,y) \rightarrow \mathtt{pC}(u,v)$ (pC *preservation w.r.t.* iIN)

[9] As such, EQ has to do with the "contemporaneousness" of causal paths, and not simultaneity of observation.

Proof. From $\texttt{iIN}(u,x)$, by (d16), $\texttt{pC}(x,a) \rightarrow \texttt{pC}(u,a)$ then, from $\texttt{pC}(x,y)$, we have $\texttt{pC}(u,y)$. From $\texttt{iIN}(v,y)$, by (d16), $\texttt{pC}(a,y) \rightarrow \texttt{pC}(a,v)$ then, from $\texttt{pC}(u,y)$, we have $\texttt{pC}(u,v)$. □

t15 $\texttt{IN}(x,x)$ (IN*: Reflexivity*)
Proof. Directly from (d17), (t11) and the reflexivity of P (a1). □

t16 $\texttt{IN}(x,y) \wedge \texttt{IN}(y,z) \rightarrow \texttt{IN}(x,z)$ (IN*: Transitivity*)
Proof. From $\texttt{IN}(x,y)$, by (d17), if $\texttt{ITEM}(u) \wedge \texttt{P}(u,y)$ then there exists v s.t. $\texttt{ITEM}(v) \wedge \texttt{P}(v,y) \wedge \texttt{iIN}(u,v)$. Then, from $\texttt{IN}(y,z)$, by (d17), there exists w s.t. $\texttt{ITEM}(w) \wedge \texttt{P}(w,z) \wedge \texttt{iIN}(v,w)$. The thesis follows from the iIN transivity (t12). □

t17 $\texttt{P}(x,y) \rightarrow \texttt{IN}(x,y)$ (P *specialises* IN)
Proof. Consider u s.t. $\texttt{ITEM}(u) \wedge \texttt{P}(u,x)$. By the transitivity of P (a3), from $\texttt{P}(u,x) \wedge \texttt{P}(x,y)$, we have $\texttt{P}(u,y)$ and, by (t11), $\texttt{P}(u,y) \wedge \texttt{iIN}(u,u)$. □

t18 $\texttt{P}(z,x) \wedge \texttt{P}(y,u) \wedge \texttt{IN}(x,y) \rightarrow \texttt{IN}(z,u)$ (IN *preservation w.r.t.* P)
Proof. Directly from (t16) and (t17). □

t19 $\texttt{IN}(x,z) \wedge \texttt{IN}(y,z) \wedge \texttt{SUM}(s,x,y) \rightarrow \texttt{IN}(s,z)$ (IN *preservation w.r.t.* SUM)
Proof. By contradiction. From $\neg\texttt{IN}(s,z)$, by (d17), $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \neg\texttt{iIN}(u,v)))$, i.e., by (d16), $\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow ([1]\ \exists a(\texttt{pC}(a,v) \wedge \neg\texttt{pC}(a,u)) \vee [2]\ \exists a(\texttt{pC}(v,a) \wedge \neg\texttt{pC}(u,a))))$. If u is part of x or y, trivially $\neg\texttt{IN}(x,z)$ or $\neg\texttt{IN}(y,z)$ and then we have a contradiction. Assume then that u overlaps both x and y.
Consider [1]. From $\neg\texttt{pC}(a,u)$, by (t8), the hypothesis that u overlaps both x and y, the existence of the mereological product (that follows from (a5) and the fact that u overlaps both x and y), and the fact that $\texttt{P}(u,s) \wedge \texttt{SUM}(s,x,y)$, we have [1.1] $\neg\texttt{pC}(a,u \times x)$ or [1.2] $\neg\texttt{pC}(a,u \times y)$.
Consider [1.1], i.e., $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \exists a(\texttt{pC}(a,v) \wedge \neg\texttt{pC}(a,u \times x)))$, i.e., by (d16), $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \neg\texttt{iIN}(v,u \times x)))$. Consider now a q s.t. $\texttt{qP}(q,u \times x)$. From $\texttt{qP}(q,u \times x) \wedge \neg\texttt{iIN}(v,u \times x)$, by (t13), $\neg\texttt{iIN}(v,q)$, then $\exists q(\texttt{qP}(q,x) \wedge \texttt{ITEM}(q) \wedge \forall v(\texttt{P}(v,z) \rightarrow \neg\texttt{iIN}(v,q)))$, i.e., by (d17), $\neg\texttt{IN}(x,z)$. Contradiction.
Consider [1.2], i.e., $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \exists a(\texttt{pC}(a,v) \wedge \neg\texttt{pC}(a,u \times y)))$. Following the reasoning done for [1.1] we obtain $\neg\texttt{IN}(y,z)$. Contradiction.
Consider [2]. From $\neg\texttt{pC}(u,a)$, by (t8), the hypothesis that u overlaps both x and y, the existence of the mereological product, and the fact that $\texttt{P}(u,s) \wedge \texttt{SUM}(s,x,y)$, we have [2.1] $\neg\texttt{pC}(u \times x,a)$ or [2.2] $\neg\texttt{pC}(u \times y,a)$.
Consider [2.1], i.e., $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \exists a(\texttt{pC}(a,v) \wedge \neg\texttt{pC}(u \times x,a)))$, i.e., by (d16), $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \neg\texttt{iIN}(v,u \times x)))$. We obtain a contradiction following what done for the case [1.1].
Consider [2.2], i.e., $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \exists a(\texttt{pC}(a,v) \wedge \neg\texttt{pC}(u \times y,a)))$, i.e., by (d16), $\exists u(\texttt{P}(u,s) \wedge \forall v(\texttt{P}(v,z) \rightarrow \neg\texttt{iIN}(v,u \times y)))$. We obtain a contradiction following what done for the case [1.2]. □

t20 $\texttt{pC}(x,y) \wedge \texttt{IN}(u,x) \wedge \texttt{IN}(v,y) \rightarrow \texttt{pC}(u,v)$ (pC *preservation w.r.t.* IN)
Proof. Consider a,b s.t. $\texttt{qP}(a,u)$ and $\texttt{qP}(b,v)$. Given (d15), we need to prove that $\texttt{C}(a,b)$. From $\texttt{IN}(u,x)$ and $\texttt{qP}(a,u)$, by (d17), there exists $\bar{a}$ s.t. $\texttt{P}(\bar{a},x) \wedge \texttt{iIN}(a,\bar{a})$. From $\texttt{IN}(v,y)$ and $\texttt{qP}(b,v)$, by (d17), there exists $\bar{b}$ s.t. $\texttt{P}(\bar{b},y) \wedge \texttt{iIN}(b,\bar{b})$. From $\texttt{P}(\bar{a},x) \wedge \texttt{P}(\bar{b},y) \wedge \texttt{pC}(x,y)$, by (t7), $\texttt{pC}(\bar{a},\bar{b})$. From $\texttt{pC}(\bar{a},\bar{b}) \wedge \texttt{iIN}(a,\bar{a}) \wedge \texttt{iIN}(b,\bar{b})$, by (t14), $\texttt{pC}(a,b)$ that, given the atomicity of a and b, implies $\texttt{C}(a,b)$. □

t21 $\nvdash \texttt{EQ}(x,y) \rightarrow x = y$ (EQ*: No Collapse on Numerical Identity*)
Proof. Consider $\texttt{dC}(1,2)$, $\texttt{dC}(1,4)$, $\texttt{dC}(3,4)$, $\texttt{dC}(3,2)$ where $1,2,3,4$ are all different. We have $\texttt{EQ}(1,3)$ but $1 \neq 3$. □

t22 $\mathtt{EQ}(x,x)$ (EQ*: Reflexivity*)
Proof. Directly from (d18) and (t15). □

t23 $\mathtt{EQ}(x,y) \to \mathtt{EQ}(y,x)$ (EQ*: Symmetry*)
Proof. Directly from (d18). □

t24 $\mathtt{EQ}(x,y) \wedge \mathtt{EQ}(y,z) \to \mathtt{EQ}(x,z)$ (EQ*: Transitivity*)
Proof. Directly from (d18) and (t16). □

t25 $\mathtt{P}(a,x) \wedge \mathtt{EQ}(b,a) \to \mathtt{EQ}(x,(x{-}a){+}b)$ (EQ *preservation w.r.t.* EQ*-Parts Substitution*)
Proof. By (d18), we prove that [A] $\mathtt{IN}(x,(x{-}a){+}b)$ and [B] $\mathtt{IN}((x{-}a){+}b,x)$.
Consider [A]. By (t22), $\mathtt{EQ}(x{-}a,x{-}a)$ and, by hypothesis, $\mathtt{EQ}(a,b)$. By (d18), $\mathtt{IN}(x{-}a,x{-}a)$ and $\mathtt{IN}(a,b)$. From $\mathtt{IN}(x{-}a,x{-}a)$ and $\mathtt{P}(x{-}a,(x{-}a){+}b)$, by (t18), $\mathtt{IN}(x{-}a,(x{-}a){+}b)$. From $\mathtt{IN}(a,b)$ and $\mathtt{P}(b,(x{-}a){+}b)$, by (t18), $\mathtt{IN}(a,(x{-}a){+}b)$. From $\mathtt{IN}(x{-}a,(x{-}a){+}b)$ and $\mathtt{IN}(a,(x{-}a){+}b)$, by (t19) and by observing that $\mathtt{P}(a,x) \to x = (x{-}a){+}a$, $\mathtt{IN}(x,(x{-}a){+}b)$.
Consider [B]. Following what done for the case [A], from $\mathtt{IN}(x{-}a,x{-}a)$ and $\mathtt{P}(x{-}a,x)$, by (t18), $\mathtt{IN}(x{-}a,x)$. From $\mathtt{IN}(b,a)$ and $\mathtt{P}(a,x)$, by (t18), $\mathtt{IN}(b,x)$. From $\mathtt{IN}(x{-}a,x)$ and $\mathtt{IN}(b,x)$, by (t19), $\mathtt{IN}((x{-}a){+}b,x)$. □

In the last step of the construction of temporal individuals, *times* are defined as maximal sums of temporally equivalent sums of quanta. We follow a procedure which is standard in mathematics: first we introduce the sum of quanta x obtained by summing up all the entities temporally equivalent to a given entity a, see (d19) and (t26). This simulates the set-theoretical construction of the equivalence class $[a]_{\mathtt{EQ}} = \{z \in D : \mathtt{EQ}(z,a)\}$ (where D is our domain). Then, we collect under times (TME) all the EQ equivalent classes, see (d20), i.e., TME simulates the quotient set $D/\mathtt{EQ}$.

d19 $\mathtt{EC}(x,a) := \forall z(\mathtt{qP}(z,x) \leftrightarrow \exists w(\mathtt{EQ}(w,a) \wedge \mathtt{qP}(z,w)))$ (*Maximal Sum of* EQ*s*)

d20 $\mathtt{TME}(x) := \exists y(\mathtt{EC}(x,y))$ (*Time*)

t26 $\mathtt{EC}(x,a) \leftrightarrow x = \sigma y\langle \mathtt{EQ}(y,a)\rangle$ (*Fusion-based definition of* EC)
Proof. By the definitions of fusion and EC, we need to prove that $\forall u(\mathtt{qP}(u,x) \leftrightarrow \exists v(\mathtt{EQ}(v,a) \wedge \mathtt{qP}(u,v)))$ if and only if $\forall u(\mathtt{O}(u,x) \leftrightarrow \exists v(\mathtt{EQ}(v,a) \wedge \mathtt{O}(v,u)))$.
Consider ($\to$). Assume $\mathtt{O}(u,x)$, by (a6) and the definition of overlap, $\exists q(\mathtt{qP}(q,u) \wedge \mathtt{qP}(q,x))$. From $\mathtt{qP}(q,x)$, by the hypothesis, $\exists v(\mathtt{EQ}(v,a) \wedge \mathtt{qP}(q,v))$. Then we have that $\mathtt{qP}(q,v)$ and $\mathtt{qP}(q,u)$, i.e., $\mathtt{O}(v,u)$. Assume $\exists v(\mathtt{EQ}(v,a) \wedge \mathtt{O}(v,u))$, by (a6) and the definition of overlap, $\exists q(\mathtt{qP}(q,v) \wedge \mathtt{qP}(q,u))$. From $\exists v(\mathtt{EQ}(v,a) \wedge \mathtt{qP}(q,v))$, by the hypothesis, $\mathtt{qP}(q,x)$ that, together with $\mathtt{qP}(q,u)$, implies $\mathtt{O}(u,x)$.
Consider ($\leftarrow$). Assume $\mathtt{qP}(u,x)$, then $\mathtt{O}(u,x)$, and then, by the hypothesis, $\exists v(\mathtt{EQ}(v,a) \wedge \mathtt{O}(v,u))$ that, by the atomicity of u, implies $\mathtt{qP}(u,v)$. Assume $\exists v(\mathtt{EQ}(v,a) \wedge \mathtt{qP}(u,v))$, then $\exists v(\mathtt{EQ}(v,a) \wedge \mathtt{O}(v,u))$ and, by the hypothesis, $\mathtt{O}(u,x)$, and because u is atomic, $\mathtt{qP}(u,x)$. □

3.2. Time

We now turn to the structure of time by introducing a precedence relation (tpC) and a parthood relation (tP) on times; see (d21) and (d22). These new notions are trivial restrictions of pC and P to times. Concerning overlap between times (tO), (d23) and (t28) demonstrate that it is equivalent to define this notion starting from P-part or tP-part.

It is now possible to check what constraints are satisfied by tP and tpC, among the ones considered by van Benthem [20] as relevant for the characterisation of temporal structures based on periods. Theorems (t29)-(t33) show that $\mathtt{tP}$ is an extensional mereology closed under product, while (t34) and (t35) show that $\mathtt{tpC}$ is a partial order. (t36), (t37), and (t38) correspond to the constraints called, respectively, *separatedness*, *monotonicity*, and MOND in [20]. Van Benthem assumes that these constraints are sufficient to (minimally) characterise a structure of periods.[10] Theorems (t40)-(t43) better characterise the nature of the time resulting from our construction: namely, boundedness, linearity and directness do not hold and times are not necessarily convex.

d21 $\mathtt{tpC}(x,y) := \mathtt{TME}(x) \wedge \mathtt{TME}(y) \wedge \mathtt{pC}(x,y)$ (*Macro Causal Precedence btw Times*)

d22 $\mathtt{tP}(x,y) := \mathtt{TME}(x) \wedge \mathtt{TME}(y) \wedge \mathtt{P}(x,y)$ (*Parthood between Times*)

d23 $\mathtt{tO}(x,y) := \mathtt{TME}(x) \wedge \mathtt{TME}(y) \wedge \mathtt{O}(x,y)$ (*Overlap between Times*)

t27 $\mathtt{TME}(x) \wedge \mathtt{qP}(a,x) \wedge s = \sigma y\langle \mathtt{EQ}(y,a)\rangle \rightarrow \mathtt{tP}(s,x)$

Proof. First, note that, from $s = \sigma y\langle \mathtt{EQ}(y,a)\rangle$, by (t26), we have $\mathtt{EC}(s,a)$ and then, by (d20), $\mathtt{TME}(s)$. From $\mathtt{TME}(x)$, by (t22) and (d19), we have $\mathtt{EC}(x,x)$. We prove that $\mathtt{EC}(s,a) \wedge \mathtt{EC}(x,x) \wedge \mathtt{qP}(a,x) \rightarrow \mathtt{P}(s,x)$, i.e., given (t1), that all the quantum parts of s are parts of x. Consider q such that $\mathtt{qP}(q,s)$. From $\mathtt{EC}(s,a)$, by (d19), $\exists w(\mathtt{EQ}(w,a) \wedge \mathtt{qP}(q,w))$. From $\mathtt{qP}(a,x) \wedge \mathtt{EQ}(w,a)$, by (t23) and (t25), $\mathtt{EQ}((x{-}a){+}w,x)$. We prove that $\mathtt{EQ}((x{-}a){+}w,x) \wedge \mathtt{EC}(x,x) \rightarrow \mathtt{P}((x{-}a){+}w,x)$. Consider a c such that $\mathtt{qP}(c,(x{-}a){+}w)$. From $\mathtt{EQ}((x{-}a){+}w,x) \wedge \mathtt{qP}(c,(x{-}a){+}w) \wedge \mathtt{EC}(x,x)$, by (d19), $\mathtt{qP}(c,x)$. This proves that $\mathtt{P}((x{-}a){+}w,x)$ and then, from $\mathtt{qP}(q,w)$, by (t3) and (a3), $\mathtt{qP}(q,x)$. This proves that $\mathtt{P}(s,x)$. The thesis follows directly from (d22). □

t28 $\mathtt{tO}(x,y) \rightarrow \exists z(\mathtt{tP}(z,x) \wedge \mathtt{tP}(z,y))$ ($\mathtt{tO}$*'s relation with* $\mathtt{tP}$)

Proof. From the hypotheses, by (d23) and (a6), $\exists a(\mathtt{qP}(a,x) \wedge \mathtt{qP}(a,y))$. By (a5) and (t22), there exists $z = \sigma u\langle \mathtt{EQ}(u,a)\rangle$ and, by (t27), $\mathtt{tP}(z,x) \wedge \mathtt{tP}(z,y)$. □

t29 $\mathtt{TME}(x) \rightarrow \mathtt{tP}(x,x)$ ($\mathtt{tP}$*: Reflexivity*)

Proof. It follows trivially from (a1). □

t30 $\mathtt{tP}(x,y) \wedge \mathtt{tP}(y,x) \rightarrow x = y$ ($\mathtt{tP}$*: Antisymmetry*)

Proof. It follows trivially from (a2). □

t31 $\mathtt{tP}(x,y) \wedge \mathtt{tP}(y,z) \rightarrow \mathtt{tP}(x,z)$ ($\mathtt{tP}$*: Transitivity*)

Proof. It follows trivially from (a3). □

t32 $\mathtt{TME}(x) \wedge \mathtt{TME}(y) \wedge \neg\mathtt{tP}(x,y) \rightarrow \exists z(\mathtt{tP}(z,x) \wedge \neg\mathtt{tO}(z,y))$ (*Strong Suppl. btw* $\mathtt{TME}$*s*)

Proof. From the hypotheses, by (a4) and (d22), we have that $\exists z(\mathtt{P}(z,x) \wedge \neg\mathtt{O}(z,y))$ and then, by (a6), $\exists a(\mathtt{qP}(a,x) \wedge \neg\mathtt{O}(a,y))$. By (a5) and (t22), there exists $s = \sigma u\langle \mathtt{EQ}(u,a)\rangle$ and, by (t27), $\mathtt{tP}(s,x)$. It remains to be proved that $\neg\mathtt{O}(s,y)$. By contradiction, assume that $\mathtt{O}(s,y)$. By (d1) and (a6), $\exists b(\mathtt{qP}(b,s) \wedge \mathtt{qP}(b,y))$. From the definition of the fusion s, we have that $\mathtt{EQ}(a,b)$ and, by the unicity of the fusion, $s = \sigma u\langle \mathtt{EQ}(u,b)\rangle$. Then, from $\mathtt{TME}(y) \wedge \mathtt{qP}(b,y) \wedge s = \sigma u\langle \mathtt{EQ}(u,b)\rangle$, by (t27), $\mathtt{tP}(s,y)$ and then $\mathtt{qP}(a,y)$ against $\neg\mathtt{O}(a,y)$. □

t33 $\mathtt{tO}(x,y) \rightarrow \exists z(\mathtt{TME}(z) \wedge \mathtt{PRD}(z,x,y))$ (*Existence of the* $\mathtt{TME}$ *Product*)

Proof. From the hypothesis, by (d23), $\mathtt{O}(x,y)$, and thus, by (a5) and the definition of product, $\exists z(\mathtt{PRD}(z,x,y))$. We need to prove that $\mathtt{TME}(z)$. To do so, we prove $\mathtt{EC}(z,z)$, i.e., [1] $\forall k(\mathtt{qP}(k,z) \rightarrow \exists w(\mathtt{EQ}(w,z) \wedge \mathtt{qP}(k,w)))$ and [2] $\forall k(\exists w(\mathtt{EQ}(w,z) \wedge \mathtt{qP}(k,w)) \rightarrow \mathtt{qP}(k,z))$. Consider [1]. It follows from (t22) and the hypotheses. Consider [2]. By (d4), $\mathtt{P}(z,x)$. Then from $\mathtt{P}(z,x) \wedge \mathtt{EQ}(w,z)$, by (t25), $\mathtt{EQ}(x,(x{-}z){+}w)$.

[10] To be precise, strong supplementation of $\mathtt{tP}$ is not among the requirements listed by van Benthem.

From $\mathtt{qP}(k,w)$, by the definition of sum, $\mathtt{qP}(k,(x-z)+w)$, thus from $\mathtt{TME}(x) \wedge \mathtt{EQ}(x,(x-z)+w) \wedge \mathtt{qP}(k,(x-z)+w)$, by (d19) and (d20), $\mathtt{qP}(k,x)$. By the definition of the product, we also have that $\mathtt{P}(z,y)$. Following the same reasoning we also have that $\mathtt{qP}(k,y)$ and then, by the definition of the product, $\mathtt{qP}(k,z)$. □

t34 $\neg\mathtt{tpC}(x,x)$ ($\mathtt{tpC}$*: Irreflexivity*)
Proof. It follows trivially from (t4). □

t35 $\mathtt{tpC}(x,y) \wedge \mathtt{tpC}(y,z) \rightarrow \mathtt{tpC}(x,z)$ ($\mathtt{tpC}$*: Transitivity*)
Proof. It follows trivially from (t5). □

t36 $\mathtt{tpC}(x,y) \rightarrow \neg\mathtt{tO}(x,y)$ ($\mathtt{tpC}$*: Separatedness*)
Proof. By contradiction assume that there exists z s.t. $\mathtt{qP}(z,x) \wedge \mathtt{qP}(z,y)$. By (a7) we have $\neg\mathtt{C}(z,z)$ against (d15). □

t37 $(\mathtt{tpC}(x,y) \wedge \mathtt{tP}(z,x) \rightarrow \mathtt{tpC}(z,y)) \wedge$
$(\mathtt{tpC}(x,y) \wedge \mathtt{tP}(z,y) \rightarrow \mathtt{tpC}(x,z))$ ($\mathtt{tpC}$*: Monotonicity*)
Proof. Directly from the definitions of $\mathtt{tpC}$ and $\mathtt{tP}$ by (t7) and (a1). □

t38 $(\mathtt{tpC}(y,x) \wedge \mathtt{tpC}(z,x) \wedge \mathtt{SUM}(s,y,z) \wedge \mathtt{TME}(s) \rightarrow \mathtt{tpC}(s,x)) \wedge$
$(\mathtt{tpC}(x,y) \wedge \mathtt{tpC}(x,z) \wedge \mathtt{SUM}(s,y,z) \wedge \mathtt{TME}(s) \rightarrow \mathtt{tpC}(x,s))$ ($\mathtt{tpC}$: MOND)
Proof. Directly from the definitions of $\mathtt{tpC}$ and $\mathtt{tP}$ by (t8). □

t39 $\mathtt{TME}(\mathtt{u})$ (the Universe is a $\mathtt{TME}$)
Proof. By (a11) $\mathtt{ITEM}(\mathtt{u})$. By the definition of $\mathtt{u}$ we have that $\mathtt{EC}(\mathtt{u},\mathtt{u})$ because $\mathtt{u}$ contains all the existing quanta. □

t40 $\nvdash \mathtt{TME}(x) \rightarrow \exists y(\mathtt{tpC}(y,x) \vee \mathtt{tpC}(x,y))$ ($\mathtt{tpC}$*: Existence Predecessor/Successor*)
Proof. By (t39) and the definition of the universe; consider $x = \mathtt{u}$. □

t41 $\nvdash \exists z(\mathtt{tpC}(x,z) \wedge \mathtt{tpC}(y,z)) \wedge \exists z(\mathtt{tpC}(z,x) \wedge \mathtt{tpC}(z,y))$ ($\mathtt{tpC}$*: Directness*)
Proof. By (t39) and the definition of the universe; consider $x = \mathtt{u}$. □

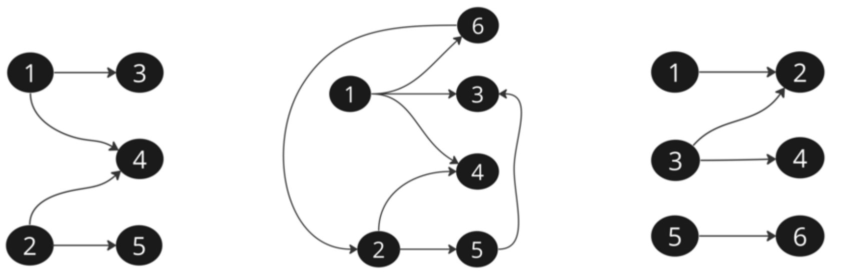
t42 $\nvdash \mathtt{TME}(x) \wedge \mathtt{TME}(y) \rightarrow \mathtt{tpC}(x,y) \vee \mathtt{tpC}(y,x) \vee \mathtt{tO}(x,y)$ ($\mathtt{tpC}$*: Linearity*)
Proof. Consider the case $\mathtt{dC}(0,1)$, $\mathtt{dC}(0,2)$, $\mathtt{dC}(1,3)$, $\mathtt{dC}(1,4)$, $\mathtt{dC}(2,5)$, $\mathtt{dC}(2,6)$. We have $\mathtt{TME}(1)$ and $\mathtt{TME}(2)$ but 1 and 2 are not in an $\mathtt{tpC}$ or $\mathtt{tO}$ relation. □

t43 $\nvdash \mathtt{tpC}(x,y) \wedge \mathtt{tpC}(y,z) \wedge \mathtt{tP}(x,u) \wedge \mathtt{tP}(z,u) \rightarrow \mathtt{tP}(y,u)$ ($\mathtt{tpC}$*: Convexity*)
Proof. Consider the situation $\mathtt{dC}(0,1)$, $\mathtt{dC}(0,2)$, $\mathtt{dC}(1,3)$, $\mathtt{dC}(1,4)$. We have $\mathtt{TME}(0)$, $\mathtt{TME}(1)$, $\mathtt{TME}(3+4)$, $\mathtt{TME}(0+3+4)$. Furthermore we have, $\mathtt{tpC}(0,1)$, $\mathtt{tpC}(1,3+4)$, $\mathtt{tP}(0,0+3+4)$, $\mathtt{tP}(3+4,0+3+4)$ but $\neg\mathtt{tP}(1,0+3+4)$. □

4. An Outline of a General Approach

In this section we investigate whether and how the approach and the results discussed in Sect. 3 can be generalised. Specifically, we focus on weaker theories in which temporal relations rest on the causal structure without requiring causal connections.

Let us start by considering (a13)'s weight in our construction of time. Fig. 2(a) (where arrows represent $\mathtt{dC}$ relations) is a model of EMMO$\setminus$(a13). Given our assumptions concerning $\mathtt{dC}$, quantum 1 ends when 3 and 4 begin (temporally), and 2 ends when 4 and 5 begin. Intuitively, 1 precedes 5 and 2 precedes 3 (temporally), however by (d15), $\neg\mathtt{pC}(1,5)$ and $\neg\mathtt{pC}(2,3)$. Thus, if (a13) is discarded, $\mathtt{pC}$ does not seem general enough to capture temporal precedence. Notably, in the model the precedence between 1 and 5 is supported by the fact that both 1 and 2 directly cause 4, despite the lack of a causal connection between 1 and 5 themselves. Models in which temporal comparability depends on similar causal structures (rather than simple causal connections) are excluded by (a13), which enforces causal connections when the relevant structures are present. Hence, a generalisation of $\mathtt{pC}$ seems in order. Our idea is to capture the (temporal) relation holding between 1 and 5 and between 2 and 3, i.e., a classic relation of (temporal) *meet*, and use this relation to generalise $\mathtt{pC}$ in theories where (a13) does not hold.[11] However, the general temporal precedence we want to arrive at can be seen as the transitive closure of meet; this poses a challenge, since in FOL the transitive closure of a relation cannot be defined using normal means.



(a) No Locality (b) Unintended Model (c) Axioms' independence

Figure 2. Generalising our construction

We start by considering the theory $\mathfrak{T} = \text{AGEM} \cup \{(\text{a7})\text{–}(\text{a10})\}$.[12] The *meet* relation $\mathtt{M}(x,y)$, where $\mathtt{M}(x,y)$ stands for "the quantum x ends when the quantum y begins", is defined in (d25) which uses the notion of *accordion* ($\mathtt{ACC}$) introduced in (d24). (d24) pinpoints comparable interactions by defining an accordion as a non atomic item of "depth 1", such that all the couples of quanta in an accordion can be partially synchronised: both the quanta end or start together or one starts when the other ends. In Fig. 2(a), 1+4, 1+2+4, and 1+2+4+5 are all accordions. It is easy to see that $\mathfrak{T} \vdash \mathtt{dC}(x,y) \rightarrow \mathtt{M}(x,y)$ while the vice versa does not hold, e.g., in Fig. 2(a), $\mathtt{M}(1,5) \wedge \neg\mathtt{dC}(1,5)$ (in EMMO $\mathtt{M}$ and $\mathtt{dC}$ collapse).

Contra our intended interpretation of meet, the asymmetry and the intransitivity of $\mathtt{M}$ does not hold in $\mathfrak{T}$: see the model in Fig. 2(b) where $\mathtt{M}(2,6) \wedge \mathtt{M}(6,2)$ and $\mathtt{M}(2,5) \wedge \mathtt{M}(5,3) \wedge \mathtt{M}(2,3)$. Accordingly, we extend $\mathfrak{T}$ into $\mathfrak{T}_m$ by adding said axioms[13] and we simulate the transitivity closure of $\mathtt{M}$ in $\mathfrak{T}_m$ via (d26)-(d28).

d24 $\mathtt{ACC}(x) \coloneqq \mathtt{CSTR}(x) \wedge$ (*Accordion*)
$\neg\exists wv(\mathtt{qP}(w,x) \wedge \mathtt{qP}(v,x) \wedge ((\mathtt{C}(w,v) \wedge \neg\mathtt{dC}(w,v)) \vee (\mathtt{C}(v,w) \wedge \neg\mathtt{dC}(v,w))))$

d25 $\mathtt{M}(x,y) \coloneqq \exists z(\mathtt{ACC}(z) \wedge \mathtt{qSRC}(x,z) \wedge \mathtt{qSNK}(y,z))$ (*Quantum Meet*)

d26 $\mathtt{bMC}(x) \coloneqq \forall yz(\mathtt{qP}(y,x) \wedge \mathtt{M}(z,y) \rightarrow \mathtt{qP}(z,x))$ (*Backward Meet Closure*)

d27 $\bar{\mathtt{M}}^*(x,y) \coloneqq \mathtt{Q}(x) \wedge \exists z(\mathtt{M}(z,y)) \wedge \forall z(\mathtt{bMC}(z) \wedge \mathtt{qP}(y,z) \rightarrow \mathtt{qP}(x,z))$

d28 $\mathtt{M}^*(x,y) \coloneqq \bar{\mathtt{M}}^*(x,y) \wedge (x \neq y \vee \exists z(z \neq x \wedge \bar{\mathtt{M}}^*(x,z) \wedge \bar{\mathtt{M}}^*(z,y)))$ (*Transitive* $\mathtt{M}$*-Closure*)

[11] This approach is supported by the fact that, upon first recognition, it does not seem possible to construct a more general notion of precedence directly from $\mathtt{pC}$ or $\mathtt{C}$.

[12] $\mathfrak{T}$ does away with EMMO's naturalism-committing axioms while preserving $\mathtt{C}$ and $\mathtt{dC}$'s core characteristics.

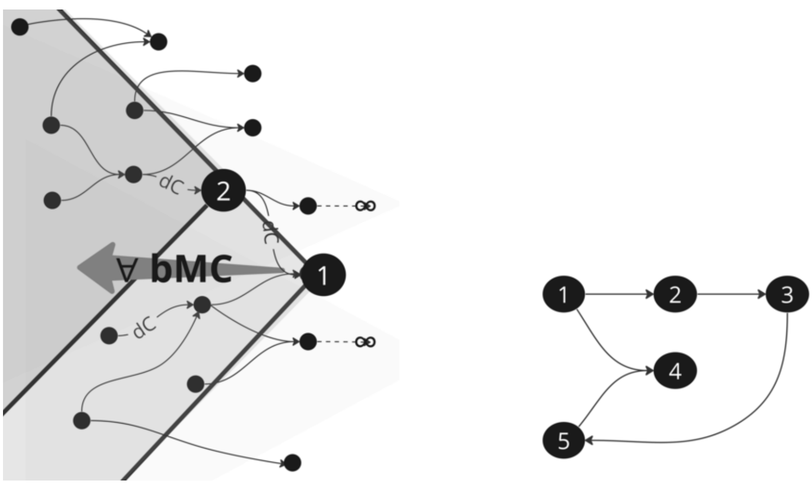
[13] No axiom discarded from EMMO (i.e., (a11)-(a13)) holds in $\mathfrak{T}_m$. Consider the model of $\mathfrak{T}_m$ in Fig. 2(c), where arrows stand for $\mathtt{dC}$ relations.

From a constructive perspective, (d26) requires to select some quanta as a starting point and, relatively to these, identifies M-chains by iteratively clustering quanta following meet relations backwards. For our purposes, scenarios involving a single starting quantum are particularly interesting; however, (d26) simply singles out entities meeting the closure criteria: bMCs can include infinite chains of quanta connected by M, in either direction, and, in general, the recursion might originate from a sum of (possibly infinite) starting quanta. This last case can be understood in terms of the fusion of all the intended bMCs generated starting from all the individual quanta making up the considered sum.

The first two conjuncts in (d27) assure that $\bar{\mathtt{M}}^*(x,y)$ holds between quanta, and that y is met by at least a quantum. The heavy-lifting is done by the last conjunct which establishes that all the bMCs having y as a quantum part must also include x,[14] i.e., given the construction of the bMCs, that y is connected to x via a sequence of M-steps. As shown in figure 3(a), the universal quantifier ensures that only the quanta reached by the recursive closure from a specific quantum are picked; consequently, x, as well as the entirety of the "minimal" bMC hinged on x, are always included in the "minimal" bMC hinged on y.

(d27) requires a refinement because it enforces $\bar{\mathtt{M}}^*$'s reflexivity, while we are looking for a relation allowing reflexivity only for quanta part of M-loops (which can be present in $\mathfrak{T}_m$, see below): all the quantum parts of a M-loop are $\mathtt{M}^*$-related. (d28) solves the issue.

Assuming that our construction correctly simulates the transitive closure of a relation, $\mathtt{M}^*$ can be considered a generic relation of (temporal) precedence among quanta. (t45) shows that $\mathtt{M}^*$ is transitive, and $\mathtt{M}(x,y) \rightarrow \mathtt{M}^*(x,y)$ as per (t46); consequently, $\mathtt{M}(x,y) \land \mathtt{M}^*(y,z) \rightarrow \mathtt{M}^*(x,z)$ holds. On the other hand, $\mathtt{M}^*$ is not irreflexive, asymmetric, and loop-less, and $\mathtt{M}(x,y) \leftrightarrow (\mathtt{M}^*(x,y) \land \neg\exists z(\mathtt{M}^*(x,z) \land \mathtt{M}^*(z,y)))$ does not hold: consider the model of $\mathfrak{T}_m$ in Fig. 3(b), where arrows stand for dC relations; In this model we have that $\mathtt{M}^*(2,2)$, that $\mathtt{M}^*(2,5) \land \mathtt{M}^*(5,2)$, and that $\mathtt{M}(5,2) \land \mathtt{M}^*(5,2) \land \mathtt{M}^*(5,3) \land \mathtt{M}^*(3,2)$. Finally, (t47) provides a minimal existential condition for bMCs.



(a) $\mathtt{M}^*$'s construction (b) $\mathtt{M}^*$'s characteristics

Figure 3. The $\mathtt{M}^*$ relation

[14]It is worth reiterating that a given quantum q might be part of an infinite number of bMCs.

t44 $\bar{\mathtt{M}}^*(x,y) \wedge \bar{\mathtt{M}}^*(y,z) \rightarrow \bar{\mathtt{M}}^*(x,z)$

Proof. Consider c such that $\mathtt{bMC}(c) \wedge \mathtt{qP}(z,c)$. From $\bar{\mathtt{M}}^*(y,z)$, by (d27), $\mathtt{qP}(y,c)$. From $\mathtt{bMC}(c) \wedge \mathtt{qP}(y,c)$ and $\bar{\mathtt{M}}^*(x,y)$, by (d27), $\mathtt{qP}(x,c)$. It remains to prove that $\mathtt{Q}(x)$ and $\exists b(\mathtt{M}(b,z))$. The first follows directly from $\bar{\mathtt{M}}^*(x,y)$, the latter from $\bar{\mathtt{M}}^*(y,z)$. □

t45 $\mathtt{M}^*(x,y) \wedge \mathtt{M}^*(y,z) \rightarrow \mathtt{M}^*(x,z)$

Proof. From the hypothesis, by (d28) and (t44), $\bar{\mathtt{M}}^*(x,z)$. If $x \neq z$ we are done. Assume $x = z$, then from the hypotheses, $x \neq y$ and $y \neq z$, therefore, by (d28), $\exists a(a \neq x \wedge \bar{\mathtt{M}}^*(x,a) \wedge \bar{\mathtt{M}}^*(a,y))$ and $\exists b(b \neq y \wedge \bar{\mathtt{M}}^*(y,b) \wedge \bar{\mathtt{M}}^*(b,z))$ and, by (t44), $\exists a(a \neq x \wedge \bar{\mathtt{M}}^*(x,a) \wedge \bar{\mathtt{M}}^*(a,z))$. □

t46 $\mathtt{M}(x,y) \rightarrow \mathtt{M}^*(x,y)$

Proof. Trivially we have $\mathtt{Q}(x) \wedge \exists z(\mathtt{M}(z,y))$. Consider c such that $\mathtt{bMC}(c) \wedge \mathtt{qP}(y,c)$, from the hypothesis, by (d26), $\mathtt{qP}(x,c)$. This proves $\bar{\mathtt{M}}^*(x,y)$. Given the asymmetry of $\mathtt{M}$ we have that $x \neq y$ and then the thesis by (d28). □

t47 $\mathtt{Q}(q) \rightarrow \exists a(\mathtt{bMC}(a) \wedge \mathtt{qP}(q,a))$

Proof. If $\exists u(\mathtt{M}(u,q))$ then, by the fact that $\mathtt{M}(x,y) \rightarrow \mathtt{M}^*(x,y)$, $\exists u(\mathtt{M}^*(u,q))$ and, by (a5), $a = q + \sigma u\langle \mathtt{M}^*(u,q)\rangle$ exists. Trivially $\mathtt{qP}(q,a)$. By contradiction assume that $\neg\mathtt{bMC}(a)$. By (d26)-(d28), $\exists cd(\mathtt{qP}(c,a) \wedge \neg\mathtt{qP}(d,a) \wedge \mathtt{Q}(d) \wedge \mathtt{M}^*(d,c))$. From $\mathtt{qP}(c,a) \wedge \mathtt{M}^*(d,c)$, by the construction of a and the transitivity of $\mathtt{M}^*$, $\mathtt{M}^*(d,q)$ and then, again by the construction of a, $\mathtt{qP}(d,a)$. Contradiction. If $\neg\exists u(\mathtt{M}(u,q))$, it is trivial to verify that $\mathtt{bMC}(q)$. □

We can now follow the construction of time described in Sect. 3, substituting $\mathtt{C}$ in (d15) with $\mathtt{M}^*$, and $\mathtt{ITEM}$ in (d16) with the analogue obtained by substituting $\mathtt{dC}$ with $\mathtt{M}$ in (d10). As such, the provided results have a broader application scope. However, it has to be carefully investigated which of the formulas considered in Sect. 3 are theorems of $\mathfrak{T}_m$, and how they are related to specific axiomatic commitments.

Notably, the strategy employed to simulate the transitive closure of $\mathtt{M}$ via (d26)-(d28) might enjoy general applicability. For what concerns us here, it paves the way to theories employing an asymmetric and intransitive primitive relation on quanta sharing the intuitive interpretation of $\mathtt{dC}$. Not only a relation of direct causation seems more appropriate from a naturalistic point of view (at least given EMMO's theoretical assumptions), but the move also allows for more fine-grained formal choices.

Arguably, the generalisations put forward in this section make our results exploitable in ontology engineering with minimal entry requirements, as, specifically, the core relations are weakly characterised, and the approach is neutral with respect to many aspects (e.g., the nature of the relata). That said, the various topics touched on in this section require a more careful examination, which is deferred to future work.

5. Concluding Remarks

In this paper we took EMMO's discrete mereocausal framework as a starting point and provided a construction of time (temporal individuals and structure) which satisfies the minimal formal requirements put forward by van Benthem. Having done that, we showed how our results could be generalised, establishing the groundwork for a systematic investigation of the connections between (discrete) causal and temporal structures. As such, our work covered a blind spot in the (formal) analysis of causal relational theories of time. From a more application-oriented point of view, the generalisation made our results easily exploitable in ontology engineering, as well as in the alignment of (foundational) ontologies, where the latter provided the occasion for this work. In the process, we developed reusable technical machinery, i.e., a strategy to simulate the transitive closure of a relation in FOL via mereology, which, if effective, might enjoy general application, beyond the field of applied ontology.

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