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# Dynamics of a Stochastic SIRI Epidemic Model

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**Abstract.** Curz Vargas et al studied the dynamics of a deterministic SIRI epidemic model, but it did not take into account the influence of environmental noise on system parameters, in the paper, we consider that the important parameters are disturbed by Ornstein-Uhlenbeck process and we can get a stochastic SIRI epidemic model. First of all, we study the existence and uniqueness of positive solution of system. The second, we also get the stationary distribution of the model . In the end, we get expression of the density function of the stochastic model. Compared with this deterministic model, the dynamical analysis of the system is more reasonable.

**Keywords.** Stochastic SIRI epidemic model; Stationary distribution; Ornstein-Uhlenbeck process; Probability density function

## 1. Introduction

Curz Vargas et al[1]studied an infectious diseases models with relapse about global stability, the follow model is studied,

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta^0 SI - \mu S, \\ \dfrac{dI}{dt} = \beta^0 SI - (\alpha+\kappa+\mu)I + \gamma R, \\ \dfrac{dR}{dt} = \kappa I - (\mu+\gamma)R. \end{cases} \tag{1}$$

Above the parameters are positive constants. $S$ is susceptible population, $I$ is infected people , $R$ is the recovered population. The constant $\Lambda$ is the recruitment rate of $S$ ,$\mu$ is the natural death rate of population . $\beta^0$ is the disease transmission coefficient. $\alpha$ is the disease related death rate . $\kappa$ is infectious individuals becomes normal individuals and $\gamma$ is the normal individuals are reverted to the infectious state. For system (1), the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ ,

$E^* = (S^*, I^*, R^*)$ with$S^* = \frac{\Lambda}{\mu R_0}, I^* = \frac{\mu}{\beta}(R_0-1), R^* = \frac{\kappa\mu}{\beta(\gamma+\mu)}(R_0-1)$ ,Where $R_0$ , the basic reproduction number, is [1]

$$R_0 = \frac{(\mu+\gamma)\beta\Lambda}{\mu(\kappa\mu+(\mu+\gamma)(\alpha+\mu))}. \tag{2}$$

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And

$$\Gamma_0 = \left\{(S,I,R) \in \mathbb{R}^3_+ : S \geq 0, I \geq 0, R \geq 0, S+I+R \leq \Lambda/\mu\right\}. \tag{3}$$

However, the above model still falls short in studying infectious disease dynamics because it does not take into account the influence of environmental noise on system parameters[2][3]. In the paper $\beta^0(t)$ is affected by the mean-reverting OrnsteinCUhlenbeck process and its representation is as follows, and $\delta$, $\sigma$ are positive constants ,where $\delta^0$ means the speed of reversion and $\sigma^2 > 0$ is the intensity of fluctuation:

$$d\beta = -\delta[\beta^0(t) - \bar{\beta}]dt + \sigma dB(t), \tag{4}$$

For(2), we consider $p(t) = \beta^0(t) - \bar{\beta}$, we get

$$\begin{cases} dp = -\delta m dt + \sigma dB(t), \\ dS = [\Lambda - (p+\bar{\beta})SI - \mu S]dt, \\ dI = [(p+\bar{\beta})SI - (\alpha+\kappa+\mu)I + \gamma R]dt, \\ dR = [\kappa I - (\mu+\gamma)R]dt. \end{cases} \tag{5}$$

and we can get a region:

$$\Gamma = \left\{(p,S,I,R) \in \mathbb{R} \times \mathbb{R}^3_+ : S+I+R \leq \frac{\Lambda}{\mu}\right\} \tag{6}$$

In Section 2 we give the conditions for existence and uniqueness of a global solution of system (3). and in Section 3, give the conditions for existence of an ergodic stationary distribution of system (3). Section 4 we will study the density function of the model . Section 5 we give the conclusion for the paper.

## 2. Existence and uniqueness of a global solution

To prove that there is a unique positive solution to system(3), we will give the following theorem:

**Theorem 2.1.** *For system (3), it exists a unique solution* $(p(t), S(t), I(t), R(t))$ *on* $t \geq 0$ *for any initial value and the initial value*$(p(0), S(0), I(0), R(0)) \in \Gamma$ *a.s..*

*Proof.* The system (3) , it is satisfy the local Lipschitz condition, and there is a unique local solution. we will define a $C^2$ Lyapunov function $V$: $\mathbb{R} \times \mathbb{R}^3_+ \to \mathbb{R}_+$as follows:

$$V = S - 1 - \ln S + I - 1 - \ln I + R - 1 - \ln R + \frac{p^2}{2}. \tag{7}$$

The remaining evidence is similar to the literature[4], we will omit it.

## 3. stationary distribution

Define

$$\mathscr{R}_0^S = \mathscr{R}_1 - \frac{\Lambda^2}{\mu^3}\frac{\sigma}{2\sqrt{\pi\delta}}. \tag{8}$$

while$\mathscr{R}_1 = \frac{(\mu+\gamma)\bar{\beta}\Lambda}{\mu(\kappa\mu+(\mu+\gamma)(\alpha+\mu))}$

**Theorem 3.1.** *Assume that* $R_0^s > 1$ *, then the stochastic system (3)admits at least one ergodic stationary distribution* $\rho(\cdot)$*on* $\Gamma$.

*Proof.* we proof it need three steps: the first we need construct stochastic Lyapunov functions; the second a compact set is constructed; in the end five the existence and ergodicity of the solution of system (3).

**Step 1. (Construct stochastic Lyapunov functions):** A $C^2$ function $H(p,S(t),I(t),R(t)) : \Gamma \to \mathbb{R} = V - V_{min}$ is defined by

$$\begin{aligned} V(p(t),S(t),I(t),R(t)) &= M[-\ln I - \frac{\Lambda}{\mu^2}\ln S - \frac{\gamma k}{(\mu+\gamma)^2}\ln R] - \ln S - \ln R - \ln(\frac{A}{\mu} - S - I - R) + \frac{p^2}{2} \\ &:= MV_1 + V_2 \end{aligned}$$

where $V_1 = -\ln I - \frac{\Lambda}{\mu^2}\ln S - \frac{\gamma k}{(\mu+\gamma)^2}\ln R$ and $V_2 = -\ln S - \ln R - \ln(\frac{A}{\mu} - S - I - R) + \frac{p^2}{2}$, $M$ is a sufficiently large number and satisfies the expression :

$$-M(\mathscr{R}_0^S - 1) + \mu + (\mu+\gamma) + \frac{\sigma^2}{2} + \sup\left\{-\frac{1}{2}\delta p^2 + p\frac{\Lambda}{\mu} + Mp\frac{\Lambda}{\mu}\right\} \le -2 \tag{9}$$

Employing the *Itô's* formula to $V_1$,$V_2$ and combining the ergodic theorem[5]and (9), we obtain

$$\int_{-\infty}^{\infty}(p\vee 0)\,\kappa(x)dx = \int_0^{\infty}\frac{\sqrt{\delta}x}{\sqrt{\pi}\sigma}e^{-\frac{\delta x^2}{\sigma^2}}dx = \frac{\sigma}{2\sqrt{\pi\delta}}\int_0^{\infty}e^{-\left(\frac{\sqrt{\delta}x}{\sigma}\right)^2}d\left(\frac{\sqrt{\delta}x}{\sigma}\right)^2 = \frac{\sigma}{2\sqrt{\pi\delta}}, \quad \text{a.s.} \tag{10}$$

$$\begin{aligned} LH &= M[-(p+\bar{\beta})S + (\alpha+\kappa+\mu) - \gamma\frac{R}{I} - \frac{\Lambda^2}{S\mu^2} + (p+\bar{\beta})\frac{\Lambda}{\mu^2}I + \mu I] \\ &\quad - \frac{\Lambda}{S} + (p+\bar{\beta})I + \mu - \kappa\frac{I}{R} + (\mu+\gamma) - \frac{\Lambda - \mu(S+I+R) - \alpha I}{\frac{\Lambda}{\mu} - S - I - R} - \delta p^2 + \frac{\sigma^2}{2} \\ &\le -2 + (M + \frac{\Lambda}{\mu^2})\bar{\beta}I - \frac{\Lambda}{S} - k\frac{I}{R} - \frac{\alpha I}{\frac{A}{\mu} - S - I - R} \\ &\quad - \frac{1}{2}\delta p^2 + M\frac{\Lambda^2}{\mu^3}[(p\vee 0) - \int_{-\infty}^{\infty}(p\vee 0)\kappa(x)dx] \\ &:= F(p,S,I,R) + +M\frac{\Lambda^2}{\mu^3}[(p\vee 0) - \int_{-\infty}^{\infty}(p\vee 0)\kappa(x)dx] \end{aligned} \tag{11}$$

**Step 2. (Construct a compact set):** Then, we construct a compact set$\mathbb{D} \in \Gamma$as follows

$$\mathbb{D}_\varepsilon = \left\{ (p,S,I,R) \in \Gamma_1 | S \geq \varepsilon, I \geq \varepsilon, R \geq \varepsilon^2, S+I+R \leq \frac{A}{d} - \varepsilon^3, |p| \leq \frac{1}{\varepsilon} \right\}, \tag{12}$$

then,

$$F(m,S,I,R) \begin{cases} -2+(M+\frac{\Lambda}{\mu^2})\bar{\beta}\frac{\Lambda}{\mu} - \frac{\Lambda}{S} \to -\infty, \text{as} S \to 0^+, \\ -2+(M+\frac{\Lambda}{\mu^2})\bar{\beta} I \to -2, \text{as} I \to 0^+, \\ -2+(M+\frac{\Lambda}{\mu^2})\bar{\beta}\frac{\Lambda}{\mu} - k\frac{I}{R} \to -\infty, \text{as} R \to 0^+, \\ -2+(M+\frac{\Lambda}{\mu^2})\bar{\beta}\frac{\Lambda}{\mu} - \frac{\alpha I}{\frac{\Lambda}{\mu} - S - I - R} \to -\infty, \text{as}(S,I,R) \to +\infty, \\ -2+(M+\frac{\Lambda}{\mu^2})\bar{\beta}\frac{\Lambda}{\mu} - \frac{1}{2}\delta p^2 \to -\infty, \text{as} p \to 0^+ \text{or} + \infty. \end{cases} \tag{13}$$

Clearly, we obtain that for a sufficient small$\varepsilon$ ,$F(p,S,I,R) \leq -1$ and$F(p,S,I,R) \leq A$ for any $(m,S,I,R) \in \mathbb{D}$.

**Step 3. (Existence and ergodicity):** Making the use of Fatou's lemma [7][8][9]and the new method [3], it is easy to get

$$\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau, (p(\tau), S(\tau), I(\tau), R(\tau)) \mathbb{D}_\varepsilon) d\tau \geq \frac{1}{A+1} > 0 \quad \text{a.s.} \tag{14}$$

This completes the proof.

## 4. Density function of the stochastic model

In the section let $X_1 = p - p^*, X_2 = S - S^*, X_3 = I - I^*, X_4 = R - R^*$.The linearized system is as follows:

$$\begin{cases} dX_1 = -\delta X_1 dt + \sigma dB(t), \\ dX_2 = (-a_{21}X_1 - a_{22}X_2 - a_{23}X_3)\, dt, \\ dX_3 = (a_{21}X_1 + a_{32}X_2 - a_{33}X_3 + a_{34}X_3)\, dt, \\ dX_4 = (a_{43}X_3 - a_{44}X_4)\, dt. \end{cases} \tag{15}$$

While $a_{21} = S^*I^*, a_{22} = \beta^*I^* + \mu, a_{23} = \beta^*S^*, a_{32} = \beta^*I^*, a_{33} = (\alpha + k + \mu) - \beta^*S^*, a_{34} = \gamma, a_{43} = k, a_{44} = \mu + \gamma$.

**Theorem 4.1.** *Let $(X_1, X_2, X_3, X_4)$ be the solution of the system (3) with any initial value $(X_1(0), X_2(0), X_3(0), X_4(0)) \in \mathbb{R} \times \mathbb{R}^4_+$ . If $R^s_0 > 1$ ,then there is a local normal density function $\Phi(X_1, X_2, X_3, X_4)$ and $\Sigma$ is satisfy the following function:*

$$\Phi(X_1, X_2, X_3, X_4) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(X_1, X_2, X_3, X_4)\Sigma^{-1}(X_1, X_2, X_3, X_4)^T}. \tag{16}$$

*While*

$$\Sigma = (a_{21}a_{43}C_2)^2 J_1^{-1}J_2^{-1}J_3^{-1}J_4^{-1}\Sigma_1 \left(J_1^{-1}J_2^{-1}J_3^{-1}J_4^{-1}\right)^T, \tag{17}$$

$$\begin{aligned}
&C_0 = a_{32}+a_{23}+a_{33}-a_{22}, C_1 = C_0-(a_{23}+a_{33}), C_2 = -\left(\tfrac{C_0C_1+a_{44}C_0}{a_{43}}\right) \neq 0,\\
&C_3 = -C_0C_2a_{23}+C_2(C_0+a_{44})^2-C_1C_2(C_0+a_{44})+C_2(C_2a_{43}+C_1^2),\\
&C_4 = C_2a_{43}a_{23}+[C_1C_2-C_2(C_0+a_{44})]a_{43}+C_1(C_2a_{43}+C_1^2),
\end{aligned}$$

*and*

$$J_1 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&0&0&1 \end{pmatrix}, J_2 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, J_3 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&\frac{C_1}{a_{43}}&1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} C_2a_{21}a_{43} & a_{43}[C_1C_2-C_2(C_0+a_{44})]-C_2a_{43}(a_{23}-a_{22}) & C_3 & C_4\\ 0 & C_2a_{43} & C_1C_2-C_2(C_0+a_{44}) & C_2a_{43}+C_1^2\\ 0&0&C_2&C_1\\ 0&0&0&1 \end{pmatrix},$$

$$\Sigma_1 = \begin{pmatrix} \frac{d_2d_3-d_1d_4}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} & 0 & -\frac{d_3}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} & 0\\ 0 & \frac{d_3}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} & 0 & -\frac{d_1}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)}\\ -\frac{d_3}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} & 0 & \frac{d_1}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} & 0\\ 0 & -\frac{d_1}{2\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} & 0 & \frac{d_1d_2-d_3}{2d_4\left(d_1d_2d_3-d_3^2-d_1^2d_4\right)} \end{pmatrix}.$$

*The parameters in* $\Sigma_1$ *are as follows:*

$$\begin{aligned}
&d_1 = \delta+a_{22}+a_{33}+a_{44},\\
&d_2 = a_{23}a_{32}+a_{22}a_{44}+a_{22}a_{33}+a_{33}a_{44}+(a_{22}+a_{33}+a_{44})\delta-a_{34}a_{43},\\
&d_3 = \delta(a_{23}a_{32}+a_{22}a_{44}+a_{22}a_{33}+a_{33}a_{44}-a_{34}a_{43})+a_{23}a_{32}a_{44}-a_{34}a_{43}a_{22},\\
&d_4 = \delta(a_{23}a_{32}a_{44}-a_{34}a_{43}a_{22}).
\end{aligned}$$

*Proof.* For system (15), it also has the from:

$$dX = AXdt+ZdB(t), \tag{18}$$

while $X = (X_1,X_2,X_3,X_4)^T$, $Z = diag(\sigma,0,0,0)$,

$$A = \begin{pmatrix} -\delta&0&0&0\\ -a_{21}&-a_{22}&-a_{23}&a_{24}\\ a_{21}&a_{32}&-a_{33}&0\\ 0&0&-a_{43}&-a_{44} \end{pmatrix}. \tag{19}$$

and we consider the following equation:

$$\varphi(\lambda) = \lambda^4+d_1\lambda^3+d_2\lambda^2+d_3\lambda+d_4, \tag{20}$$

where

$$\begin{aligned}
d_1 &= \delta + a_{22} + a_{33} + a_{44},\\
d_2 &= a_{23}a_{32} + a_{22}a_{44} + a_{22}a_{33} + a_{33}a_{44} + (a_{22} + a_{33} + a_{44})\delta - a_{34}a_{43},\\
d_3 &= \delta(a_{23}a_{32} + a_{22}a_{44} + a_{22}a_{33} + a_{33}a_{44} - a_{34}a_{43}) + a_{23}a_{32}a_{44} - a_{34}a_{43}a_{22},\\
d_4 &= \delta(a_{23}a_{32}a_{44} - a_{34}a_{43}a_{22}).
\end{aligned}$$

According to the RouthCHurwitz criterion, we obtain that $d_1d_2 - d_3 > 0, d_1d_2d_3 - d_3^2 - d_1^2 d_4 > 0$. And system (3) has a unique probability density function, according to literature[10][11] and it satisfies the following FokkerCPlanck equation:

$$\begin{aligned}
&-\frac{\sigma^2}{2}\frac{\partial^2}{\partial X_1^2}\Phi + \frac{\partial}{\partial X_1}(-\delta X_1\Phi) + \frac{\partial}{\partial X_2}\left[(-a_{21}X_1 - a_{22}X_2 - a_{23}X_3)\Phi\right]\\
&+\frac{\partial}{\partial X_3}\left[(a_{21}X_1 + a_{32}X_2 - a_{33}X_3 - a_{34}X_4)\Phi\right] + \frac{\partial}{\partial X_4}\left[(a_{43}X_3 - a_{44}X_4)\Phi\right] = 0,
\end{aligned} \tag{21}$$

and

$$\Phi(x) = l_0 e^{-\frac{1}{2}XQX^T}. \tag{22}$$

And $l_0$ is a positive constant and $\int_{\mathbb{R}_+^3}\Phi(x)dX = 1$. And $B$ need accord with the following equation:

$$BZ^2B + A^TB + BA = 0. \tag{23}$$

here $B$ is positive definite, let $\Sigma = B^{-1}$, then

$$Z^2 + A\Sigma + \Sigma A^T = 0. \tag{24}$$

by calculation, $A_1 = J_4J_3J_2J_1AJ_1^{-1}J_2^{-1}J_3^{-1}J_4^{-1}$, Thus Eq. (24) can be expressed as follows:

$$J_4J_3J_2J_1Z^2J_1^TJ_2^TJ_3^TJ_4^T + A_1J_4J_3J_2J_1\Sigma J_1^TJ_2^TJ_3^TJ_4^T + J_4J_3J_2J_1\Sigma J_1^TJ_2^TJ_3^TJ_4^TA_1^T = 0. \tag{25}$$

Through calculating,

$$J_4J_3J_2J_1Z^2J_1^TJ_2^TJ_3^TJ_4^T = (C_2a_{21}a_{43})^2Z^2 \text{ and } J_4J_3J_2J_1\Sigma J_1^TJ_2^TJ_3^TJ_4^T = (C_2a_{21}a_{43})^2\Sigma_1, \tag{26}$$

where

$$\Sigma_1 = \begin{pmatrix}
\frac{d_2d_3-d_1d_4}{2(d_1d_2d_3-d_3^2-d_1^2d_4)} & 0 & -\frac{d_3}{2(d_1d_2d_3-d_3^2-d_1^2d_4)} & 0\\
0 & \frac{d_3}{2(d_1d_2d_3-d_3^2-d_1^2d_4)} & 0 & -\frac{d_1}{2(d_1d_2d_3-d_3^2-d_1^2d_4)}\\
-\frac{d_3}{2(d_1d_2d_3-d_3^2-d_1^2d_4)} & 0 & \frac{d_1}{2(d_1d_2d_3-d_3^2-d_1^2d_4)} & 0\\
0 & -\frac{d_1}{2(d_1d_2d_3-d_3^2-d_1^2d_4)} & 0 & \frac{d_1d_2-d_3}{2d_4(d_1d_2d_3-d_3^2-d_1^2d_4)}
\end{pmatrix}.$$

In the end, we use the Lemma 4.2 of [12], and $\Sigma_1$ is a positive definite matrix, thus this completes the proof.

## 5. Conclusion

Curz Vargas et al studied a deterministic model,but it does not take into account the influence of environmental noise on system parameters, in the paper, we consider that the important parameters are disturbed by Ornstein-Uhlenbeck process, which is more reasonable. We study a stochastic model and consider the disease transmission coefficient is affect by Ornstein-Uhlenbeck process. We prove some dynamic behaviors like solution of the global, and we get the condition of stationary distribution. Compared with this deterministic model, the dynamical analysis of the system is more reasonable.

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