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Generalized Countable Fuzzy Semi-Compactness in L-Topological Spaces

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Abstract. In this paper, the generalized countable fuzzy semi-compactness is defined in LTS, and its weak topological invariance and topological generation are proved. When L is complete Heyting algebra, the union of two generalized countable fuzzy semi-compactness L-set is generalized countable fuzzy semi-compactness; the intersection of a generalized countable fuzzy semi-compactness L-set and a generalized semi-closed countable L-set is generalized countable fuzzy semi-compactness.

Keywords. L-topological space; generalized countable fuzzy semi-compactness; generalized semi-open countable; generalized semi-closed countable

1. Introduction

In 1976, the concept of fuzzy compactness is introduced in $[0,1]$ –TPS($[0,1]$ –topological Spaces) by reference[1]. In 1988, [2] extended it to LTS, where L is a completely allocated DeMorgan algebra. [3] proposed a new definition of fuzzy compactness in LTS. [4] studies the countably compactness of L-set.[5]gives the Generalized semi-open L-sets and generalized semi-closed L-sets. [6]gives the concept of generalized fuzzy semi-compactness, properties of generalized fuzzy semi-compactness and some equivalent characterizations.[7]-[14]many experts have studied the related properties of compactness and semi-compactness in L-topological Spaces.[15]-[19]experts have studied some properties of compactness by means of operators. This paper gives the definition of generalized countable fuzzy semi-compactness, Some of its properties are studied. The remaining concepts and notations not described in the text can be found in [2]. For convenience, we will hereafter refer to $L-$ topological space as LTS for short.

2. Related works

In this paper, the compactness of LTS is extended on the basis of [6], and some related properties of [6] are studied. On this basis, the weak topological invariance and topological generatability of generalized countable fuzzy semi-compactness are also studied.

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3. Preliminary Knowledge

In this part, we will review some primary concepts of generalized fuzzy semi-compactness.

Definition 3.1[5]Hypothesis $\left(L^X,\delta\right)$ is an LTS, $B\in L^X$ .Then B is the generalized semi-closed $L-$set, if the semi-open countable $L-$set U satisfying $B\le U$ there is $cl\left(B\right)\le U$. B is called generalized semi-open if B' is generalized semi-closed.

$GSO\left(X\right)$ is denoted as the sets of the all generalized semi-open $L-$sets on X and $GSC\left(X\right)$ is denoted as the sets of the all generalized semi-closed $L-$sets on X .

Definition 3.2[6]Hypothesis $\left(L^X,\delta\right)$ is an LTS, $M\in L^X$. If for each family $P\subset GSO\left(X\right)$ there is

$$\bigcap_{x\in X}\left(M'\left(x\right)\cup\bigcup_{D\in P}D\left(x\right)\right)\le\bigcup_{V\in 2^{(P)}}\bigcap_{x\in X}\left(M'\left(x\right)\cup\bigcup_{D\in V}D\left(x\right)\right).$$

Then M is called generalized fuzzy semi-compact.

Definition 3.3[7]Hypothesis $\left(L^X,\alpha\right)$ and $\left(L^Y,\beta\right)$ are LF topological space, $f:\left(L^X,\alpha\right)\to\left(L^Y,\beta\right)$ is $L-$value Zadeh- type function, if $\forall H\in\beta$ have $f^{-1}\left(H\right)\in\alpha$, f is called continuous.

Lemma3.1[8] Hypothesis $\left(L^X,\omega_L\left(F\right)\right)$ is an LTS induced by the distinct topological space $\left(X,F\right)$. Hypothesis U is the semi-open set in $\left(X,F\right)$, then χ_U is the semi-open set in $\left(L^X,\omega_L\left(F\right)\right)$ .If R is the semi-open set in $\left(L^X,\omega_L\left(F\right)\right)$, then for $\mathrm{b}\in L$, $R\left(\mathrm{b}\right)$ is the semi-open set in $\left(X,F\right)$.

For the subset $P\subset L^X$, $2^{(P)}$ is denoted as the set of all finite subfamilies of P .

4. Generalized countable fuzzy semi-compactness

Definition 4.1 Hypothesis $\left(L^X,\delta\right)$ is an LTS, $M\in L^X$. Hypothesis for every countably family $P\subset GSO\left(X\right)$ there is

$$\bigcap_{x\in X}\left(M'\left(x\right)\cup\bigcup_{D\in P}D\left(x\right)\right)\le\bigcup_{V\in 2^{(P)}}\bigcap_{x\in X}\left(M'\left(x\right)\cup\bigcup_{D\in V}D\left(x\right)\right).$$

Then M is denoted generalized countable fuzzy semi-compact.

Definition 4.2 Hypothesis $\left(L^X,\delta\right)$ is an LTS, $c\in L-\{1\}$, $M\in L^X$ .A countable family $P\subset GSO\left(X\right)$ is called generalized semi-open countable $c-$*shading* of M , hypothesis for every $x\in X$ there is $\left(M'\left(x\right)\cup\bigcup_{D\in P}D\left(x\right)\right)\le c$. P is called generalized semi-open countable strong $c-shading$ of M hypothesis for any $x \in X$ there is $\bigcap_{x\in X}\left(M'(x) \cup \bigcup_{D\in P} D(x)\right) \leq c$.

The generalized semi-open countable strong $c-shading$ of M is the generalized semi-open countable $c-shading$ of M.

Definition 4.3. Hypothesis $\left(L^X, \delta\right)$ is an LTS, $c \in L-\{1\}$, $M \in L^X$ .A countable family $Q \in GSC(X)$ is called generalized semi-closed countable $c-remote$ family of M, hypothesis for every $x \in X$ there is $\left(M(x) \cap \bigcap_{B\in Q} B(x)\right) \geq c$. Q is called generalized semi-closed countably strong $c-remote$ family of M, if $\bigcup_{x\in X}\left(M(x) \cap \bigcap_{B\in Q} B(x)\right) \geq c$.

The generalized semi-closed countably strong $c-remote$ family of M is the generalized semi-closed countably $c-remote$ family of M.

By Definition 4.1 and order inversing involution ,we will introduce theorem4.1.

Theorem 4.1. Hypothesis $\left(L^X, \delta\right)$ is an LTS, $M \in L^X$ .Then M is generalized countably fuzzy semi-compact if and only if for any countably family $Q \in GSC(X)$, there is $\bigcup_{x\in X}\left(M(x) \cap \bigcap_{B\in Q} B(x)\right) \geq \bigcap_{F\in 2^{(Q)}} \bigcup_{x\in X}\left(M(x) \cap \bigcap_{B\in F} B(x)\right)$.

Theorem 4.2. Hypothesis M is generalized fuzzy semi-compact, then it is generalized countable fuzzy semi-compact.

Proof. If M is generalized fuzzy semi-compact, by definition 3.2, for every family $P \subset GSO(X)$, there is

$$\bigcap_{x\in X}\left(M'(x) \cup \bigcup_{D\in P} D(x)\right) \leq \bigcup_{V\in 2^{(P)}} \bigcap_{x\in X}\left(M'(x) \cup \bigcup_{D\in V} D(x)\right).$$

There is certainly $P \subset GSO(X)$ countable subset $V \subset GSO(X)$ meet

$$\bigcap_{x\in X}\left(M'(x) \cup \bigcup_{D\in V} D(x)\right) \leq \bigcup_{C\in 2^{(V)}} \bigcap_{x\in X}\left(M'(x) \cup \bigcup_{D\in C} D(x)\right).$$

Prove that M is generalized countably fuzzy semi-compact.

By Definitions 4.1 and 4.2 we will introduce theorem4.3.

Theorem 4.3 Hypothesis $\left(L^X, \alpha\right)$ is an LTS, $M \in L^X$. Then M is generalized countably fuzzy semi-compact if and only if for every $c \in L-\{1\}$,every generalized semi-open countably strong $c-shading$ P of M has finite subfamily D is generalized semi-open countably strong $c-shading$ of M.

By Definitions 4.1 and 4.3 we will introduce theorem4.4.

Theorem 4.4 Hypothesis $\left(L^X, \alpha\right)$ is an LTS, $M \in L^X$ .Then M is generalized countably fuzzy semi-compact if and only if for every $c \in L-\{0\}$,every generalized semi-closed countably strong $c-remote$ family K of M has finite subfamilies C is generalized semi-closed countably $a-remote$ family of M.

5. Properties of generalized countable fuzzy semi-compactness

Definition 5.1 Hypothesis $\left(L^X,\delta\right)$ and $\left(L^Y,\mu\right)$ are LTS, $f:\left(L^X,\delta\right)\to\left(L^Y,\mu\right)$ is an homomorphism said to be a continuous order homomorphism if for any countably closed set H in $\left(L^Y,\mu\right)$, $f^{-1}(H)$ is countably closed set in $\left(L^X,\delta\right)$.If $L_1=L_2=L$, f is a *Zadeh*- type mapping, f is said to be the $L-$ continuum mapping from $\left(L^X,\delta\right)$ to $\left(L^Y,\mu\right)$.

Definition 5.2 Hypothesis $\left(L^X,\delta\right),\left(L^Y,\mu\right)$ is an LTS, $f:\left(L^X,\delta\right)\to\left(L^Y,\mu\right)$ as the one-to-one mapping, f and f^{-1} are $L-$ continuous , says f is $L-$ homeomorphism mapping. The property that remains invariant under $L-$ homeomorphism mapping is called weak topological invariance.

Definition 5.3 Hypothesis $\left(L^X,\omega_L(T)\right)$ is an LTS induced by the distinct topological spaces (X,T). Hypothesis U is the generalized semi-open countable set in, then χ_U is the generalized semi-open countable set of $\left(L^X,\omega_L(F)\right)$.If A is the generalized semi-open countable set of $\left(L^X,\omega_L(F)\right)$,then for $a\in L$, $A(a)$ is the generalized semi-open countable set in (X,T).

Theorem 5.1 Hypothesis $\left(L^X,\delta\right)$ and $\left(L^Y,\mu\right)$ are LTS, $f:\left(L^X,\delta\right)\to\left(L^Y,\mu\right)$ is continuous $L-$ value Zadeh- type function and $M\in L^X$ .Then $f(M)$ is generalized countably fuzzy semi-compact set in $\left(L^Y,\mu\right)$ when M is the generalized countably fuzzy semi-compact set in $\left(L^X,\delta\right)$.

Proof. Let P be countable family of $f(M)$, then $f^{-1}(P)$ is a countable family of M .Is defined by Definition 4.1 has

$$\bigcap_{y\in Y}\left(f(M)'(y)\cup\bigcup_{D\in P}D(y)\right)=\bigcap_{x\in X}\left(M'(x)\cup\bigcup_{D\in P}f^{-1}(D)(x)\right)$$

$$\le\bigcup_{V\in 2^{(P)}}\bigcap_{x\in X}\left(M'(x)\cup\bigcup_{A\in V}f^{-1}(D)(x)\right)=\bigcup_{V\in 2^{(P)}}\bigcap_{y\in Y}\left(f(M)'(y)\cup\bigcup_{D\in V}D(y)\right).$$

That is $f(M)$ is the generalized countably fuzzy semi-compact set in $\left(L^Y,\mu\right)$.

Corollary 1 Generalized countable fuzzy semi-compact in $L-$topological spaces is weakly topologically invariant.

Theorem 5.2 Makes L is a complete Heyting algebra.Hypothesis both M and N are generalized countably fuzzy semi-compact, then $M\vee N$ is generalized countably fuzzy semi-compact.

Proof. For every countable family $U\in GSC(X)$, given by theorem 4.1 have

$$\bigcup_{x\in X}\left((M\cup N)(x)\cap\bigcap_{B\in U}B(x)\right)$$

$$= \left\{ \bigcup_{x\in X} M(x) \cap \bigcap_{B\in U} B(x) \right\} \cup \left\{ \bigcup_{x\in X} \left(N(x) \cap \bigcap_{B\in U} B(x) \right) \right\}$$
$$\geq \left\{ \bigcap_{F\in 2^{(U)}} \bigcup_{x\in X} \left(M(x) \cap \bigcap_{B\in F} B(x) \right) \right\} \cup \left\{ \bigcap_{F\in 2^{(U)}} \bigcup_{x\in X} \left(N(x) \cap \bigcap_{B\in F} B(x) \right) \right\}$$
$$= \bigcap_{F\in 2^{(U)}} \bigcup_{x\in X} \left((M \cup N)(x) \cap \bigcap_{B\in F} B(x) \right).$$

Therefore $M \vee N$ is generalized countably fuzzy semi-compact.

Theorem 5.3 Hypothesis M is the generalized countably fuzzy semi-compact $L-$set, $N \in GSC(X)$,then $M \wedge N$ is the generalized countably fuzzy semi-compact $L-$set.

Proof. Since M is the generalized countably fuzzy semi-compact $L-$set,for every countably family $U \in GSC(X)$, given by theorem 4.1 have

$$\bigcup_{x\in X} \left((M \cap N)(x) \cap \bigcap_{B\in U} B(x) \right) = \bigcup_{x\in X} \left(M(x) \cap \bigcap_{B\in U \cup \{N\}} B(x) \right)$$
$$\geq \bigcap_{F\in 2^{(U\cup\{N\})}} \bigcup_{x\in X} \left(M(x) \cap \bigcap_{B\in F} B(x) \right)$$
$$= \left\{ \bigcap_{F\in 2^{(U)}} \bigcup_{x\in X} \left(M(x) \cap \bigcap_{B\in F} B(x) \right) \right\} \cap \left\{ \bigcap_{F\in 2^{(U)}} \bigcup_{x\in X} \left(M(x) \cap \left(N(x) \cap \bigcap_{B\in F} B(x) \right) \right) \right\}$$
$$= \bigcap_{F\in 2^{(U)}} \bigcup_{x\in X} \left((M \cap N)(x) \cap \bigcap_{B\in F} B(x) \right).$$

Therefore $M \wedge N$ is the generalized countably fuzzy semi-compact $L-$set.

Theorem 5.4 Hypothesis $\left(L^X, \omega_L(F)\right)$ is an LTS induced by the distinct topological spaces (X,F), $T \subset X$, then T is generalized countably fuzzy semi-compact in (X,F) if and only if χ_T is generalized countably fuzzy semi-compact in $\left(L^X, \omega_L(F)\right)$.

Proof. $\Leftarrow$. Hypothesis Γ is generalized semi-open countable family of $\left(L^X, \omega_L(F)\right)$,

let $\bigcap_{x\in X} \left((\chi_T)'(x) \cup \bigcup_{B\in\Gamma} B(x) \right) = a$.

When $a = 0$, There is obviously

$$\bigcap_{x\in X} \left((\chi_T)'(x) \cup \bigcup_{B\in\Gamma} B(x) \right) \leq \bigcup_{\Upsilon\in 2^{(\Gamma)}} \bigcap_{x\in X} \left((\chi_T)'(x) \cup \bigcup_{B\in\Upsilon} B(x) \right).$$

Let $a \neq 0$, then for $b < a$ have $b < \bigcap_{x\in X} \left((\chi_T)'(x) \cup \bigcup_{B\in\Gamma} B(x) \right)$.

Thus $\left\{ B_{(b)} \mid B \in \Gamma \right\}$ is generalized semi-open countable covering of T . By T is generalized countable fuzzy semi-compact of (X,F), exist $\Upsilon \in 2^{(\Gamma)}$ make $\left\{ B_{(b)} \mid B \in \Upsilon \right\}$ is generalized half-open countable covering of T .

Then $b \leq \bigcap_{x\in T} \left(\bigcup_{B\in\Upsilon} B \mid T(x) \right)$. Thus

$$b \le \underset{x\in T}{\cap}\left(\underset{B\in\Upsilon}{\cup}(B\mid T)\right)(x) = \underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\left(\underset{B\in\Upsilon}{\cup}B(x)\right)\right)$$

$$\le \underset{\Upsilon\in 2^{(\Gamma)}}{\cup}\ \underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\left(\underset{B\in\Upsilon}{\cup}B(x)\right)\right)$$, Thus have

$$\underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\underset{B\in\Gamma}{\cup}B(x)\right) = a = \cup\{b\mid b<a\} \le \underset{T\in 2^{(\Gamma)}}{\cup}\ \underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\left(\underset{B\in\Upsilon}{\cup}B(x)\right)\right)$$

Therefore χ_T generalized countable fuzzy semi-compact of $\left(L^X,\omega_L(F)\right)$.

$\Rightarrow$ .Hypothesis B be any generalized semi-open countable covering of T ,then $\{\chi_J \mid J\in B\}$ is the generalized semi-open countable sets in $\left(L^X,\omega_L(F)\right)$ and

$$\underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\underset{J\in B}{\cup}\chi_J(x)\right)=1 .$$

By χ_T is the generalized countably fuzzy semi-compact in $\left(L^X,\omega_L(F)\right)$ known,

$$\underset{\Upsilon\in 2^{(B)}}{\cup}\ \underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\left(\underset{J\in\Upsilon}{\cup}\chi_J(x)\right)\right)=1 .$$

There are $\Upsilon\in 2^{(B)}$ make $\underset{x\in X}{\cap}\left((\chi_T)'(x)\cup\underset{J\in\Upsilon}{\cup}\chi_J(x)\right)=1$.

So $\underset{x\in T}{\cap}\left(\underset{J\in\Upsilon}{\cup}(\chi_J\mid T)(x)\right)=1$, thus $\{J\mid T\mid J\in\Upsilon\}$ is generalized semi-open countable covering of T , namely Υ is generalized semi-open countable covering of T .

6. Conclusion

In this paper, the generalized countable fuzzy semi-compactness is defined in LTS. This definition does not depend on the structure of L and does not require distributivity. As a generalization of the compactness of L-topological Spaces, some good properties of the generalized countably fuzzy semi-compactness are proved. In the future, can continue to study some properties such as equivalent characterization and good generalization.

Reference

[1] Lowen R. Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl.,56(1976)621-633.

[2] Wang GJ. L-Fuzzy Topological Space Theory [M] Xi'an: Shaanxi Normal University Press, 1988.

[3] Shi FG. Fuzzy compactness in L-topological spaces[J]. Fuzzy Sets and Systems, 2007, 58:1486-1495.

[4] Shi FG. Countable compactness and the Lindelof property of L-fuzzy sets [J].Iranian Journal of Fuzzy Systems, 2004,1:79-88.

[5] Yang GQ, Li HY, Xu ZG. Generalized semi-open L-sets and generalized semi-closed L-sets[J]. Journal of Liaoning Normal University(Natural Science Edition), 2006, 29(4):194-196.

[6] Xu ZG, Liu MG. Generalized fuzzy semi-compactness in L-topological spaces[J]. Journal of Liaoning Normal University(Natural Science Edition), 20 21, 44(4):450-453.

[7] Jiang JP. The countably pairwise ultra-compactness in L-bitopological spaces[J].Fuzzy Systems and

Mathematics, 2015, 29(3):75-78.

[8] Shi FG. Semi-compactness in L-topological spaces[J]. International Journal of Mathematics and Mathematical Sciences, 2005, 12: 1869-1878.

[9] Pan W, Xu ZG, Zhao Y. Countable semi-compactness L-fuzzy topological spaces[J]. Fuzzy Systems and Mathematics, 2015, 29(4):71-75.

[10] Han HX, Meng GW. Local semi-compactness of L-topological spaces[J]. College Mathematics, 2010, 26(4): 76-79.

[11] Shi FG, Li RX. Compactness in L-fuzzy topological spaces[J]. Hacettepe journal of Mathematics and Statistics, 2011,40 (6):767-774.

[12] Shi FG. A new definition of fuzzy compactness[J]. Fuzzy Sets and Systems, 2007,158:1486-1495.

[13] Zhang J, Shi FG, Zheng CY. On L-fuzzy topological spaces[J]. Fuzzy Sets and Systems, 2005,149:473-484.

[14] Shi FG, Li RX. Semi-compactness in L-fuzzy topological spaces[J]. Annals of Fuzzy Mathematics and Informatics, 2011, 2:163-169.

[15] Benkhaled H, Elleuch A, Jeribi A.Demicompactness Properties for Uniformly Continuous Cosine Families[J].Mediterranean Journal of Mathematics, 2022, 156-169.

[16] Slim C, Aref J, Bilel K. Demicompactness perturbation in Banach algebras and some stability results[J].Georgian Math, 2023, 30(1):53-63.

[17] Hedi B,Elleuch A,Jeribi A.Relative Demicompactness Properties for Exponentially Bounded C 0-Semigroups[J].Russian Mathematics, 2023, 1-7.

[18] Krichen B,Trabelsi B.B-Weyl and Drazin invertible operators linked by weak pseudo \(S_0\)-demicompactness [J] Ricerche di Matematica, 2021, 1-18.

[19] Wasi HA,Rajihy Y.On various properties of (;m, n;)-semi-compactness in bitopological spaces [J].Journal of Interdisciplinary Mathematics,2021,1119-1122.