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Assisted Normative Reasoning with Aristotelian Diagrams

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Abstract. We design a framework for assisted normative reasoning based on Aristotelian diagrams and algorithmic graph theory which can be employed to address heterogeneous tasks of deductive reasoning. Here we focus on two problems of normative determination: we show that the algorithms used to address these problems are computationally efficient and their operations are traceable by humans. Finally, we discuss an application of our framework to a scenario regulated by the GDPR.

Keywords. Algorithmic graph theory, Aristotelian diagrams, assisted reasoning, automated deduction, normative reasoning

1. Introduction

The growing interest in formal analyses and computational applications of *Aristotelian diagrams* [1,2,3,4,6,8,9] encourages the development of new methods for their use in assisted reasoning. If compared to full-fledged logical systems, Aristotelian diagrams *trade expressiveness for transparency*: the inferences performed over these diagrams (i) are simple and limited, (ii) can be automated via algorithms whose computational complexity is low, (iii) are sufficient to address relevant reasoning problems from everyday life, and (iv) are traceable by individuals without a background in formal logic.

This article provides a novel and general framework for assisted reasoning with Aristotelian diagrams, based on a systematic interaction between logic and graph theory (Section 2). We illustrate how this framework can be used to represent two problems of *normative determination* [7]. The *first problem* (*RP*1) consists in checking whether the truth of some statements in a set X determines the truth-value of all statements in a set Y in light of a normative theory Γ. The *second problem* (*RP*2) generalizes the first and

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consists in checking whether X can be expanded to a set X' via stronger assumptions so that the truth of some statements in X' determines the truth-value of all statements in Y in light of Γ. Aristotelian diagrams are here explored via algorithms from graph theory (Section 3): *RP*1 and *RP*2 are solved in linear and cubic time, respectively, and linear space with respect to the size of a diagram. Our framework is highly flexible and many other reasoning problems can be dealt with in a modular way. Moreover, the algorithms employed can be related to inferences visually made by humans on diagrams and are thus transparent. Transparency is a crucial feature for legal applications, such as decisions about rights and obligations [5], and is a goal pursued in this area also by other traditions, such as argumentation theory [10]. Finally, as an application of our framework, we chose a scenario based on the GDPR (Section 4), one of the most discussed regulations nowadays and often used to build examples for automated legal reasoning.

2. Formal framework

Our framework consists of *three layers of analysis*. The *first* is given by a normative theory N formulated over natural language and reasoning problems on it. We move to the *second* by representing N as a formal theory Γ, which we call *Aristotelian theory*, and by representing reasoning problems on N as formal deduction problems on Γ. Finally, we move to the *third* by transforming Γ into a graph D_Γ, which we call *Aristotelian diagram*, and formulate search procedures over D_Γ to solve problems on Γ in a computationally efficient way. We will focus on the second and the third layer of analysis.

Let $\mathscr{L}$ be a language with an arbitrary vocabulary and $St(\mathscr{L})$ the set of its statements, namely the set of expressions of $\mathscr{L}$ describing *states-of-affairs*. We denote a function f with domain X and codomain Y as $f : X \longrightarrow Y$ if f is *total* and as $f : X \rightharpoonup Y$ if f is *possibly partial*. An interpretation of $\mathscr{L}$ is a function $I : St(\mathscr{L}) \longrightarrow \{0,1\}$.

Definition 1 (Aristotelian relation). *An* Aristotelian relation *between* $\phi, \psi \in St(\mathscr{L})$ *is a relation that we call either* dominance *(*`dom`*) or* contrariety *(*`cty`*) or* subcontrariety *(*`sty`*) or* contradiction *(*`ctd`*) or* equivalence *(*`equ`*), where:*

- $\mathtt{dom}(\phi,\psi)$ *iff* $\forall I(I(\phi)=1 \Rightarrow I(\psi)=1) \land \exists I(I(\psi)=1 \land I(\phi)=0)$*;*
- $\mathtt{cty}(\phi,\psi)$ *iff* $\forall I(I(\phi)=1 \Rightarrow I(\psi)=0)$*;*
- $\mathtt{sty}(\phi,\psi)$ *iff* $\forall I(I(\phi)=0 \Rightarrow I(\psi)=1)$*;*
- $\mathtt{ctd}(\phi,\psi)$ *iff* $\forall I(I(\phi)=1 \Leftrightarrow I(\psi)=0)$*;*
- $\mathtt{equ}(\phi,\psi)$ *iff* $\forall I(I(\phi)=1 \Leftrightarrow I(\psi)=1)$.[2]

Definition 2 (Aristotelian theory). *Let* $\mathsf{AR} = \{\mathtt{dom}, \mathtt{cty}, \mathtt{sty}, \mathtt{ctd}, \mathtt{equ}\}$. *An* Aristotelian theory *over* $\mathscr{L}$ *is* $\Gamma = \langle \Sigma, f \rangle$ *where* $\Sigma \subseteq St(\mathscr{L})$ *is a finite set and* $f : \Sigma \times \Sigma \rightharpoonup \mathsf{AR}$.

[2] Notice that we are here modifying and expanding the standard theory of Aristotelian relations: the four *traditional* Aristotelian relations are ‘contrariety’, ‘subcontrariety’, ‘contradiction’ and ‘subalternation’. Subalternation is just the converse of dominance: ψ is a subalternate of ϕ iff ϕ dominates ψ. By contrast, equivalence is not connected to any traditional Aristotelian relation, yet sometimes employed to pre-process theories [2].

f is a possibly partial function in Γ since we want to represent reasoning problems where one has *incomplete* information about which Aristotelian relations hold in Σ. Yet, for brevity, here we only deal with theories that are *symmetry-closed*, in the following sense:[3]

Definition 3 (Symmetry-closed theory)**.** *An Aristotelian theory* $\Gamma = \langle \Sigma, f \rangle$ *is* symmetry-closed *iff, for any* $x \in \mathsf{AR} \setminus \{\mathtt{dom}\}$ *and* $\phi, \psi \in \Sigma$, $f(\phi, \psi) = x$*, entails* $f(\psi, \phi) = x$*.*

Definition 4 (Reasoning problems on theories)**.** *A reasoning problem on a theory* $\Gamma = \langle \Sigma, f \rangle$ *is* $RP = \langle \Gamma, X, Y, \mathfrak{R} \rangle$*, where* $X, Y \subseteq \Sigma$ *and* $\mathfrak{R}$ *is a relation among the truth-values of statements in (a set defined via)* X *and statements in (a set defined via)* Y*.*

We focus on two reasoning problems of normative determination [7]. The first one is:

> $RP1 = \langle \Gamma, X, Y, \mathfrak{R}_1 \rangle$, where $\mathfrak{R}_1$ indicates that within Γ the truth of some formulas in X determines the truth-value of all formulas in Y, i.e.,
> $\exists \phi_1, ..., \phi_n \in X \forall I_1, I_2((I_1(\phi_j) = I_2(\phi_j) = 1 \text{ for } 1 \leq j \leq n) \rightarrow \forall \psi \in Y(I_1(\psi) = I_2(\psi)))$.

To define the second one, we introduce the notion of a *dominance ancestor*.

Definition 5 (Dominance ancestors)**.** *Take* $\Gamma = \langle \Sigma, f \rangle$ *and* $X \subseteq \Sigma$*. The set of* dominance ancestors *of* X*, i.e.* $\mathsf{DAS}(X)$*, is the set of all* $\xi \in \Sigma \setminus X$ *s.t. for some* $\theta_1, ..., \theta_n \in \Sigma$ *(*$n > 1$*) and* $\phi \in X$*, it holds that* $\theta_1 = \xi$*,* $\theta_n = \phi$ *and, for* $1 \leq i < n$*,* $f(\theta_i, \theta_{i+1}) = \mathtt{dom}$*.*

Note that there may be contradictions between statements in $\mathsf{DAS}(X)$. We can now proceed to the second kind of reasoning problem:

> $RP2 = \langle \Gamma, X, Y, \mathfrak{R}_2 \rangle$, where $\mathfrak{R}_2$ indicates that within Γ the truth of some formulas in $X^* = X \cup \mathsf{DAS}(X)$ determines the truth-value of all formulas in Y, i.e.,
> $\exists \phi_1, ..., \phi_n \in X^* \forall I_1, I_2((I_1(\phi_j) = I_2(\phi_j) = 1 \text{ for } 1 \leq j \leq n) \rightarrow \forall \psi \in Y(I_1(\psi) = I_2(\psi)))$.

In *RP*2 we can regard X as conveying factual information available on a scenario and $X^* = X \cup \mathsf{DAS}(X)$ as an integration of such information with further (and stronger) hypotheses. Notice that a positive answer to *RP*1 is also a positive answer to *RP*2 but not the other way around. Thus, *RP*2 is especially relevant when the answer to *RP*1 is negative.

A reasoning problem on a theory Γ will be addressed via search procedures on the Aristotelian diagram corresponding to Γ (which can be easily proven to be unique up to isomorphism), as per the definition below. By doing this we move from the second to the third layer of analysis in our framework.

Definition 6 (Aristotelian diagram)**.** *The Aristotelian diagram corresponding to a theory* $\Gamma = \langle \Sigma, f \rangle$ *is* $D_\Gamma = \langle V, l, E, r \rangle$*, where:*

- V *is the set of* vertices *of* D_Γ *s.t.* $|V| = |\Sigma|$*;*
- $l : V \longrightarrow \Sigma$ *is a bijective function assigning a statement to each vertex;*
- $E \subseteq V \times V$ *is the relation constituting the set of edges of* D_Γ *s.t., for* $v, u \in V$*,* $(v, u) \in E$ *iff* $f(l(v), l(u)) \in \mathsf{AR}$*.*
- $r : E \longrightarrow \mathsf{AR}$ *is a function assigning an Aristotelian relation to each edge s.t., for* $v, u \in V$*,* $r(v, u) = f(l(v), l(u))$*.*

[3]Restricting our attention to symmetry-closed theories does not affect the computational complexity of the methods illustrated in Section 3, given that a theory Γ can be transformed into its symmetry-closure Γ' in linear time with respect to $|\Theta|$, where Θ is the subset of $\Sigma \times \Sigma$ for which f is defined.

Let $\mathbf{size}(D_\Gamma) = |V| + |E|$: this is always finite, by Def. 2 and Def. 6. Diagram D_Γ allows for graphically displaying theory Γ; thus, one can visually reconstruct the solutions to a given reasoning problem on Γ provided by an algorithm, as discussed in Section 4.

3. Search procedures

We will now present two search procedures over D_Γ, $SP1$ and $SP2$, that can be respectively used to provide a solution to the reasoning problems $RP1$ and $RP2$ described in Section 2. These procedures solve the two problems in a computationally efficient way.

Given D_Γ, X, and Y, the algorithm for $SP1$ solves $RP1$ using a modified breadth-first search on D_Γ, which we assume is implemented in a helper routine called SEARCH. The algorithm maintains a state for each vertex v, indicating whether $l(v)$ has already been inferred, $\neg l(v)$ has already been inferred or neither of the two. SEARCH uses a queue to temporally store those vertices that have already been encountered, but have not been processed yet, and starts by enqueuing each vertex with a label in X and recording that it has already been inferred. It then proceeds analogously to a classic breadth-first search, but observing the relation that is assigned to each edge it processes. The algorithm can also detect and report inconsistencies, i.e., contradictions inferred from X by continuing to run until the graph is exhausted.

The algorithm for $SP1$ can be implemented analogously to breadth-first search and thus has the same *time complexity*, which is *linear in the number of vertices and edges* of D_Γ, i.e., $O(\mathbf{size}(D_\Gamma))$. The *space complexity* is $O(|V|)$.

An algorithm that solves $RP2$ can reuse the SEARCH procedure from $SP1$. Indeed, it starts exactly as $SP1$. If SEARCH returns all statements in Y or their negation, we are done. Otherwise, it performs a backwards breadth-first search, starting from the vertices with label in X and using *only* those incoming edges with a `dominance` relation. Notice that for each vertex u that is newly encountered in this way, $l(u)$ is a dominance ancestor of at least one $\phi \in X$. The backwards breadth-first search is run until the graph is exhausted, thereby computing $\mathsf{DAS}(X)$. For each non-adjacent pair of vertices u, u' s.t. $l(u), l(u') \in \mathsf{DAS}(X)$, we run SEARCH to determine whether $X \cup \{l(u), l(u')\}$ has an inconsistency. Afterwards, we compute all consistent maximal subsets $S_1, S_2, \ldots, S_k \subseteq \mathsf{DAS}(X)$. The algorithm then returns S_i for some $1 \leq i \leq k$ such that $X \cup S_i$ determines the truth-value of all statements in Y via SEARCH, if exists, and otherwise returns failure.

The time complexity of the backwards breadth-first search itself is linear in $\mathbf{size}(D_\Gamma)$. As SEARCH is run for each non-adjacent pair of vertices u, u' with $l(u), l(u') \in \mathsf{DAS}(X)$, the time complexity for this part is $O(|V|^2 \cdot \mathbf{size}(D_\Gamma))$. Running SEARCH on $X \cup S_i$ for $1 \leq i \leq k$ takes $O(|V| \cdot \mathbf{size}(D_\Gamma))$ time in the worst case, since $k \leq |V|$ due to the maximality of the subsets. Thus, the overall time complexity for $SP2$ is $O(|V|^2 \cdot \mathbf{size}(D_\Gamma))$.

4. An application to the GDPR

We apply the methodology presented above to the legal rule set out in Article 9 of the GDPR[4]. This article concerns the *Processing of special categories of personal data*. Its

[4] Art. 9 of Regulation EU 2016/679 on the protection of natural persons with regard to the processing of personal data and on the free movement of such data, and repealing Directive 95/46/EC (General Data Protection

purpose is to declare that some categories of personal data are more sensitive than others and therefore the rules for processing them are stricter. The first paragraph of this Article states which specific categories of personal data may not be processed. The second paragraph provides a list of exceptions to the general prohibition, namely conditions under which the processing of the categories of personal data at issue is not prohibited. We here discuss an example, taking into account one of the exceptions, i.e. obtaining the explicit consent for each specific purpose of personal data processing.

Consider a mobile application m operating on the principle of telemedicine and including a questionnaire about the health status of its users. On the basis of the input received, the app may take action (e.g., arranging an appointment with an expert). Thus, the app is *able* to (i.e., practically can) process data concerning the health of its users. Yet, this does not entail that the app is *allowed* (i.e., has a right) to process such data.

We may wonder whether a user u explicitly gave consent to data processing for two specific purposes, $p(a)$ and $p(b)$:

$p(a)$ evaluating further diagnostic procedures;
$p(b)$ transferring their data to a doctor and arranging a personal appointment.

Moreover, we may wonder whether u's consent to the processing of data for the purposes $p(a)$ and $p(b)$ is not precluded by the relevant Member State legislation. If all these additional conditions apply to the scenario, then m complies with Article 9 and is allowed to process personal data, otherwise it is not allowed for at least one of the two purposes.

We will represent this example of normative theory (Article 9 GDPR) and the described scenario over it in our framework. It is a very simple scenario, yet it already gives a precise idea about how the framework is designed. Consider the following statements:

1 mobile app m is able to process data concerning the health of user u;
$2a$ u gave consent for the processing of their data to m for the purpose $p(a)$;
$2b$ u gave consent for the processing of their data to m for the purpose $p(b)$;
$3a$ u's consent for $p(a)$ is permitted by the Member State law;
$3b$ u's consent for $p(b)$ is permitted by the Member State law;
$4a$ m has a right to process data concerning the health of u for the purpose $p(a)$;
$4b$ m has a right to process data concerning the health of u for the purpose $p(b)$.

Let $\mathscr{L}$ be a language s.t., for $p \in \{1, 2a, 2b, 3a, 3b, 4a, 4b\}$, the BNF grammar for $St(\mathscr{L})$ is $\phi ::= p \mid \neg\phi \mid \phi \& \phi$. We can define many Aristotelian theories representing our scenario. A simple choice is a *symmetry-closed* theory $\Gamma = \langle \Sigma, f \rangle$, where, for $x \in \{a, b\}$:

- $\Sigma = \{1,\ 1\&2x,\ 1\&\neg 2x,\ 1\&2x\&3x,\ 1\&2x\&\neg 3x,\ 4x,\ \neg 4x\}$
- $f(4x, \neg 4x) = \texttt{ctd}$
- $f(1\&2x, 1\&\neg 2x) = f(1\&2x\&3x, 1\&2x\&\neg 3x) = \texttt{cty}$
- $f(1\&2x, 1) = f(1\&\neg 2x, 1) = f(1\&2x\&3x, 1\&2x) = f(1\&2x\&\neg 3x, 1\&2x) = \texttt{dom}$
- $f(1\&2x\&3x, 4x) = f(1\&2x\&\neg 3x, \neg 4x) = f(1\&\neg 2x, \neg 4x) = \texttt{dom}$

We can formulate various instances of $RP1$ and $RP2$ by taking $Y = \{4a, 4b\}$ as the set of final statements and different sets of initial statements, such as $X = \{1\}$ or $X = \{1\&2a\}$ or $X = \{1\&2a, 1\&2b\}$, etc., depending on what factual information about the scenario we imagine to be available. For instance, let $X = \{1\}$: in this case we just take to be a fact that app m is able to process u's health data. $RP1 = \langle \Gamma, \{1\}, \{4a, 4b\}, \mathfrak{R}_1 \rangle$ receives a *nega-*

Regulation). Available at: https://eur-lex.europa.eu/eli/reg/2016/679/oj *tive* answer via $SP1$, whereas $RP2 = \langle \Gamma, \{1\}, \{4a, 4b\}, \mathfrak{R}_2 \rangle$ receives a *positive* answer via $SP2$. Moreover, $SP2$ specifies all the consequences of expanding X with stronger assumptions (namely, that adding $1\&\neg 2a$ determines the falsity of $4a$, that adding $1\&2a\&3a$ determines the truth of $4a$, etc.). Human reasoners can explore theory Γ and keep track of the solutions provided by $SP1$ and $SP2$ to the given instances of $RP1$ and $RP2$ by looking at the Aristotelian diagram D_Γ in Fig. 1. This is a simple diagram (its size is 13 vertices plus 20 edges) for a simple scenario; yet, the methods provided here work for any Aristotelian diagram with a finite number of vertices and a finite number of edges, thus being able to efficiently solve reasoning problems on much more complex theories.

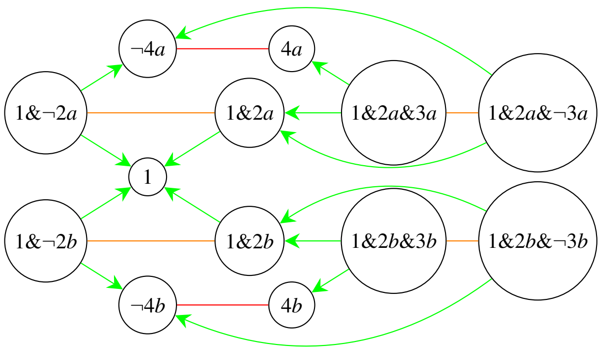

Figure 1. An Aristotelian diagram for the GDPR example. A green edge directed from a vertex v to a vertex u indicates that (the label of) v dominates (the label of) u (i.e., u subalternates v); a red edge between v and u indicates that v and u are contradictories; an orange edge between v and u indicates that v and u are contraries.

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