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The Level Cardinality of Fuzzy Module Under $\mathbb{Z}$-Module Homomorphism on $\mathbb{Z}_n$ into $\mathbb{Z}_m$ Where $gcd(n,m)$ Is Product of Primes

Manju VARGHESE[b, a], Shery FERNANDEZ[c,1]
[a] *Research Scholar, Cochin University of Science and Techonology*
[b] *Assistant Professor, Mar Thoma College for Women, Perumbavoor*
[c] *Professor, Cochin University of Science and Techonology*

Abstract. The homomorphic image of a fuzzy module of an R-module is a fuzzy module[1]. In one of our papers [2] we have proved that if $f : \mathbb{Z}_n \to \mathbb{Z}_m$ is a $\mathbb{Z}$-module homomorphism with $gcd(n,m) = p$, a prime and λ is any fuzzy module on $\mathbb{Z}_n$ with any level cardinality then the fuzzy module $f(\lambda)$ on $\mathbb{Z}_m$ has level cardinality atmost 3. In this paper, we are considering the fuzzy module homomorphism between the $\mathbb{Z}$-modules $\mathbb{Z}_n$ and $\mathbb{Z}_m$ where $n,m \in \mathbb{Z}^+$ with $gcd(n,m) = pq$, where p and q are primes and trying to find the level cardinality of the fuzzy module $f(\lambda)$ on $\mathbb{Z}_m$ when λ is a fuzzy module on $\mathbb{Z}_n$.

Keywords. Fuzzy module, level submodule, Fuzzy module homomorphism

1. Introduction

The idea of fuzzy set on a nonempty set was first introduced by L A Zadeh [3] in 1965. He defined the fuzzy subset of a nonempty set X as a membership function $\lambda : X \to [0,1]$. In 1971 a milestone in the development of fuzzy group was laid by Rosenfeld [4]. The level set or a-cut [1] of a fuzzy set λ for $a \in [0,1]$ is defined as $\lambda_a = \{x/x \in X, \lambda(x) \geq a\}$. In 1975 Negoita and Ralescu [5] came up with the concept of fuzzy module. The module homomorphism is a mapping between modules which preserves the module structure. The image of a fuzzy module of an R-module is a fuzzy module under module homomorphism [1]. In our previous paper [2] we have studied the level cardinalities of image of fuzzy modules of $\mathbb{Z}$-module $\mathbb{Z}_n$, $n \in \mathbb{Z}$. Also the level cardinalites of image of fuzzy module under $\mathbb{Z}$-module homomorphism of $\mathbb{Z}_n$ into $\mathbb{Z}_m$ when $gcd(n,m) = p$, a prime and when $gcd(n,m) = 1$. Now we are checking the level cardinality of the image of the fuzzy module on $\mathbb{Z}_n$ where f is a homomorphism of $\mathbb{Z}_n$ into $\mathbb{Z}_m$ when $gcd(n,m) = pq$, p,q are primes.

[1]Corresponding Author: Shery Fernandez, Professor, Cochin University of Science and Techonology; E-mail: sheryfernandezaok@gmail.com.

2. Preliminaries

Definition 2.1. [6] Let R be a ring. A left R-module is a set M together with

1. a binary operation $+$ on M under which M is an abelian group, and
2. an action of R on M (that is, a map $R \times M \to M$) denoted by ax, for all $a \in R$ and for all $x \in M$ which satisfies

   (a) $(a+b)x = ax+bx$, for all $a,b \in R, x \in M$
   (b) $(ab)x = a(bx)$, for all $a,b \in R, x \in M$ and
   (c) $a(x+y) = ax+ay$, for all $a \in R, x,y \in M$
   If the ring R has a unity $'1'$ we impose the additional axiom:
   (d) $1.x = x$, for all $x \in M$

Definition 2.2. [7] Let R be a ring, M and N be R-modules. An R-module homomorphism from M to N is a map $f : M \to N$ which respects addition and scalar multiplication of these modules and satisfies the following axioms

1. $f(m+n) = f(m) + f(n)$ for all $m,n \in M$
2. $f(rm) = rf(m)$ for all $m \in M$ and $r \in R$

Definition 2.3. [8] Let R be a ring and let M be an R-module, then a fuzzy module on M is a map $\lambda : M \to [0,1]$ satisfying the following conditions

1. $\lambda(m_1+m_2) \geq \min\{\lambda(m_1), \lambda(m_2)\}, \ \forall m_1, m_2 \in M$
2. $\lambda(-m_1) = \lambda(m_1) \ \forall m_1 \in M$
3. $\lambda(rm_1) \geq \lambda(m_1) \ \forall m_1 \in M, \ r \in R$
4. $\lambda(0) = 1$

Definition 2.4. [9] Let μ and λ be two fuzzy modules of an R-module M, then λ is called a fuzzy submodule of μ if $\lambda \subseteq \mu$ (i.e $\lambda(m) \leq \mu(m) \ \forall m \in M$)

Definition 2.5. [1] Let f be a mapping from X into Y and let λ be a fuzzy subset on X then the fuzzy subset $f(\lambda)$ on Y is defined by $\forall y \in Y$,

$$f(\lambda)(y) = \begin{cases} \vee\{\lambda(x)/x \in M, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is called the image of λ under f, where $\vee$ denotes the maximum or supremum.

Theorem 2.6. *Let f be an $\mathbb{Z}$-module homomorphism of $\mathbb{Z}_n$ into $\mathbb{Z}_m$ where $n,m \in \mathbb{Z}$ then*

1. $f(0) = 0$.
2. $f(h) = hf(1)$ *for all $h \in \mathbb{Z}$ i.e, The module homomorphism is determined by the value of $f(1) \in \mathbb{Z}_m$.*
3. *order of $f(h) \in \mathbb{Z}_m$ divides order of h for all $h \in \mathbb{Z}_n$.*
4. *If H be a submodule of $\mathbb{Z}_n$ then $f(H)$ is a submodule of $\mathbb{Z}_m$.*

Theorem 2.7. *[2] The level cardinality of any fuzzy module of an $\mathbb{Z}$-module $\mathbb{Z}_n$ where $n = p_1^{r_1}.p_2^{r_2}\dots p_k^{r_k}$, $p_i's$ are distinct primes and $r = r_1+r_2+\cdots+r_k$ is less than or equal to $r+1$.*

Theorem 2.8. *[10] Let $gcd(n,m) = d$ and suppose that d divides l, then the linear congruence $na \equiv l$ (mod m) has exactly d solutions modulo m and the solutions are $t, t+\frac{m}{d}, t+\frac{2m}{d}, \ldots, t+\frac{(d-1)m}{d}$ where t is the solution, unique modulo $\frac{m}{d}$, of the linear congruence $\frac{a}{d}x \equiv \frac{b}{d}$ (mod $\frac{m}{d}$).*

Remark 2.9. From (2) of theorem 2.6, we have the $\mathbb{Z}$-module homomorphism is determined by the value of $f(1) = a \in \mathbb{Z}_m$ and also we have $na \equiv 0$ (mod m). So the number of homomorphisms from $\mathbb{Z}_n \to \mathbb{Z}_m$ is equal to the possible values of a. By theorem 2.8 there are $gcd(n,m) = d$ possible values for a and they are $0, \frac{m}{d}, 2\frac{m}{d}, \ldots, (d-1)\frac{m}{d}$. So there are $'d'$ $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}_n \to \mathbb{Z}_m$, where $gcd(n,m) = d$ and the module homomorphisms are $f_k(x) = \frac{m}{d}kx$ (mod m), $k = 0, 1, \ldots, d-1$.

Theorem 2.10. *[2] Let M_1 and M_2 be R-modules and f be an R-module homomorphism of M_1 into M_2. If λ is a fuzzy module on M_1 then $f(\lambda)$ is a fuzzy module on M_2.*

Theorem 2.11. *[2] Let $f : \mathbb{Z}_n \to \mathbb{Z}_m$ be an $\mathbb{Z}$-module homomorphism with $gcd(n,m) = p$, a prime and let λ be any fuzzy module on $\mathbb{Z}_n$ then the level cardinality of $f(\lambda)$ is atmost* 3.

3. Fuzzy module homomorphism

Theorem 3.1. *Let $f : \mathbb{Z}_n \to \mathbb{Z}_m$ be an $\mathbb{Z}$-module homomorphism with $gcd(n,m) = pq$, product of primes p and q and let λ be any fuzzy module on $\mathbb{Z}_n$ then the level cardinality of $f(\lambda)$ is atmost* 4.

Proof. By remark 2.9 the only $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_n$ into $\mathbb{Z}_m$ are $f_k(x) = \frac{m}{pq}kx$ (mod m) where $k = 0, 1, 2, \ldots, pq-1$, as $gcd(n,m) = pq$. There are pq homomorphisms. By theorem 2.10 $f_k(\lambda)$ is a fuzzy module on $\mathbb{Z}_m$. Now the $\mathbb{Z}$-module homomorphisms are divided into 4 according to the value of $k = 0$ and values of $gcd(pq,k)$. Let in the prime factorisation of n, the highest power of p is r_1 and q is r_2. Without loss of generality we can assume that $p < q$.

I $k = 0$. It is the trivial homomorphism, $f_0(x) = 0$ for all $x \in \mathbb{Z}_n$, then only $0 \in \mathbb{Z}_m$ has preimage in $\mathbb{Z}_n$ under f_0. Let λ be any fuzzy module on $\mathbb{Z}_n$ then the fuzzy module $f_0(\lambda)$ is defined by,

$$f_0(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 0 & \text{if } y \in <1> \setminus <0> \end{cases}$$

Hence the level cardinality of $f_0(\lambda)$ on $\mathbb{Z}_m$ is 2.

II $gcd(k,pq) = p$.
When $gcd(k,pq) = p$ we can write the homomorphisms $f_k(x) = \frac{m}{pq}kx$ (mod m) as $f_k(x) = \frac{m}{q}k'x$ (mod m) where $k = k'p$ and $gcd(k',q) = 1$. So there are $\phi(q) = q-1$ such k' and hence $q-1$ homomorphisms. In these homomorphisms $f_k(x) = 0$ if x is a multiple q or 0. Hence the submodule $<q> = \{0, q, 2q, \ldots, (\frac{n}{q}-1)q\}$ of $\mathbb{Z}_n$ of order $\frac{n}{q}$ is mapped

to 0 in $\mathbb{Z}_m$ under these f_ks. So the elements in $\mathbb{Z}_n$ having non zero images under these f_ks has orders which are not factors of $\frac{n}{q}$. The possible orders of elements in $\mathbb{Z}_n$ with non zero images under these f_k are lq^{r_2} where $l \mid (\frac{n}{q^{r_2}})$. Also by theorem 2.6 order of $f_k(h)$ divides both order of h and m for all $h \in \mathbb{Z}_n$, hence $|f(h)| \mid gcd(n,m) = pq$. But in the $\mathbb{Z}$-module homomorphisms $f_k(x) = \frac{m}{q}k'x(\bmod m)$ where $k = k'p$ and $gcd(k',q) = 1$, the order of $\frac{m}{q}k'x(\bmod m)$ is 1 or q for all $x \in \mathbb{Z}_n$. So if $x \in <q>$ *i.e* order of x, $|x| = \frac{n}{q}$ or its divisor then $\frac{m}{q}k'x(\bmod m) = 0 \in \mathbb{Z}_m$ and if $x \in <1> \setminus <q> \subset \mathbb{Z}_n$ or $|x| = lq^{r_2}$ such that $l \mid (\frac{n}{q^{r_2}})$ then $\frac{m}{q}k'x(\bmod m) = a \in <\frac{m}{q}> \setminus <0>$ in $\mathbb{Z}_m$ and the submodule $<\frac{m}{q}> = \{0, \frac{m}{q}, 2\frac{m}{q}, \ldots, (q-1)\frac{m}{q}\}$ of $\mathbb{Z}_m$ of order q has only preimage under these f_k's.

$$f_k(x) = \begin{cases} 0 & \text{if } |x| = \frac{n}{q} \text{ or its divisors} \\ a & \text{if } |x| = lq^{r_2} \text{ such that } l \mid (\frac{n}{q^{r_2}}) \text{ and } a \text{ is any element in } <\frac{m}{q}> \setminus <0> \subset \mathbb{Z}_m \end{cases}$$

Now let λ be a fuzzy module on $\mathbb{Z}_n$ then $f_k(\lambda)(y) = 0$ for all $y \in <1> \setminus <\frac{m}{q}> \subset \mathbb{Z}_m$.

1. If $\lambda(0) = t \neq 0$ where $t = \vee\{\lambda(x) \,/\, x \in \mathbb{Z}_n, |x| = lq^{r_2}, l \mid (\frac{n}{q^{r_2}})\}$ then $f_k(\lambda)$ have level submodule of order q, Also $f_k(\lambda)$ on $\mathbb{Z}_m$ is

$$f_k(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <\frac{m}{q}> \\ 0 & \text{if } y \in <1> \setminus <\frac{m}{q}> \end{cases}$$

   Hence the level cardinality of $f_k(\lambda)$ is 2.
2. If $\lambda(0) \neq t \neq 0$ where $t = \vee\{\lambda(x) \,/\, x \in \mathbb{Z}_n, |x| = q^{r_2}, l \mid (\frac{n}{q^{r_2}})\}$ then $f_k(\lambda)$ have level submodules of order 1 and q, Also $f_k(\lambda)$ on $\mathbb{Z}_m$ is

$$f_k(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ t & \text{if } y \in <\frac{m}{q}> \setminus <0> \\ 0 & \text{if } y \in <1> \setminus <\frac{m}{q}> \end{cases}$$

   Hence the level cardinality of $f_k(\lambda)$ is 3.

So the level cardinality of $f_k(\lambda)$ is either 2 or 3 when $gcd(k, pq) = p$.

III $gcd(k, pq) = q$. This is similar to **II** $gcd(k, pq) = p$, as the homomorphisms $f_k(x) = \frac{m}{pq}kx\ (\bmod m)$ can be written as $f_k(x) = \frac{m}{p}k'x\ (\bmod m)$ where $k = k'q$ and $gcd(k', p) = 1$ and the submodule $<\frac{m}{p}> = \{0, \frac{m}{p}, 2\frac{m}{p}, \ldots, (p-1)\frac{m}{p}\}$ of $\mathbb{Z}_m$ of order p has only preimage under these f_k's. So the level cardinality of $f_k(\lambda)$ is either 2 or 3 when $gcd(k, pq) = q$.

IV $gcd(k, pq) = 1$. There are $\phi(pq) = (p-1)(q-1)$ such k and hence $(p-1)(q-1)$ homomorphisms. The $\mathbb{Z}$-module homomorphisms are $f_k(x) = \frac{m}{pq}kx\ (\bmod m)$ with $gcd(k, pq) = 1$. So $f_k(x) = 0$ if and only if x is a multiple of pq or 0 and hence the submodule of $\mathbb{Z}_n$, $<pq> = \{0, pq, 2pq, \ldots, (\frac{n}{pq} - 1)pq\}$ of order $\frac{n}{pq}$ is mapped to 0 in $\mathbb{Z}_m$ under these f_k. So the elements of $\mathbb{Z}_n$ under f_k having non zero images will have order either lp^{r_1} with $l \mid (\frac{n}{p^{r_1}})$ or lq^{r_2} with $l \mid (\frac{n}{q^{r_2}})$. Also by theorem 2.6 order of $f_k(h)$ divides both order of h and m for all $h \in \mathbb{Z}_n$, hence $|f(h)| \mid gcd(n,m) = pq$. So the possible values of order of $f(h)$ are $1, p, q, pq$. Hence the elements in $\mathbb{Z}_m$ of order $1, p, q, pq$ can only have preimages *i.e* the submodule $< \frac{m}{pq} >= \{0, \frac{m}{pq}, 2\frac{m}{pq}, \ldots, (pq-1)\frac{m}{pq}\}$ of $\mathbb{Z}_m$ of order pq only have preimage in $\mathbb{Z}_n$ under these f_k and $f_k(\lambda)(y) = 0$ for all $y \in < 1 > \setminus < \frac{m}{pq} >$ for every fuzzy module λ on $\mathbb{Z}_n$, if $m \neq pq$. When $m = pq$, the $\mathbb{Z}$-module homomorphism is ONTO. If $x \in < pq >$ *i.e* $|x| = \frac{n}{pq}$ or its divisors then $\frac{m}{pq}kx = 0 \in \mathbb{Z}_m$, if $x \in < p > \setminus < pq >$ or $|x| = lq^{r_2}$ with $l \mid (\frac{n}{pq^{r_2}})$ then $\frac{m}{pq}kx \pmod{m} = a_1$ is any element in $< \frac{m}{q} > \setminus < 0 >$, if $x \in < q > \setminus < pq >$ or $|x| = lp^{r_1}$ with $l \mid (\frac{n}{p^{r_1}q})$ then $\frac{m}{pq}kx \pmod{m} = a_2$ is any element in $< \frac{m}{p} > \setminus < 0 >$ and if $x \in < 1 > \setminus (< p > \cup < q >)$ or $|x| = lp^{r_1}q^{r_2}$ with $l \mid (\frac{n}{p^{r_1}q^{r_2}})$ then $\frac{m}{pq}kx \pmod{m} = a_3$ is any element in $< \frac{m}{pq} > \setminus (< \frac{m}{p} > \cup < \frac{m}{q} >)$. Then

$$f_k(x) = \begin{cases} 0 & \text{if } |x| = \frac{n}{pq} \text{ or its divisor} \\ a_1 & \text{if } |x| = lq^{r_2} \text{ with } l \mid (\frac{n}{pq^{r_2}}) \\ a_2 & \text{if } |x| = lp^{r_1} \text{ with } l \mid (\frac{n}{p^{r_1}q}) \\ a_3 & \text{if } |x| = lp^{r_1}q^{r_2} \text{ with } l \mid (\frac{n}{p^{r_1}q^{r_2}}) \end{cases}$$

Now let λ be a fuzzy module on $\mathbb{Z}_n$, then the fuzzy module $f_k(\lambda)$ on $\mathbb{Z}_m$ is defined as follows

1. If $\lambda(0) = t_1 = t_2 \neq 0$ where $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lq^{r_2},\ l \mid (\frac{n}{pq^{r_2}})\}$ or $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lp^{r_1},\ l \mid (\frac{n}{p^{r_1}q})\}$ and $t_2 = \vee\{\lambda(x_2) \,/\, x_2 \in \mathbb{Z}_n,\ |x_2| = lp^{r_1}q^{r_2},\ l \mid (\frac{n}{p^{r_1}q^{r_2}})\}$ then $f_k(\lambda)$ have level submodule of order pq, Also $f_k(\lambda)$ on $\mathbb{Z}_m$ is

$$f_k(\lambda)(y) = \begin{cases} 1 & \text{if } y \in < \frac{m}{pq} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{m}{pq} > \end{cases}$$

   Hence the level cardinality of $f_k(\lambda)$ is 2 if $m \neq pq$ and has level cardinality 1 if $m = pq$.
2. If $\lambda(0) \neq t_1 = t_2 \neq 0$ where $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lq^{r_2},\ l \mid (\frac{n}{pq^{r_2}})\}$ or $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lp^{r_1},\ l \mid (\frac{n}{p^{r_1}q})\}$ and $t_2 = \vee\{\lambda(x_2) \,/\, x_2 \in \mathbb{Z}_n,\ |x_2| = lp^{r_1}q^{r_2},\ l \mid (\frac{n}{p^{r_1}q^{r_2}})\}$ then $f_k(\lambda)$ have level submodules of orders 1 and pq, Also $f_k(\lambda)$ on $\mathbb{Z}_m$ is

$$f_k(\lambda)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in < \frac{m}{pq} > \setminus < 0 > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{m}{pq} > \end{cases}$$

   Hence the level cardinality of $f_k(\lambda)$ is 3 if $m \neq pq$ and has level cardinality 2 if $m = pq$.
3. If $\lambda(0) = t_1 \neq t_2 \neq 0$ where $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lq^{r_2},\ l \mid (\frac{n}{pq^{r_2}})\}$ or $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lp^{r_1},\ l \mid (\frac{n}{p^{r_1}q})\}$ and $t_2 = \vee\{\lambda(x_2) \,/\, x_2 \in \mathbb{Z}_n,\ |x_2| = lp^{r_1}q^{r_2},\ l \mid (\frac{n}{p^{r_1}q^{r_2}})\}$ then $f_k(\lambda)$ have level submodules of orders p or q and pq, Also $f_k(\lambda)$ on $\mathbb{Z}_m$ is

$$f_k(\lambda)(y) = \begin{cases} 1 & \text{if } y \in < \frac{m}{p} > \text{ or } y \in < \frac{m}{q} > \\ t_2 & \text{if } y \in < \frac{m}{pq} > \setminus < \frac{m}{p} > \text{ or } y \in < \frac{m}{pq} > \setminus < \frac{m}{q} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{m}{pq} > \end{cases}$$

Hence the level cardinality of $f_k(\lambda)$ is 3 if $m \neq pq$ and has level cardinality 2 if $m = pq$.

4. If $\lambda(0) \neq t_1 \neq t_2 \neq 0$ where $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n,\ |x_1| = lq^{r_2},\ l \mid (\frac{n}{pq^{r_2}})\}$ or $t_1 = \vee\{\lambda(x_1) \,/\, x_1 \in \mathbb{Z}_n, |x_1| = lp^{r_1}, l \mid (\frac{n}{p^{r_1}q})\}$ and $t_2 = \vee\{\lambda(x_2) \,/\, x_2 \in \mathbb{Z}_n, |x_2| = lp^{r_1}q^{r_2}, l \mid (\frac{n}{p^{r_1}q^{r_2}})\}$ then $f_k(\lambda)$ have level submodules of orders 1, p or q and pq, Also $f_k(\lambda)$ on $\mathbb{Z}_m$ is

$$f_k(\lambda)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in < \frac{m}{p} > \setminus < 0 > \text{ or } y \in < \frac{m}{q} > \setminus < 0 > \\ t_2 & \text{if } y \in < \frac{m}{pq} > \setminus < \frac{m}{p} > \text{ or } y \in < \frac{m}{pq} > \setminus < \frac{m}{q} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{m}{pq} > \end{cases}$$

Hence the level cardinality of $f_k(\lambda)$ is 4 if $m \neq pq$ and has level cardinality 3 if $m = pq$.

Hence the level cardinality of the fuzzy module $f(\lambda)$ on $\mathbb{Z}_m$ is atmost 4 where f is an $\mathbb{Z}$-module homomorphism of $\mathbb{Z}_n$ into $\mathbb{Z}_m$.

□

Example 3.2. The $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_{315}$ into $\mathbb{Z}_{150}$ are $f_k(x) = 10kx \pmod{150}$ $\forall x \in \mathbb{Z}_{105}$ $\;k = 0, 1, 2, \ldots, 14$, by remark 2.9 and since $gcd(315, 150) = 15$. By theorem 2.7 the maximum level cardinality of fuzzy module on $\mathbb{Z}_{315}$ is 5. Consider a fuzzy module on $\mathbb{Z}_{315}$ with level cardinality 5 and for the $\mathbb{Z}$-module homomorphisms f_k, where $k = 0, 1, 2, \ldots 14$.

$$Let\ \ \lambda(x) = \begin{cases} 1 & \text{if } x \in < 0 > \\ 1/2 & \text{if } x \in < 63 > \setminus < 0 > \\ 1/3 & \text{if } x \in < 21 > \setminus < 63 > \\ 1/4 & \text{if } x \in < 7 > \setminus < 21 > \\ 1/5 & \text{if } x \in < 1 > \setminus < 7 > \end{cases}$$

then

Cases	$k=0$	$gcd(k,15)=3$	$gcd(k,15)=5$	$gcd(k,15)=1$
Number of homomorphisms	1	$\phi(5)=4$	$\phi(3)=2$	$\phi(15)=8$
Homomorphisms	f_0	f_3, f_6, f_9, f_{12}	f_5, f_{10}	$f_1, f_2, f_4, f_7, f_8, f_{11}, f_{13}, f_{14}$
Fuzzy module	$f_0(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 0 & \text{if } y \in <1> \setminus <0> \end{cases}$	$f_3(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 1/2 & \text{if } y \in <30> \setminus <0> \\ 0 & \text{if } y \in <1> \setminus <30> \end{cases}$	$f_5(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 1/4 & \text{if } y \in <50> \setminus <0> \\ 0 & \text{if } y \in <1> \setminus <50> \end{cases}$	$f_1(\lambda)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 1/2 & \text{if } y \in <30> \setminus <0> \\ 1/4 & \text{if } y \in <10> \setminus <30> \\ 0 & \text{if } y \in <1> \setminus <10> \end{cases}$

4. Conclusion

In this paper, We have proved that the level cardinalities of fuzzy module $f(\lambda)$ is atmost 4, where f is an $\mathbb{Z}$- module homomorphism of fuzzy modules of the $\mathbb{Z}$-module $\mathbb{Z}_n$ into the $\mathbb{Z}$-module $\mathbb{Z}_m$ where $n, m \in \mathbb{Z}^+$, $gcd(n,m) = pq$, p and q are primes. In our future work, we are trying to extend this result in the case of fuzzy module homomorphism from $\mathbb{Z}_n$ into $\mathbb{Z}_m$ $n, m \in \mathbb{Z}$, $gcd(n,m) = p^r q^s$

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