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Ecosystem Stability Analysis and Numerical Simulation via Three Improved Lotka-Volterra Models

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Abstract. This study analyzed the stability of predator-prey ecosystem (including the sensitivity of the initial value and coefficient) via the original Lotka-Volterra model and three improved models. The first improved model considered the impact of hunting and internal competition on the ecosystem, the second introduced the second prey to the original model, while the third added the second-type predator and the second-type prey to the original system. The performed numerical simulation of the original and three improved models proved instability of the ecosystem represented by the original Lotka-Volterra model and the stability of those represented by all three improved models.

Keywords. Lotka-Volterra model, predator-prey ecosystem, dynamic system, stability analysis

1. INTRODUCTION

The Lotka-Volterra (LV) model, introduced by Lotka in 1925 and later refined by Volterra, has been originally focused on simulating the predator-prey relationship between species [1]. This model was found applicable to enterprise competition, coexistence, and competition among marine organisms, etc. Therefore, it found application in competition analysis research, such as species competition research in ecosystems and competition between enterprises or industries [2-3].

While the original LV model comprised one type of prey and one type of predator, it also has some drawbacks that do not consider the inhibitory effect of limited resources and environment on population growth, its applicability to multiprey-single predator, single prey-multipredator, and multiprey-multipredator systems is problematic due to a lack of accurate theoretical systems and numerical simulations of application scenarios. Thus, the theoretical data and the real data are far from different. Therefore, the LV model needs to be adapted to the above ecosystems, which typically have more than two species and feature some diversity [4-5].

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This study introduces three improved LV models simulating (i) multiprey-predator, (ii) prey-multipredator, and (iii) multiprey-multipredator ecosystems.

2. RELATED WORKS

The LV model can simulate scenarios involving predation and competition. Regarding predation, Liu et al. conducted a dynamic analysis of a stage-structured predator-prey model with harvesting power and multiple time delays. They assumed that multiple time delays were introduced into the model system due to the delayed maturation of the prey population and the delayed pregnancy of the predator population [6].

Many researchers have applied the LV model to competitive analysis. Thus, Han et al. used the LV model to study the cooperation between enterprises to resist market competition, simulating various situations, including the market equilibrium obtained through control conditions [7]. Zhang applied a recurrent neural network (RNN) to realize the competitive layer model (CLM) and proposed implementing the LV RNN. The contributions of this paper included establishing the necessary and sufficient conditions for the minimum point of the CLM energy function, verifying the convergence of the LV RNN model, and proving that the stable attractor subset of the LV RNN was precisely equal to the minimum point set of the CLM energy function in the nonnegative direction [8]. Guharay and Chang used a Bayesian simulation algorithm to study whether competition between living species can be detected statistically. Their study (mainly aimed at mammals, plants, birds, marine organisms, and reptiles) proved that birds valued habitats more than marine organisms and assessed the competitive process in detail [9]. Chen and Lin established a mathematical model of the interaction and coexistence mechanisms of two competitive ecosystems with nonlinear intraspecific regulation. They discussed the joint dynamic effects of ripening delay and harvesting effort on population dynamics and the local stability of the model near the internal balance by analyzing the relevant characteristic equations [10].

Several researchers have attempted to improve the LV model. Liu et al. proposed an improved model with a logistic blocking effect. Based on the analysis of the positive equilibrium point's local stability, the improved system's global stability criterion was proposed for predator-prey systems [11]. Hung et al. developed a sales forecasting model to analyze the interaction between convenience and budget retail formats. This improved model mitigated the impact of self-growth, internal competition, and external competition among species neglected in the traditional LV model, making the model more consistent with the actual conditions of the sales market [12]. Mao et al. established the Grey LV model to quantitatively analyse and predict the impact of online payment systems of Chinese commercial banks on the development of third-party online payment systems. In the modeling process, the continuous model is discretized using gray theory, and the model parameters are estimated using the least squares method [13]. Ray and Chaudhuri studied the system by considering the changes in biological parameters and the time cycle of the harvesting effort [14]. Ma et al. used the LV model to perform voltage stability analysis of power systems based on bifurcation theory [15]. Chiang et al. discussed the innovative growth of Taiwan's personal computer (PC) shipments, analyzing the dynamic competitive relationship between PC categories using the LV model to explain the actual diffusion phenomenon in the competitive market [16]. Luo et al. studied the collinear competition between urban rail and bus public transport based on the population competition model. They used the ecological theory and methods to assess the collinear competition various between public transport modes [17]. Manjunath and Raina performed the stability and bifurcation analysis of the Lotka-Volterra time-delayed system [18]. Gao considered the LV model of three species with discrete delay, studied the stability of positive equilibrium and the existence of Hopf bifurcation, and then obtained the direction and stability criteria of the periodic bifurcation solution using the normal form theory and the central manifold theorem [19]. Wang and Gui studied the existence of periodic solutions for a class of three-species periodic LV predator-prey systems with impulsive perturbations and obtained sufficient and realistic conditions by using the Mawhin extension theorem of the degree of coincidence [20]. El Arabi et al. performed the numerical simulation of the SIR and the Lotka-Volterra models used to describe the dynamics of predator-prey biological systems, the evolution of an infectious disease in any population, and bacterial growth in a given environment. These models formed a system of non-linear and coupled equations, which required special numerical processing [21]. Bouaine and Rachik established an improved spatiotemporal discrete LV model, including different species occupying a three-dimensional area and obeying any complex marine food chain. The method's effectiveness was validated by numerical simulation [22].

3. ORIGINAL MODEL

Before establishing the model, it was assumed that the resources in this ecosystem were infinite and could accommodate an unlimited number of living creatures. According to the above assumptions, it was concluded that the growth rate of prey continuously increased and was affected by its number. With the increased number of prey, the growth rate gradually accelerated, and the prey growth model could be expressed as follows:

$$\frac{dx}{dt} = \alpha x, \alpha > 0 \tag{1}$$

After the introduction of predators, the number of prey will decrease with the number of predators, which will slow down the growth rate of predators. Therefore, the growth rate model of predators can be described as

$$\frac{dy}{dt} = -\beta y, \beta > 0 \tag{2}$$

The growth rate of prey will be affected by the number of predators. It will slow down with increasing predation intensity, and the growth rate of predators will increase with increasing predation intensity. Combining Eqs. (1) and (2) yields the original LV prey-predator model:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \lambda xy = (\alpha - \lambda y)x \\ \frac{dy}{dt} = -\beta y + \gamma xy = (-\beta + \gamma x)y \end{cases} \tag{3}$$

The notations involved in the original LV model are listed in Table 1.

Table 1. Notations in the original LV model.

Notations	Meaning
$x(t)$	number of the first-type predators
$y(t)$	number of the first-type preys
α	growth rate of preys without predators
β	Mortality of predators without preys
λ	predation parameter
γ	γ/λ is the conversion factor

4. IMPROVED MODELS

4.1 The First Improved Model

The first improved model considers artificial fishing, hunting, and internal competition on the original LV model. In the ecosystem, the internal competition between hunting species will cause the growth rate of this species to decrease. This improved model can be adapted to the above conditions as follows:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \lambda xy - r_1 x - r_3 x^2 \\ \frac{dy}{dt} = -\beta y + \gamma xy - r_2 y - r_4 y^2 \end{cases} \tag{4}$$

The notations in the first improved model are listed in Table 2.

Table 2. Notations in the first improved LV model.

Parameter	Meaning
r_1	hunting coefficient of prey
r_2	hunting coefficient of predator
r_3	internal competition coefficient of prey
r_4	internal competition coefficient of predator

4.2 The Second Improved Model

The second improved model considers that there is one predator and two types of prey in an ecosystem, and competition exists between the two types of prey, assuming that there is no fishing in this model. The competition between two types of prey will slow down their growth. According to the above conditions, an improved model of two types of prey and one type of predator can be obtained, as shown in the following equation set:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \lambda xy - \lambda_2 xp - r_3 x^2 \\ \frac{dp}{dt} = \mu p - \lambda_3 px - \lambda_1 py - r_5 p^2 \\ \frac{dy}{dt} = -\beta y + \gamma xy + \gamma_1 py - r_4 y^2 \end{cases} \tag{5}$$

The notations in the second improved model are listed in Table 3.

Table 3. Notations in the second improved LV model.

Parameter	Meaning
p(t)	number of the second-type prey
λ_1	predation parameter
λ_2	competition coefficient between prey
λ_3	internal competition coefficient of prey
γ_1	γ_1/λ_3 is the conversion factor
r_5	internal competition coefficient of prey

4.3 The Third Improved Model

The third improved model covers a system of two types of predators and two types of prey. Compared to the second improved model, one more type of predator is introduced. Both types of predators will hunt on both types of prey, so there will be competition between predators, reducing the growth of predators' population. At the same time, the internal competition between the two types of predators will also inhibit their population growth. A model can be established according to the above conditions, as follows:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \lambda xy - \lambda_2 xp - \lambda_4 xq - r_3 x^2 \\ \frac{dp}{dt} = \mu p - \lambda_3 px - \lambda_1 py - \lambda_5 pq - r_5 p^2 \\ \frac{dy}{dt} = -\beta y + \gamma xy + \gamma_1 py - \gamma_4 yq - r_4 y^2 \\ \frac{dq}{dt} = -\delta q + \gamma_2 xq + \gamma_3 pq - \gamma_5 yq - r_6 q^2 \end{cases} \tag{6}$$

The notations in the third improved model are listed in Table 4.

Table 4. Notations in the third improved LV model.

Parameter	Meaning
q(t)	number of the second-type predators
λ_4 , λ_5	predation parameter
γ_2	γ_2/λ_4 is the conversion factor
γ_3	γ_3/λ_5 is the conversion factor
γ_4 ,γ_5	competition coefficients between predators
r_6	internal competition coefficient of predator

5 NUMERICAL SIMULATION

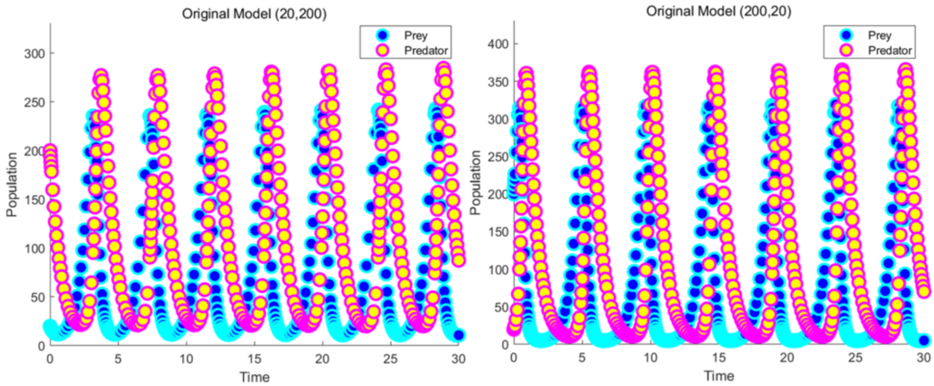
The numerical simulation via the original and three proposed improved LV models was performed by choosing appropriate coefficients to verify the respective ecosystems' stability.

5.1 Original LV Model

Only one predator and one prey are considered in the original model, and there is no internal fishing competition. For $\alpha = 2, \beta = 1.5, \lambda = 0.02, \gamma = 0.02$, the following set is obtained.

$$\begin{cases} \frac{dx}{dt} = 2x - 0.02xy \\ \frac{dy}{dt} = -1.5y + 0.02xy \end{cases} \tag{7}$$

To test whether the model is sensitive to the initial values of predators and prey, we set two groups of initial values, $(x_0, y_0) = (20,200)$ and $(x_0, y_0) = (200,20)$, respectively. According to the model and initial values, we can obtain the quantitative change of predators and prey, as shown in Fig. 1 (a) and (b).



(a) (b)

Fig. 1. The original LV model: (a) x=20; y=200; (b) x=200; y=20.

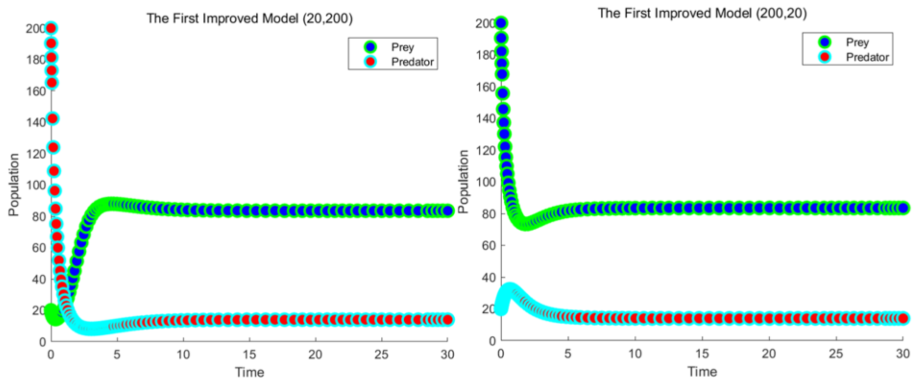
According to the results in Fig. 1, under the two initial conditions of $x_0 > y_0$ and $x_0 < y_0$, the changes in the populations of prey and predators present an oscillating state, indicating that the ecosystem represented by the original model is unstable.

5.2 The First Improved LV Model

For the first improved model, only one type of predator and one type of prey are considered, and internal competition and hunting (fishing) coexist. For $\alpha = 2, \beta = 1.5, \lambda = 0.02, \gamma = 0.02, r_1 = 0.05, r_2 = 0.03, r_3 = 0.02, r_4 = 0.01$, the following set is obtained.

$$\begin{cases} \frac{dx}{dt} = 2x - 0.02xy - 0.05x - 0.02x^2 \\ \frac{dy}{dt} = -1.5y + 0.02xy - 0.03y - 0.01y^2 \end{cases} \tag{8}$$

The initial values are selected as $(x_0, y_0) = (20,200)$ and $(x_0, y_0) = (200,20)$. The numerical simulation results are shown in Fig. 2.



(a) (b)

Fig. 2. The first improved LV model: (a) x=20; y=200; (b) x=200; y=20.

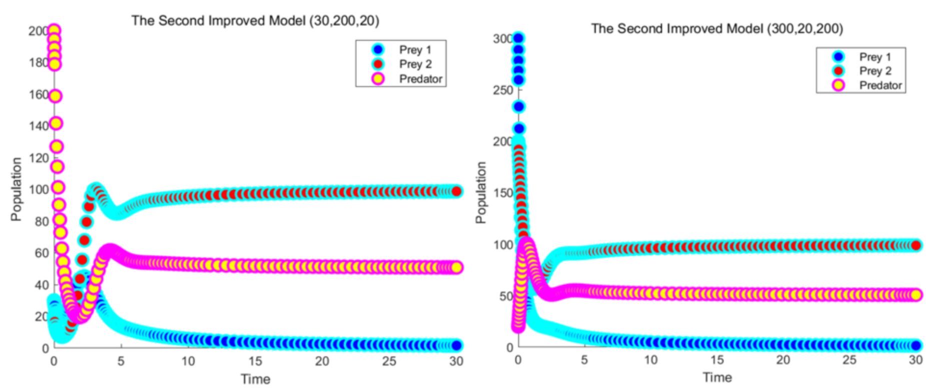
According to the results in Fig. 2, for the first improved model considering hunting and internal competition, under the two initial conditions of $x_0 > y_0$ and $x_0 < y_0$, the population of predator and prey gradually tends to be stable without oscillation, indicating that the ecosystem represented by the first improved model is stable.

5.3 The Second Improved LV Model

One type of predator and two types of prey are considered for the second improved model, and internal competition exists. Taking the parameters $\alpha = 2, \beta = 1.5, \mu = 3, \lambda = 0.02, \lambda_1 = 0.01, \lambda_2 = 0.01, \lambda_3 = 0.02, \gamma = 0.02, \gamma_1 = 0.02, r_3 = 0.02, r_4 = 0.01$, and $r_5 = 0.02$, the following equation set can be obtained.

$$\begin{cases} \frac{dx}{dt} = 2x - 0.02xy - 0.01xp - 0.02x^2 \\ \frac{dp}{dt} = 3p - 0.01px - 0.02py - 0.02p^2 \\ \frac{dy}{dt} = -1.5y + 0.02xy + 0.02py - 0.01y^2 \end{cases} \quad (9)$$

In the numerical simulation of the second improved model, $(x_0, y_0, p_0) = (30,200,20)$ and $(x_0, y_0, p_0) = (300,20,200)$ are selected for the initial value sensitivity test. The numerical simulation results are shown in Fig. 3.



(a) (b)

Fig. 3. The second improved LV model: (a) x=30; y=200; p=20; (b) x=300; y=20; p=200.

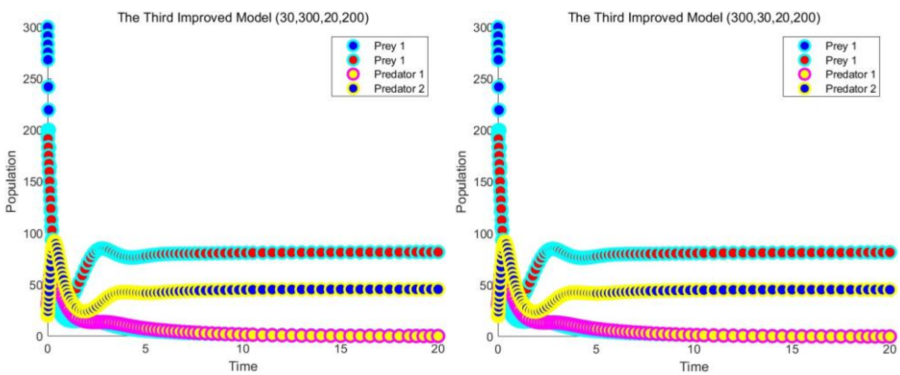
According to the results in Fig. 3, for the second improved model considering hunting (fishing) and internal competition, under the two initial conditions of $x_0 > y_0, p_0 > y_0$ and $x_0 < y_0, p_0 < y_0$, the population of predators and two types of prey gradually tend to be stable without oscillation, indicating that the ecosystem represented by the second improved model is stable.

5.4 The Third Improved LV Model

Two types of predators and two types of prey are considered in the third model, and internal competition exists. Take $\alpha = 2, \beta = 1.5, \mu = 3, \lambda = 0.02, \lambda_1 = 0.01, \lambda_2 = 0.01, \lambda_3 = 0.02,\ \gamma = 0.02,\ \gamma_1 = 0.02, r_3 = 0.02,\ \gamma_4 = 0.01, \gamma_5 = 0.01,\ r_4 = 0.01, r_5 = 0.02, r_6 = 0.01$. The equation set (10) can be obtained.

$$\begin{cases} \frac{dx}{dt} = 2x - 0.02xy - 0.01xp - 0.03xq - 0.02x^2 \\ \frac{dp}{dt} = 3p - 0.02px - 0.02py - 0.03pq - 0.02p^2 \\ \frac{dy}{dt} = -1.5y + 0.02xy + 0.02py - 0.01yq - 0.01y^2 \\ \frac{dq}{dt} = -2q + 0.03xq + 0.03pq - 0.01yq - 0.01q^2 \end{cases} \tag{10}$$

The initial values $(x_0, y_0, p_0, q_0) = (30,300,20,200)$, and $(x_0, y_0, p_0, q_0) = (300,30,20,200)$ were selected for the initial value sensitivity test. The numerical simulation results are shown in Fig. 4.



(a) (b)

Fig. 4. The third improved LV model: (a) x=30; y=300; p=20; q =200; (b) x=300; y=30; p=20; q =20.

According to the results in Fig. 4, for the third improved LV model considering hunting and internal competition, under the two initial conditions of $x_0 > y_0, p_0 > y_0, x_0 > q_0, y_0 > q_0$ and $x_0 < y_0, p_0 < y_0, x_0 < q_0, y_0 < q_0$, the population of two types of predators and two types of prey gradually tends to be stable without oscillation, indicating that the respective ecosystem is stable.

6 Equilibrium Points and Stability Analysis

After the model's establishment, the system's equilibrium point can be obtained according to the model, and the stability of the equilibrium point can be judged [21]. In solving the equilibrium point, we must set each equation to zero. Taking the original model as an example, the calculation process is expressed as follows:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \lambda xy = (\alpha - \lambda y)x = 0 \\ \frac{dy}{dt} = -\beta y + \gamma xy = (-\beta + \gamma x)y = 0 \end{cases} \tag{11}$$

According to formula (7), two equilibrium points (0,0) and $(\frac{\beta}{\gamma}, \frac{\alpha}{\lambda})$ can be obtained.

Then, let $U = \frac{dx}{dt}, V = \frac{dy}{dt}$ and calculate the Jacobian row matrix of the system, via Eq. (12).

$$\mathcal{F} = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha - \lambda y & -\lambda x \\ \gamma y & -\beta + \gamma x \end{pmatrix} \tag{12}$$

After obtaining the Jacobian matrix, it is necessary to check whether the equilibrium solution is stable. The value of ω can be used as the judgment basis (see [22]). The characteristic equation has the following form:

$$|\omega I - \mathcal{F}| = \begin{vmatrix} \omega - \alpha + \lambda y & \lambda x \\ -\gamma y & \omega + \beta - \gamma x \end{vmatrix} \tag{13}$$

6.1 Equilibrium Points

The equilibrium points of the original model are $(0,0)$ and $(75,100)$. The equilibrium points of the first improved model are $(0,0), (0,-153), (83.5,14)$, and $(97.5,0)$. The equilibrium points of the original model are $(0,0,0), (100,0,0), (0,-150,0), (0,0,150), (0,50,100), (25,0,150)$ and $\left(\frac{3375}{43}, \frac{800}{43}, \frac{250}{43}\right)$. The equilibrium points of the third improved LV model are listed in Table 5 (all negative equilibrium points are meaningless in the present work) [18].

Table 5. Equilibrium Points of the Third Improved Model

variable	E_1	E_2	E_3	E_4	E_5	E_6	E_7
x	0	100	0	0	0	25	0
y	0	0	-150	0	50	0	0
p	0	0	0	150	100	150	0
q	0	0	0	0	0	0	-200
variable	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}
x	0	800/11	0	1375/19	-200/30	$30\sqrt{6} + 70$	$70 - 30\sqrt{6}$
y	-350	0	0	400/19	0	$30\sqrt{6} - 230$	$-30\sqrt{6} - 230$
p	50	0	900/11	250/19	2700/31	$-30\sqrt{6} - 20$	$30\sqrt{6} - 20$
q	300	200/11	500/11	0	1300/31	$180 - 30\sqrt{6}$	$30\sqrt{6} + 180$

6.2 Stability Analysis

This paper mainly analyses the stability about the two equilibrium points of the original LV model and $\left(\frac{3375}{43},\frac{800}{43},\frac{250}{43}\right)$ of the second improved LV model. The calculation process of the equilibrium point $(0,0)$ is expressed in Eqs. (14) and (15):

$$\mathcal{F}(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1.5 \end{pmatrix} \tag{14}$$

$$|\omega I - \mathcal{F}(0,0)| = \begin{vmatrix} \omega - 2 & 0 \\ 0 & \omega + 1.5 \end{vmatrix} = 0 \tag{15}$$

According to formula (15), we can obtain $\omega_1 = -1.5, \omega_2 = 2$. Therefore, (0,0) is a saddle point and not a stable equilibrium point.

The calculation process of the equilibrium point (75,100) is given in Eqs. (16) and (17).

$$\mathcal{F}(75,100) = \begin{pmatrix} \alpha - 100\lambda & -75\lambda \\ 100\gamma & -\beta + 75\gamma \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 2 & 0 \end{pmatrix} \tag{16}$$

$$|\omega I - \mathcal{F}(75,100)| = \begin{vmatrix} \omega & 1.5 \\ -2 & \omega \end{vmatrix} = 0 \tag{17}$$

According to formula (17), we can obtain $\omega_1 = -\sqrt{3}i, \omega_2 = \sqrt{3}i$. Therefore, $(75,100)$ is a central point and not a stable equilibrium point.

The equilibrium point $\left(\frac{3375}{43},\frac{250}{43},\frac{800}{43}\right)$ is calculated in Eq. (18) and Eq. (19).

$$\mathcal{F}\left(\frac{3375}{43},\frac{250}{43},\frac{800}{43}\right) = \begin{pmatrix} -1.57 & -0.78 & -1.57 \\ -0.06 & 1.61 & 0.12 \\ 0.37 & 0.37 & -0.19 \end{pmatrix} \tag{18}$$

$$\left|\omega I - \mathcal{F}\left(\frac{3375}{43},\frac{250}{43},\frac{800}{43}\right)\right| = \begin{vmatrix} \omega + 1.57 & 0.78 & 1.57 \\ 0.06 & \omega - 1.61 & -0.12 \\ -0.37 & -0.37 & \omega + 0.19 \end{vmatrix} = \omega^3 + 0.15\omega^2 - 2.0456\omega - 1.494 \tag{19}$$

According to formulas (18) and (19), we get $\omega_1 = 1.65, \omega_2 = -0.90 - 0.31i, \omega_3 = -0.90 + 0.31i$.

Therefore, $\left(\frac{3375}{43},\frac{250}{43},\frac{800}{43}\right)$ is a stable equilibrium point.

7 CONCLUSIONS

This study aimed to extend the Lotka-Volterra model to ecosystems containing more than one type of predator and prey, as well as internal competition. The first improved model considered hunting and internal competition, the second one considered two types of prey and one type of predator, and the third one covered two types of prey and two types of predators. The performed numerical computation proved that appropriate human activities play a role in maintaining the stability of the ecosystem, while biological diversity is conducive to improving this stability. From Fig. 2 to 4 and numerical simulations, the three improved LV models tend to stabilize their respective ecosystems. In future research, the LV model will be further modified to consider climate and terrain images in the stability analysis of biological systems. At the same time, in model verification and solution, actual data will be collected for their numerical simulation.

REFERENCES

[1] A. J. Lotka. Elements of Physical Biology, Williams and Wilkins, Baltimore, 1925: 168-177.

[2] J. Chattopadhyay and O. Arino (1999). A predator-prey model with disease in the prey. Nonlinear Analysis: Theory, Methods & Application, 36(6), 747-766.

[3] L.Yu and S. Bin (2022). Research on the competition between high-speed rail and air express transportation based on the Lotka-Volterra model. Railway Transportation and Economy, 2022, 44 (03): 73-78+91. (in Chinese).

[4] N. L. Christensen, A.M. Bartuska, J. H. Brown, et al. (1996). The report of the Ecological Society of America Committee on the scientific basis for ecosystem management. Ecological Applications, 6(3), 665-691.

[5] D. Qiu, X. Ma, J. Yan, et al. (2020). Biogeomorphological processes and structures facilitate seedling establishment and distribution of annual plants: implications for coastal restoration. Science of The Total Environment, 143842.

[6] C. Liu, Q. Zhang, Y. Li, and W. Yue (2014). Dynamical analysis of a stage-structured prey-predator model with harvest effort and multiple time delays. Proc. 33rd Chinese Control Conference.

[7] R. W. Han et al. (2021). Research on the cooperation between large and medium-sized enterprises based on the Lotka-Volterra model. Science and Technology Economic Market, 2021 (07): 99-102. (in Chinese).

[8] Z. Yi, (2010). Foundations of implementing the competitive layer model by the Lotka-Volterra Recurrent Neural Networks, in IEEE Transactions on Neural Networks, vol. 21, no. 3, pp. 494-507, March 2010.

[9] S. Guharay and K. Chang (2015). Application of Bayesian simulation framework in quantitatively measuring presence of competition in living species, In Proc. 2015 Winter Simulation Conference (WSC), 2015, pp. 4033-4044.

[10] L. L. Chen and Z. Z. Lin (2008). The effect of habitat destruction on the multispecies competition system with allee effects, In Proc. 2008 Fourth International Conference on Natural Computation, 2008, pp. 628-632, doi: 10.1109/ICNC.2008.740.

[11] W. Liu, C. Fu, F. Zeng, and T. Xuan (2014). Stability analysis for an improved Volterra predator-prey model, Proc. 4th IEEE International Conference on Information Science and Technology, 2014, pp. 345-348.

[12] H. C. Hung, Y.C. Chiu, H.C. Huang, M. C. Wu (2017). An enhanced application of Lotka-Volterra model to forecast the sales of two competing retail formats. Computers & Industrial Engineering, 109(), 325-334.

[13] S. Mao, M. Zhu, X. Wang, and X. Xiao (2020). Grey Lotka-Volterra model for the competition and cooperation between third-party online payment systems and online banking in China. Applied Soft Computing, 106501.

[14] S. Saha Ray and K. S. Chaudhuri (1989). Lotka-Volterra prey-predator model with harvesting and environmental perturbations. Ecological Modelling, 47(3-4), 283-290. doi:10.1016/0304-3800(89)90006-9.

[15] Y. Ma, S. Lv, X. Zhou, and Z. Gao (2018). Voltage stability analysis of power system based on bifurcation theory. 2018 IEEE International Conference on Mechatronics and Automation (ICMA).

[16] S. Y. Chiang, G. G. Wong, Y. Li, and H. C. Yu (2008). A dynamic competition analysis on the personal computer shipments in Taiwan using the Lotka-Volterra model. 2008 IEEE Asia-Pacific Services Computing Conference, Yilan, Taiwan, 2008, pp. 1412-1417, doi: 10.1109/APSCC.2008.296.

[17] Q. Y. Luo, B. F. Sun, and Y. P. Zhong (2010). Collinear competition between rail transit and bus transit based on population competition model, 2010 WASE International Conference on Information Engineering, Beidai, China, 2010, pp. 313-315, doi: 10.1109/ICIE.2010.251.

[18] S. Manjunath and G. Raina (2014) Stability and bifurcation analysis of the Lotka–Volterra time delayed system, The 26th Chinese Control and Decision Conference (2014 CCDC), Changsha, China, 2014, pp. 2056-2063, doi: 10.1109/CCDC.2014.6852506.

[19] Q. Gao (2009). Stability and Hopf bifurcations of a three-species symbiosis model with delays, 2009 International Workshop on Chaos-Fractals Theories and Applications, Shenyang, China, 2009, pp. 272-276, doi: 10.1109/IWCFTA.2009.63.

[20] K. Wang and Z. Gui (2012). Existence and simulations of periodic solution of a three-species the Lotka-Volterra predator-prey system with impulsive perturbations, 2012 IEEE International Conference on Computer Science and Automation Engineering (CSAE), Zhangjiajie, China, 2012, pp. 179-183, doi: 10.1109/CSAE.2012.6272934.

[21] I. El Arabi, A. Chafi, and S. K. Alami (2019). Numerical simulation of the SIR and the Lotka-Volterra models used in biology, 2019 International Conference on Intelligent Systems and Advanced Computing Sciences (ISACS), Taza, Morocco, 2019, pp. 1-4, doi: 10.1109/ISACS48493.2019.9068876.

[22] A. Bouaine and M. Rachik (2017). Adjustment submersible cages for optimal fish farming: improved spatiotemporal Lotka-Volterra model for *n* species in partitioned area. International Journal of Dynamics and Control, 6(3), 919-928. doi:10.1007/s40435-017-0329-3.