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A Model-Theoretic Approach to Belief Revision in Multi-Agent Belief Logic and Its Syntactic Characterizations

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Abstract. Belief change studies how an agent modifies her beliefs on receiving new information. However, so far most research on belief change works on beliefs represented in propositional logic. There have been many works on integrating belief revision with reasoning about actions, and some works extending belief change from propositional logic to epistemic logics. In this paper, we study revision on beliefs of a third person represented with the multi-agent KD45 logic. Our formal technique is analogous to that of distance-based belief revision in propositional logic: to revise a KB by a formula, select from models of the formula those that are closest to models of the KB. To this end, a challenge is that in modal logics, a formula may have infinitely many Kripke models. To tackle this, we propose a variant of Moss' canonical formulas called alternating canonical formulas, treat them as models for formulas, and define a notion of distance between them, based on the Hausdorff distance between two sets. We show that our revision satisfies all of the AGM postulates. To give syntactic characterizations of our revision, we make use of a normal form for KD45n called alternating cover disjunctive formulas (ACDFs). We give syntactic characterizations firstly on fragments of ACDFs called proper ACDFs and alternating cover conjunctive formulas (ACCFs), and finally on the whole ACDFs.

1 Introduction

Belief change studies how an agent modifies her beliefs on receiving new information. So far research on belief change works on beliefs represented in propositional logic. Two main types of belief change are revision and update: revision concerns belief change about static environments due to partial and possibly incorrect information, whereas update concerns belief change about dynamic environments due to the performance of actions. Various guidelines for belief change have been proposed, *e.g.*, the AGM postulates for belief revision [1]. Both revision and update are guided by the principle of minimal change, where the notion of closeness between models can be based on set inclusion or cardinality. To formalize the distinction between revision and update, Katsuno and Mendelzon [17] presented model-theoretic definitions of them. Del Val [9] gave syntactic characterizations for a number of belief change operators, including the cardinality based revision of Dalal [8].

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The issue of belief change becomes more perplexing in the presence of multiple agents. In such settings, in addition to first-order beliefs, *i.e.*, beliefs about the world, there are also higher-order beliefs, *i.e.*, beliefs about agents' beliefs. Beliefs change as a result of the performance of ontic, sensing, and communication actions.

Many works have been done on integrating belief change with reasoning about actions in dynamic epistemic logics (DELs) [23] or the situation calculus [20]. DELs focus on reasoning about epistemic actions in the multi-agent case. Following Aucher [2] and Van Benthem [22], Baltag and Smets [5] presented a general framework for integrating belief revision into DELs. In line with the AGM approach of giving priority to new information, they proposed the action priority update operation: when updating a plausibility model by an action plausibility model, give priority to the action plausibility order. In the single-agent case, Shapiro *et al.* [21] and Delgrande and Levesque [10] integrated belief revision into the situation calculus, by augmenting it with a notion of plausibility over situations. In the multi-agent case, by integrating action priority update from DELs into the situation calculus, Fang and Liu [12] gave a general framework for reasoning about actions and belief change.

There have also been some works extending belief change from propositional logic to epistemic logics. Multi-agent KD45 modal logic is a logic suitable for describing multi-agent beliefs [11]. Herzig *et al.* [15] studied the update and revision of KD45n Kripke models of two agents. Aucher [3] proposed a notion of distances between KD45n models, and gave a semantic study of multi-agent belief revision incurred by private announcements. Caridroit *et al.* [6] studied private expansion and revision of KD45n models; then they [7] investigated several measures of distances between KD45n models, and used them to define the revision of a finite set of KD45n models by a formula. Miller and Muise [18] studied belief update for knowledge bases consisting of belief literals. Huang *et al.* [16] proposed a general multi-agent epistemic planner based on higher-order belief change. The idea is that the progression of knowledge bases w.r.t. actions is achieved through belief revision or update on KD45n formulas. They made use of a normal form for KD45n called alternating cover disjunctive formulas (ACDFs). They proposed syntactic higher-order belief change operators for ACDFs. However, they were not able to give semantic characterizations for their syntactic operators. Recently, Wan *et al.* [24] gave semantic characterizations for Huang *et al.* 's syntactic operators for proper ACDFs, a fragment of

ACDFs disallowing negative or disjunctive beliefs.

In this paper, we study revision on beliefs of a third person represented with KD45n. To illustrate, consider the following example where B_i means agent i believes:

Example 1 There are two persons a and b and two apples. Let p stand for a has an apple and q for b has an apple. At the beginning, a has an apple, and both a and b believe that. Also, a believes that b believes it, and b believes that a believes it. This can be written in:

$$\phi = p \wedge B_a p \wedge B_b p \wedge B_a B_b p \wedge B_b B_a p.$$

Then a ate her apple, and b got an apple. This can be expressed by:

$$\mu = \neg p \wedge q \wedge B_a \neg p \wedge B_b q.$$

How should ϕ be revised by μ? A plausible result is:

$$\neg p \wedge q \wedge B_a \neg p \wedge B_a B_b p \wedge B_b (p \wedge q) \wedge B_b B_a p.$$

Our research questions are: how to give a model-theoretic definition of the revision of a KD45n formula by another one, and how to compute the result of revision. The long-term intended application of our work is multi-agent epistemic planning. Our work differs from existing works on integrating belief change into reasoning about actions in that actions take important roles there, while we are concerned with revising a KD45n formula by another one. Our work differs from existing works on extending belief change from propositional logic to epistemic logics in that we are interested in first a semantic approach to the revision of arbitrary belief formulas and then syntactic characterizations.

Our formal technique is analogous to that of distance-based belief revision in propositional logic: to revise a KB by a formula, select from models of the formula those that are closest to models of the KB. To use this technique, a challenge is that in modal logics, a formula may have infinitely many Kripke models, and it is difficult to define clean notions of distances between Kripke models, as shown by [3, 7]. A first rescue we can think of is Moss' canonical formulas (CFs) [19]: a CF captures a Kripke model up to a given depth. Moss showed that an arbitrary modal formula is equivalent to a disjunction of a finite set of CFs. However, CFs satisfiable in KD45n contain redundant information, which complicates the definition of distances between them. To tackle this, we propose the notion of alternating canonical formulas (ACFs), and show that there is a bijection between ACFs and CFs satisfiable in KD45n. We treat ACFs as models for formulas, and define a notion of distances between ACFs, based on the Hausdorff distance between two sets. We show that our revision operator satisfies all of the AGM postulates. To give syntactic characterizations, we propose a new fragment of ACDFs without disjunctions called alternating cover conjunctive formulas (ACCFs), which subsumes ACFs and can be treated as the analogy of satisfiable terms in propositional logic. We give syntactic characterizations for our revision operator firstly on proper ACDFs and ACCFs, and finally on the whole ACDFs. Our syntactic characterization for ACDFs takes the same form as that for Dalal's propositional revision.

Table 1 gives the list of acronyms we use in this paper:

2 Preliminaries

In this section, we introduce KD45n modal logic, canonical formulas, ACDFs, and Dalal's propositional revision.

Table 1. Acronyms and their definitions

CDF	cover disjunctive formula	Def. 9
ACDF	alternating cover disjunctive formula	Def. 11
CF	canonical formula	Def. 6
ACF	alternating canonical formula	Def. 15
ACCF	alternating cover conjunctive formula	Def. 17

2.1 KD45n modal logic

Let $\mathcal{A}$ be a finite set of agents and $\mathcal{P}$ be a finite set of atoms. A *literal* is an atom p or its negation $\neg p$. A (propositional) *term* is a conjunction of literals, a *minterm* is a term where each atom appears exactly once.

Definition 1 The multi-agent modal language $\mathcal{L}_B$ is defined by $\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid B_a\phi$, with $a \in \mathcal{A}$, $p \in \mathcal{P}$.

As usual, "$\vee$" and "$\rightarrow$" are treated as abbreviations. Intuitively, $B_a\phi$ means agent a believes ϕ. We use $\hat{B}_a\phi$ to denote $\neg B_a \neg\phi$, $\top$ and $\bot$ to denote true and false respectively. $\bigwedge\Phi$ (resp. $\bigvee\Phi$) is the conjunction (resp. disjunction) of the members in Φ. The modal depth of a formula ϕ, denoted by $md(\phi)$, is the depth of nesting of modal operators in ϕ. For $a \in \mathcal{A}$, we use $\mathcal{L}_B^{-a}$ to denote the set of formulas not using B_a as an outmost modal operator.

Definition 2 A frame is a pair (W, R), where W is a non-empty set of possible worlds, and for all $a \in \mathcal{A}$, R_a is a binary relation on W. In KD45n modal logic, R_i should be:

- serial, *i.e.*, for all $w \in W$, there is $w' \in W$ s.t. wR_aw';
- Euclidean, *i.e.*, if wR_aw_1 and wR_aw_2 then $w_1R_aw_2$;
- transitive, *i.e.*, if wR_aw_1 and $w_1R_aw_2$ then wR_aw_2.

Definition 3 A Kripke model is a tuple $M = (W, R, V)$ where (W, R) is a frame and V is a valuation map that maps each $w \in W$ to a subset of $\mathcal{P}$. A *pointed* Kripke model is a pair (M, w), in which M is a Kripke model and w is a world of M, called the *actual world*.

Definition 4 Let $M = (W, R, V)$. Let $s = (M, w)$ be a pointed Kripke model, then for formulas in $\mathcal{L}_B$:

- $M, w \models p$ iff $p \in V(w)$;
- $M, w \models \neg\phi$ iff $M, w \not\models \phi$;
- $M, w \models \phi \wedge \psi$ iff $M, w \models \phi$ and $M, w \models \psi$;
- $M, w \models B_a\phi$ iff for all v such that wR_av, $M, v \models \phi$.

We say ϕ is *satisfiable* if there is a KD45n Kripke model (M, w) s.t. $M, w \models \phi$; ϕ *entails* ψ, written $\phi \models \psi$, if for any KD45n Kripke model (M, w) s.t. $M, w \models \phi$, $M, w \models \psi$; ϕ *is equivalent to* ψ, written $\phi \Leftrightarrow \psi$, if $\phi \models \psi$ and $\psi \models \phi$.

2.2 Canonical formulas

In this subsection, we introduce *canonical formulas* (CFs) proposed by Moss [19], explain why CFs satisfiable in KD45n can be treated as models for formulas, and introduce a property showing that satisfiable CFs contain redundant information.

We first introduce the *cover* modality. Intuitively, $\nabla_a\Phi$ means that each world considered possible by agent a satisfies an element of Φ, and each element of Φ is satisfied by some world considered possible by agent a. We use "$\doteq$" to mean "is defined as".

Definition 5 (Cover modality) Let $a \in \mathcal{A}$, and let Φ a finite set of formulas. $\nabla_a\Phi \doteq B_a(\bigvee\Phi) \wedge \bigwedge_{\phi\in\Phi} \hat{B}_a\phi$.

Note that in KD45, $B_a\phi \models \hat{B}_a\phi$, so $B_a\phi \Leftrightarrow \nabla_a\{\phi\}$.

Definition 6 ([19] Canonical formulas) The set E_k of depth-k canonical formulas (CFs) can be inductively defined as:

- $E_0 = \{\bigwedge_{p\in S} p \wedge \bigwedge_{p\in\mathcal{P}\setminus S} \neg p \mid S \subseteq \mathcal{P}\}$, *i.e.*, E_0 is the set of minterms;
- $E_{k+1} = \{\tau_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Phi_a \mid \tau_0 \in E_0 \textit{ and } \Phi_a \subseteq E_k\}$.

Let $\tau = \tau_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Phi_a$. We denote τ_0 by $w(\tau)$, and call it the world of τ; we denote Φ_a by $R_a(\tau)$, and call it the set of the a-children of τ.

Example 2 Let $\mathcal{A} = \{a, b\}$, and $\mathcal{P} = \{p, q\}$. We use truth assignments to represent minterms. Let $\tau_a = 00 \wedge \nabla_a\{00 \wedge \nabla_a\{00\} \wedge \nabla_b\{01\}\} \wedge \nabla_b\{01 \wedge \nabla_a\{01\} \wedge \nabla_b\{01\}\}$, and let $\tau_b = 11 \wedge \nabla_a\{10 \wedge \nabla_a\{10\} \wedge \nabla_b\{00\}\} \wedge \nabla_b\{11 \wedge \nabla_a\{10\} \wedge \nabla_b\{11\}\}$. Then $\tau = 00 \wedge \nabla_a\{\tau_a\} \wedge \nabla_b\{\tau_b\}$ is a CF and can be represented as the tree in Figure 1, where we mark *alternating paths*, *i.e.*, paths on which any adjacent agents are different, with dashed arrows.

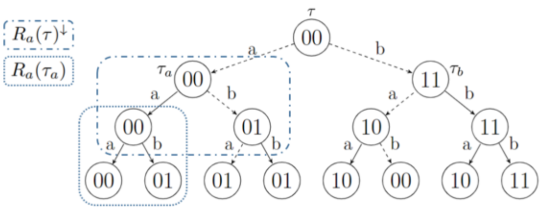

Figure 1. An example of CFs

We remark that the number and size of canonical formulas are non-elementary in the number of atoms [19].

Note that some CFs, *e.g.*, $p \wedge \nabla_a\{\}$, are unsatisfiable in KD45n. Clearly, if a CF is satisfiable, then for each $\nabla_a\Phi$ subformula, $\Phi \neq \emptyset$.

The following two propositions show that if $md(\phi) \leq k$, we can treat depth-k satisfiable CFs as the models for ϕ since any such CF entails either ϕ or its negation and ϕ is equivalent to the disjunction of such CFs entailing ϕ. A nice property is that there are finitely many such models.

Proposition 1 *[19] Let τ be a depth-k satisfiable CF, and $md(\phi) \leq k$. Then either $\tau \models \phi$ or $\tau \models \neg\phi$.*

Proposition 2 *[19] Let $k \geq md(\phi)$. Let Φ be the set of depth-k satisfiable CFs τ s.t. $\tau \models \phi$. Then $\phi \Leftrightarrow \bigvee\Phi$.*

However, satisfiable CFs contain redundant information because of transitivity of accessibility relations in KD45n. For example, in τ_a of Example 2, the 00 in $\nabla_a\{00\}$ can be copied from the 00 in $\nabla_a\{00 \wedge \ldots\}$. Below we introduce the identical children property of satisfiable CFs: for any agent a and any a-child τ_a of a satisfiable CF τ, the a-children of τ_a can be copied from τ's a-children. For example, in Figure 1, as shown by the dashed boxes, the a-children of τ_a can be copied from τ's a-children.

We need some definitions first. Fang *et al.* [13] introduced the projection operations on CFs. When a CF τ is represented as a tree, the operation $\tau^{\downarrow}$ prunes the leaves of this tree, and $\tau^{\downarrow l}$ prunes the bottom l levels of the tree.

Definition 7 Let $k > 0$. For $\tau \in E_k$, $\tau^{\downarrow} = w(\tau)$, if $k = 1$, otherwise, $\tau^{\downarrow} = w(\tau) \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a R_a(\tau)^{\downarrow}$. Here $\Phi^{\downarrow}$ is $\{\phi^{\downarrow} \mid \phi \in \Phi\}$.

Definition 8 Let $\tau \in E_k$ and $0 < l \leq k$. Then $\tau^{\downarrow l} = \tau^{\downarrow}$, if $l = 1$, otherwise, $\tau^{\downarrow l} = (\tau^{\downarrow l-1})^{\downarrow}$.

Proposition 3 *[13] Let τ be satisfiable CF with depth ≥ 2. Then, for all $a \in \mathcal{A}$ and $\tau_a \in R_a(\tau)$, $R_a(\tau_a) = R_a(\tau)^{\downarrow}$.*

Note that [13] shows the proposition for K45.

2.3 ACDFs and proper ACDFs

Hales *et al.* [14] introduced the notion of *alternating cover disjunctive formulas* (ACDFs), a special form of *cover disjunctive formulas*, and showed that in KD45n, every formula in $\mathcal{L}_B$ can be transformed into an equivalent ACDF.

Definition 9 (CDF) The set of cover disjunctive formulas (CDFs) is inductively defined as follows:

- A propositional term is a CDF;
- If ϕ_0 is a disjunction of propositional terms, and for each $a \in \mathcal{B} \subseteq \mathcal{A}$, Φ_a is a finite set of CDFs, then $\phi_0 \wedge \bigwedge_{a\in\mathcal{B}} \nabla_a\Phi_a$ is a CDF, called a CDF term;
- The disjunction of a finite set of CDF terms is a CDF.

Definition 10 (Alternating formulas) We say that a formula is $alternating$ if it has the property that modal operators of an agent do not directly occur inside those of the same agent.

Definition 11 (ACDF) We call an alternating CDF an alternating cover disjunctive formula (ACDF).

For example, $\nabla_a\{\top, \nabla_a p\}$ is a CDF but not an ACDF, while $\nabla_b\{\top, \nabla_a p\}$ is an ACDF.

Below is a characterization of satisfiability of ACDFs, which will be used to show each ACCF is satisfiable.

Proposition 4 *[24] An ACDF term $\delta = \phi_0 \wedge \bigwedge_{a\in\mathcal{B}} \nabla_a\Phi_a$ is satisfiable iff the following hold:*

1. *ϕ_0 is propositionally satisfiable;*
2. *for each $a \in \mathcal{B}$, Φ_a is not empty;*
3. *for each $a \in \mathcal{B}$, for each $\phi \in \Phi_a$, ϕ is satisfiable.*

Wan *et al.* [24] proposed a fragment of ACDFs called *proper ACDFs*. Intuitively, proper ACDFs only allow negation and disjunction for propositional formulas, *i.e.*, they disallow negative or disjunctive beliefs.

Definition 12 (Proper ACDF) An ACDF ϕ is $proper$ if in any $\nabla_a\Phi_a$ subformula, Φ_a must be a singleton, and in ϕ disjunctions can only be used for propositional formulas.

For example, $\nabla_a\{\neg p \vee q\}$, $\nabla_a\{\nabla_b\{p\}\}$ are both proper ACDFs, but $\nabla_a\{\neg p \vee \nabla_b\{p\}\}$ is not.

Below we introduce their result that any proper ACDF is equivalent to a conjunction of beliefs of propositional formulas along paths. This result will be used in the syntactic characterization of our revision for proper ACDFs.

Let p be a path of agents $a_1, a_2, \ldots, a_n$. They use $B_p\phi$ to abbreviate for $B_{a_1}B_{a_2}...B_{a_n}\phi$. In case p is the empty path, $B_p\phi$ simply represents ϕ. They call p an *alternating path* if any adjacent agents on the path are different. They showed:

Proposition 5 *[24] Any proper ACDF can be equivalently transformed to a formula of the form $\bigwedge_{p\in P} B_p\phi_p$, where P is a set of alternating paths, and each ϕ_p is propositional.*

2.4 Dalal's propositional revision

Katsuno and Mendelzon [17] presented a model-theoretic definition of revision: $\phi \circ \mu$ selects from the models of μ those that are closest to models of ϕ. Take the notion of closeness based on cardinality, *i.e.*, a model I is closer to a model M than a model J if $H(I, M) \leq H(J, M)$, where $H(I, M)$ is the Hamming distance between I and M. Then we get Dalal's revision operator $\circ_d$ [8].

Now we introduce del Val's syntactic characterization of Dalal's revision, using somewhat different notation from there. We assume that ϕ and μ are both in DNF (disjunctive normal form); we use ϕ with subscripts to denote the disjuncts of ϕ, and similarly for μ. For two propositional terms t_1 and t_2, $Dist(t_1, t_2)$ is the number of atoms that appears positively in one term but negatively in another. Let $MinDist(\phi, \mu) = \min\{Dist(\phi_i, \mu_j) \mid \phi_i \in \phi, \mu_j \in \mu\}$.

Theorem 1 *(Theorem 4 in [9])*

$$\phi \circ_d \mu \Leftrightarrow \bigvee_{\langle \phi_i, \mu_j \rangle \text{ s.t. } Dist(\phi_i, \mu_j) = MinDist(\phi, \mu)} \phi_i \circ_d \mu_j,$$

where $\phi_i \circ_d \mu_j$ is obtained from ϕ_i by removing those literals whose complement occurs in μ_j and then union with μ_j.

The above theorem says that the revision of DNF formulas can be reduced to the revision of terms, which we use a simple example to illustrate. Let $\phi = p \wedge q$ and $\mu = \neg p \wedge r$. Then models of ϕ are $\{p, q, r\}$ and $\{p, q, \neg r\}$, and models of μ are $\{\neg p, q, r\}$ and $\{\neg p, \neg q, r\}$. So the minimum distance of $\{\neg p, q, r\}$ to models of ϕ is 1, and that for $\{\neg p, \neg q, r\}$ is 2. Thus $\phi \circ_d \mu$ has a single model $\{\neg p, q, r\}$. Hence $\phi \circ_d \mu \Leftrightarrow \neg p \wedge q \wedge r$. This result can be simply obtained with the method given in Theorem 1.

3 ACFs, ACCFs and their distances

In this section, we define two new fragments of ACDFs. One is *alternating canonical formulas* (ACFs). The other is *alternating cover conjunctive formulas* (ACCFs), and ACFs are special ACCFs. We define a notion of distance between ACCFs, and show that the minimal distance between models of two ACCFs is simply equal to the distance between the two ACCFs.

3.1 ACFs

In this subsection, we introduce ACFs, and show that we can treat ACFs as models for formulas. Finally, we introduce the concept of expansions of ACFs, which will be used to show that the result of our model-theoretic definition of revision is independent of the depth of models we use as long as it is sufficiently big.

To adapt distance-based belief revision from propositional logic to epistemic logics, it is important to determine what are the models for formulas and define distances between them. Though satisfiable CFs can be treated as models for formulas, as pointed out in the last section, they contain redundant information which complicates the definition of the distance. To resolve the issue, we define *alternating canonical formulas* (ACFs) based on CFs. An ACF can be treated as a maximal alternating subformula of a CF. An ACF can be obtained from a CF as follows: For each $\nabla_a \Phi$ subformula, and for each $\phi \in \Phi$, remove its $\nabla_a \Phi'$ conjunct.

Below we give a formal definition of ACFs by defining a mapping m from satisfiable CFs. Then we will show that m is a bijection between satisfiable CFs and ACFs when $|\mathcal{A}| > 1$. We begin with a mapping m_a for $a \in \mathcal{A}$:

Definition 13 For $a \in \mathcal{A}$, a satisfiable CF τ, $m_a(\tau)$ is defined as follows: 1. If $\tau \in E_0$, $m_a(\tau) = \tau$;
2. If $\tau = \tau_0 \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i \in E_{k+1}$, $m_a(\tau) = \tau_0 \wedge \bigwedge_{i \in \mathcal{A}, i \neq a} \nabla_i m_i(\Phi_i)$, where $m_i(\Phi_i) = \{m_i(\phi) \mid \phi \in \Phi_i\}$.

Definition 14 For a satisfiable CF τ, $m(\tau)$ is defined as follows: 1. If $\tau \in E_0$, $m(\tau) = \tau$; 2. If $\tau = \tau_0 \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i \in E_{k+1}$, $m(\tau) = \tau_0 \wedge \bigwedge_{i \in A} \nabla_i m_i(\Phi_i)$.

Definition 15 (ACF) An alternating canonical formula (ACF) is $m(\tau)$ for some satisfiable CF τ.

Example 3 Recall τ in Example 2. Let $\delta = m(\tau)$, then $\delta = 00 \wedge \nabla_a\{00 \wedge \nabla_b\{01 \wedge \nabla_a\{01\}\}\} \wedge \nabla_b\{11 \wedge \nabla_a\{10 \wedge \nabla_b\{00\}\}\}$, as shown in Figure 2.

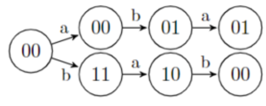

Figure 2. An example of ACFs

Note that when $|\mathcal{A}| = 1$, for any satisfiable CF τ, the modal depth of $m(\tau)$ is 0 or 1. For example, let $\mathcal{A} = \{a\}$, $\tau = \neg p \wedge \nabla_a\{p \wedge \nabla_a\{p \wedge \nabla_a\{p\}\}\}$, then τ is satisfiable and $m(\tau) = \neg p \wedge \nabla_a\{p\}$.

Proposition 6 *For any satisfiable CF τ, $\tau \models m(\tau)$.*

Proof: It is easy to prove for any $a \in \mathcal{A}$ and satisfiable CF τ, $\tau \models m_a(\tau)$, by induction on τ. Then the property holds. ■

Proposition 7 *If $|\mathcal{A}| > 1$, then for each ACF δ, there is a unique satisfiable CF τ s.t. $\tau \models \delta$ and $m(\tau) = \delta$. If $|\mathcal{A}| = 1$, then for each ACF δ s.t. $md(\delta) = 1$ and each $k \geq 1$, there is a unique depth-k satisfiable CF τ s.t. $\tau \models \delta$ and $m(\tau) = \delta$.*

Proof: The idea is to expand δ into τ by using the identical-children property of satisfiable CFs (Proposition 3), in a bottom-up approach, starting from the innermost subformulas. We will build a unique depth-k satisfiable CF, where k is given when $|\mathcal{A}| = 1$, and $k = md(\delta)$ when $|\mathcal{A}| > 1$.

When $k \leq 1$, an ACF is also a CF, so nothing needs to be done. Now let $k > 1$. Suppose that $\delta = \delta_0 \wedge \bigwedge_{a \in \mathcal{A}} \nabla_a \Phi_a$. We will construct $\tau = \delta_0 \wedge \bigwedge_{a \in \mathcal{A}} \nabla_a \Psi_a$ so that $m(\tau) = \delta$.

We begin with a notation: a set of CFs with a superscript s means the CFs have depth s. So Ψ_a can be rewritten as Ψ_a^{k-1}. Also recall $\tau^{\downarrow l}$ denotes the lth projection of τ. Now suppose $\Psi_a^{k-1} = \{\nabla_a \Psi_1 \wedge \phi_1, ..., \nabla_a \Psi_n \wedge \phi_n\}$. By Proposition 3, $\Psi_1 = ... = \Psi_n = (\Psi_a^{k-1})^{\downarrow} = \{\nabla_a \Psi_1^{\downarrow} \wedge \phi_1^{\downarrow}, ..., \nabla_a \Psi_n^{\downarrow} \wedge \phi_n^{\downarrow}\}$. We denote this set by Ψ_a^{k-2} and get $\Psi_a^{k-1} = \{\nabla_a \Psi_a^{k-2} \wedge \phi_i \mid i = 1, ..., n\}$. When we continue to do this, we will have:

- $\Psi_a^{k-1} = \{\nabla_a \Psi_a^{k-2} \wedge \phi_i \mid i = 1, ..., n\}$
- $\Psi_a^{k-2} = (\Psi_a^{k-1})^{\downarrow} = \{\nabla_a \Psi_a^{k-3} \wedge \phi_i^{\downarrow} \mid i = 1, ..., n\}$
- ...
- $\Psi_a^1 = (\Psi_a^{k-1})^{\downarrow k-2} = \{\nabla_a \Psi_a^0 \wedge \phi_i^{\downarrow k-2} \mid i = 1, ..., n\}$

Now we can get $\Psi_a^0 = (\Psi_a^{k-1})^{\downarrow k-1}$=$\{w(\phi_i) \mid i = 1, ..., n\}$, which is given by δ. Note that $\phi_i = \phi_{i0} \wedge \bigwedge_{b \in (\mathcal{A} - \{a\})} \nabla_b \Psi'_{ib}$, where each Ψ'_{ib} is a set of CFs of depth $k-2$. By induction, each Ψ'_{ib} can be rebuilt from $R_b(\delta_i)$ for $\delta_i \in R_a(\delta)$. So we get Ψ_a^0, then Ψ_a^1, Ψ_a^2, ..., and at last Ψ_a^{k-1}. Thus, we can rebuild a CF from an ACF. ■

Theorem 2 *If $|\mathcal{A}| > 1$, then m is a bijection between satisfiable CFs and ACFs, and $m(\tau) \Leftrightarrow \tau$. If $|\mathcal{A}| = 1$, then m is a surjection from satisfiable CFs to ACFs, and $m(\tau) \Leftrightarrow \tau$.*

Proof: Let τ_1, τ_2 be two satisfiable CFs s.t. $m(\tau_1) = m(\tau_2)$. By Prop. 6, $\tau_1 \models m(\tau_1)$ and $\tau_2 \models m(\tau_2)$. By Prop. 7, when $|\mathcal{A}| > 1$, $\tau_1 = \tau_2$, thus m is an injection. By Prop. 7, m is onto. Since $\tau \models m(\tau)$ and there is a unique depth-$md(\tau)$ satisfiable CF τ' s.t. $\tau' \models m(\tau)$, by Prop. 2, $m(\tau) \Leftrightarrow \tau$. ■

We have explained in Section 2.2 that if $md(\phi) \leq k$, depth-k satisfiable CFs can be treated as models for ϕ. By Theorem 2, we can also use depth-k ACFs as models for ϕ.

Notation 1 *We use S_k to denote the set of depth-k ACFs. Let $\delta \in S_k$, and $md(\phi) \leq k$. If $\delta \models \phi$, we call δ a depth-k model of ϕ. We let $S_k(\phi) = \{\delta \in S_k \mid \delta \models \phi\}$.*

Then $\phi \Leftrightarrow \bigvee S_k(\phi)$. To see this, let Φ be the set of depth-k CFs τ s.t. $\tau \models \phi$. By Proposition 2, $\phi \Leftrightarrow \bigvee \Phi$. Then by Theorem 2, $\phi \Leftrightarrow \bigvee \Phi \Leftrightarrow \bigvee S_k(\phi)$.

Now we introduce a method which will often be used in the proofs of this paper. Suppose an a-child δ_a of an ACF satisfies $\phi \in \mathcal{L}_B^{-a}$, *i.e.*, ϕ does not use B_a as an outmost modal operator. Since δ_a is not an ACF, how do we get an ACF satisfying ϕ?

Let $\delta = \delta_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a \Phi_a \in S_k$, $\delta_a \in \Phi_a$, $\phi \in \mathcal{L}_B^{-a}$, $md(\phi) \leq k-1$, and $\delta_a \models \phi$. δ_a is not an ACF. However, we can make $\delta_a^* \in S_{k-1}(\phi)$ from δ_a. Let $\delta'_a \in S_{k-1}$. We let $\delta_a^* = \delta_a \wedge \nabla_a R_a(\delta'_a)$, *i.e.*, we conjoin the ∇_a part of δ'_a to δ_a. Then $\delta_a^* \in S_{k-1}$ and $\delta_a^* \models \phi$ since $\phi \in \mathcal{L}_B^{-a}$.

Below we give an algorithm to check if an ACF entails an alternating formula. There's a similar property of CFs and the idea of our proof is similar to that one, except we deal with alternating paths.

Proposition 8 *Let $\delta \in S_k$, ϕ an alternating formula, and $md(\phi) \leq k$. Then we can check if $\delta \models \phi$ recursively:*

- *$\delta \models p$ iff p appears positively in $w(\delta)$;*
- *$\delta \models \neg\phi$ iff $\delta \not\models \phi$; $\delta \models \phi \wedge \psi$ iff $\delta \models \phi$ and $\delta \models \psi$;*
- *$\delta \models B_a\phi$ iff for all $\delta' \in R_a(\delta)$, $\delta' \models \phi$.*

Proof: Firstly we show: like Proposition 1 in Section 2.2, let $\delta \in S_k$ and $md(\phi) \leq k$, then either $\delta \models \phi$ or $\delta \models \neg\phi$. Secondly we make use of this property to prove this proposition.

Step 1: Let $\delta \in S_k$ and $md(\phi) \leq k$. By Prop. 7, there is a unique depth-k satisfiable CF τ s.t. $\tau \models \delta$. By Prop. 2, $\delta \Leftrightarrow \tau$. By Prop. 1, either $\tau \models \phi$ or $\tau \models \neg\phi$. Hence either $\delta \models \phi$ or $\delta \models \neg\phi$.

Step 2: The cases of atom and conjunction are trivial. The case of negation follows from Step 1. We now prove the case of knowledge. $\Leftarrow$: Suppose for all $\delta' \in R_a(\delta), \delta' \models \phi$. Then $\bigvee R_a(\delta) \models \phi$, so $\delta \models \nabla_a R_a(\delta) \models B_a \bigvee R_a(\delta) \models B_a\phi$. $\Rightarrow$: Suppose $\delta \models B_a\phi$. Assume there is $\delta' \in R_a(\delta)$ s.t. $\delta' \not\models \phi$. Note that $B_a\phi$ is alternating, thus $\phi \in \mathcal{L}_B^{-a}$. By using the method introduced before Proposition 8, we expand δ' to get an ACF δ'' s.t. $\delta' \models \phi$ iff $\delta'' \models \phi$. So $\delta'' \not\models \phi$. By Step 1, $\delta'' \models \neg\phi$, so $\delta' \models \neg\phi$. Then $\delta \models \nabla_a R_a(\delta) \models \hat{B}_a\neg\phi$, contradicting $\delta \models B_a\phi$. Thus for all $\delta' \in R_a(\delta)$, $\delta' \models \phi$. ■

In the above, we have explained we use depth-k ACFs as models for ϕ as long as $md(\phi) \leq k$. In this paper, we will give a model-theoretic definition of the revision of ϕ by μ, and we will show that the result is independent of the depth k of models we use as long as $k \geq \max\{md(\phi), md(\mu)\}$. To this end, we need the notion that an ACF is the expansion of another one. Expansion is defined via projection, which can be similarly defined for ACFs as for CFs.

Definition 16 Let δ and δ' be ACFs. If $\delta'^{\downarrow l} = \delta$ for some $l > 0$, we call δ' an expansion of δ.

Proposition 9 *Let $\delta \in S_k$, and $k' > k$. Let Δ be the set of all depth-k' expansions of δ. Then $\delta \Leftrightarrow \bigvee \Delta$.*

Proof: Let $\delta' \in S_{k'}$. Clearly, $\delta' \models \delta$ iff δ' is an expansion of δ. Thus $\Delta = \{\delta' \in S_{k'} \mid \delta' \models \delta\} = S_{k'}(\delta)$. So $\delta \Leftrightarrow \bigvee S_{k'}(\delta) \Leftrightarrow \bigvee \Delta$. ■

Proposition 10 *Let $md(\phi) = k$, and δ' an ACF of depth $\geq k$. Then $\delta' \models \phi$ iff there is $\delta \in S_k$ s.t. $\delta \models \phi$ and δ' is an expansion of δ.*

Proof: We have $\phi \Leftrightarrow \bigvee S_k(\phi)$. Since δ' is an ACF with depth $\geq k$, for each $\delta \in S_k(\phi)$, either $\delta' \models \delta$ or $\delta' \models \neg\delta$. Thus $\delta' \models \phi$ iff $\delta' \models \bigvee S_k(\phi)$ iff there is $\delta \in S_k(\phi)$ s.t. $\delta' \models \delta$ iff there is $\delta \in S_k$ s.t. $\delta \models \phi$ and δ' is an expansion of δ. ■

3.2 ACCFs and their distances

In this subsection, we propose a new fragment of ACDF – ACCF, and define distances between two ACCFs. Since ACFs are special ACCFs, the notion can be applied to ACFs, and we show that it has the basic properties of distance.

Definition 17 (ACCF) An alternating cover conjunctive formula (ACCF) is an ACDF without using disjunctions, even in the propositional parts of the formula, s.t. for any CDF term subformula $\phi_0 \wedge \bigwedge_{a\in\mathcal{B}} \nabla_a \Phi_a$, $\mathcal{B} \subseteq \mathcal{A}$, ϕ_0 is a satisfiable term, and each Φ_a is not empty.

For example, $\nabla_a\{p, q \wedge \nabla_b\{p \wedge q\}\}$ is an ACCF. By Prop. 4, each ACCF is satisfiable. An ACCF can be viewed as the analogy of a satisfiable term in propositional logic.

Note that ACFs are special ACCFs: First, an ACF does not use disjunctions. Second, for any CDF term subformula $\phi_0 \wedge \bigwedge_{a\in B} \nabla_a \Phi_a$ of an ACF, ϕ_0 is a minterm and is thus satisfiable, Φ_a is obtained from the corresponding part of a CF satisfiable in KD45, and is thus non-empty.

Figure 3 shows relationships between fragments we introduce in this paper, where arrows denote subset inclusions.

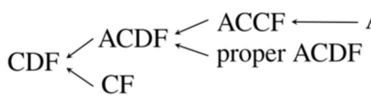

Figure 3. Relationships between fragments

To define distances between two ACCFs, we make use of the Hausdorff distance between two sets.

Definition 18 (Hausdorff distance) Let Φ and Φ' be two sets. Let $d(\phi, \phi')$ be the distance between $\phi \in \Phi$ and $\phi' \in \Phi'$. Then the Hausdorff distance between Φ and Φ' based on d is defined as:

$$d(\phi, \Phi) = min\{d(\phi, \phi')|\phi' \in \Phi\};$$
$$\mathcal{H}_d(\Phi, \Phi') = max(\{d(\phi, \Phi')|\phi \in \Phi\} \cup \{d(\phi', \Phi)|\phi' \in \Phi'\}).$$

For convenience, we use $d(\Phi, \Phi')$ to denote $\mathcal{H}_d(\Phi, \Phi')$.

Figure 4 (a) shows the intuition behind the above definition. There are two curves c_1 and c_2. The length of l_1 is the distance between w_1 and c_2, and the length of l_2 is the distance between w_2 and c_1. The Hausdorff distance between c_1 and c_2 is the maximum of the length of all such lines, which is equal to the length of l_1.

Definition 19 (Distances between ACCFs) The distance between two ACCFs is inductively defined as follows:

1. If δ and δ' are propositional terms, then $dist(\delta, \delta')$ is the number of atoms that appears positively in one term but negatively in another term.
2. If $\delta = \delta_0 \wedge \bigwedge_{a\in\mathcal{B}} \nabla_a\Phi_a$, $\delta' = \delta'_0 \wedge \bigwedge_{a\in\mathcal{B}'} \nabla_a\Phi'_a$, then

$$dist(\delta, \delta') = dist(\delta_0, \delta'_0) + \sum_{a\in\mathcal{B}\cap\mathcal{B}'} dist(\Phi_a, \Phi'_a).$$

Figure 4 (b) shows the intuition behind the above definition. There are two ACCFs δ_1 and δ_2. The distance between δ_1 and δ_2 is the sum of the distance between w_1 and w_2 and the Hausdorff distance between Φ_a and Φ'_a, Φ_b and Φ'_b, and Φ_c and Φ'_c.

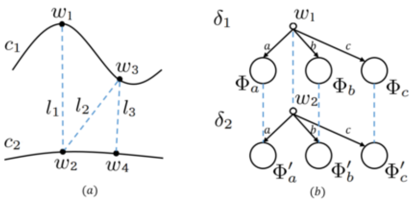

Figure 4. Intuitions of distances

Since ACFs are special ACCFs, Definition 19 applies to ACFs. Note that in Definition 19, we do not require two ACCFs have the same modal depth. Actually, the distance between ACCFs is a pseudometric, which has the same properties as a metric except allows the distance between two different points to be zero. However, in the following result, we require two ACFs have the same modal depth.

Proposition 11 *The distance between two ACFs with the same modal depth k has the following basic properties:*

1. *Nonnegativity:* $dist(\delta, \delta') \geq 0$*;*
2. *Indistinguishability:* $dist(\delta, \delta') = 0$ *iff* $\delta = \delta'$*;*
3. *Symmetry:* $dist(\delta, \delta') = dist(\delta', \delta)$*;*
4. *Subadditivity:* $dist(\delta, \delta'') \leq dist(\delta, \delta') + dist(\delta', \delta'')$.

3.3 *Mindist between formulas*

To adapt distance-based belief revision, when revising a formula ϕ by a formula μ, we select from models of μ those that are closest to models of ϕ. In this subsection, we first define the minimal distance between models of two formulas, *i.e.*, the mindist between them. Then we show the mindist between two ACCFs is simply equal to their distance. Note that $mindist$ is obtained by calculating the distances of all pairs of models of the formulas, but $dist$ can be calculated directly from the formulas. Hence we give a syntactic characterization of the semantic concept of mindist. This result will be used in the syntactic characterizations of our revision for ACCFs and ACDFs.

Definition 20 Let ϕ and ϕ' be two satisfiable formulas, and $k \geq \max(md(\phi), md(\phi'))$. We define $mindist_k(\phi, \phi')$ as

$$\min\{dist(\delta, \delta') \mid \delta \in S_k(\phi), \delta' \in S_k(\phi')\}.$$

Below we prove that $mindist_k(\phi, \phi')$ is independent of the k we use as long as $k \geq \max\{md(\phi), md(\phi')\}$, hence we can omit the subscript, and consider $S_k(\phi)$ and $S_k(\phi')$ where $k = \max\{md(\phi), md(\phi')\}$ in case we do not specify the value of k. Before giving Theorem 3, we introduce two propositions used in its proof: the first says the distance between any two depth-k' expansions of δ_1 and δ_2 is at least $dist(\delta_1, \delta_2)$; the second says for any depth-k' expansion of δ_1, there is a depth-k' expansion of δ_2 s.t. their distance is equal to $dist(\delta_1, \delta_2)$.

Proposition 12 *Let* $k \leq k'$, $\delta_1, \delta_2 \in S_k$, $\delta'_1, \delta'_2 \in S_{k'}$, δ'_i *is an expansion of* δ_i, $i = 1, 2$. *Then* $dist(\delta'_1, \delta'_2) \geq dist(\delta_1, \delta_2)$.

Proof: We prove by induction on k. Base: $k = 0$, obviously holds. If $k > 0$, $dist(w(\delta'_1), w(\delta'_2)) = dist(w(\delta_1), w(\delta_2))$. Now we prove that for each $a \in \mathcal{A}$, $dist(R_a(\delta'_1), R_a(\delta'_2)) \geq dist(R_a(\delta_1), R_a(\delta_2))$. For each $\varphi'_1 \in R_a(\delta'_1)$ and $\varphi'_2 \in R_a(\delta'_2)$, let $\delta_1^{*'}$ be an ACF rebuilt from φ'_1, by adding the outmost ∇_a subformula. Note that φ'_1, φ'_2 does not use ∇_a as an outmost operator, so it is obvious that we can construct $\delta_2^{*'} = \varphi'_2 \wedge \nabla_a R_a(\delta_1^{*'})$. Now $\delta_1^{*'}, \delta_2^{*'}$ are in $S_{k'-1}$, and $dist(\delta_1^{*'}, \delta_2^{*'}) = dist(\varphi'_1, \varphi'_2)$. Similarly, for each $\varphi_1 \in R_a(\delta_1)$ and $\varphi_2 \in R_a(\delta_2)$, we can find δ_1^*, δ_2^* in S_{k-1}, that $dist(\delta_1^*, \delta_2^*) = dist(\varphi_1, \varphi_2)$. We rebuild φ_1, φ_2 with $(\delta_1^{*'})^{\downarrow k'-k}$ so that $\delta_i^{*'}$ will be an expansion of δ_i^*, $i = 1, 2$. By induction, $dist(\delta_1^{*'}, \delta_2^{*'}) \geq dist(\delta_1^*, \delta_2^*)$, so $dist(\varphi'_1, \varphi'_2) \geq dist(\varphi_1, \varphi_2)$ for all $\varphi'_i \in R_a(\delta'_i)$ and $\varphi_i \in R_a(\delta_i)$, $i = 1, 2$. So $dist(R_a(\delta'_1), R_a(\delta'_2)) \geq dist(R_a(\delta_1), R_a(\delta_2))$. ■

Proposition 13 *Let* $k \leq k'$, $\delta_1, \delta_2 \in S_k$. *Then for any* $\delta'_1 \in S_{k'}$ *s.t.* δ'_1 *is an expansion of* δ_1, *there exists* $\delta'_2 \in S_{k'}$ *s.t.* δ'_2 *is an expansion of* δ_2, *and* $dist(\delta'_1, \delta'_2) = dist(\delta_1, \delta_2)$.

Proof: For each alternating path p of length k, for each leaf f_1 of δ_1 and each leaf f_2 of δ_2 reachable from the root by path p, let δ'_2 expand f_2 in the same way as δ'_1 expands f_1, *i.e.*, the subtree of δ'_2 rooted at f_2 is the same as that of δ'_1 rooted at f_1 except for f_1 and f_2. We show $dist(\delta'_1, \delta'_2) = dist(\delta_1, \delta_2)$ by induction on k. ■

Theorem 3 *Let* $k, k' \geq \max(md(\phi), md(\phi'))$. *Then* $mindist_k(\phi, \phi') = mindist_{k'}(\phi, \phi')$.

Proof: We first prove $mindist_k(\phi, \phi') \leq mindist_{k'}(\phi, \phi')$. By Prop. 10, any depth-k' model δ'_1 (resp. δ'_2) of ϕ (resp. ϕ') is an expansion of some depth-k model δ_1 (resp. δ_2) of ϕ (resp. ϕ'). By Prop. 12, $dist(\delta'_1, \delta'_2) \geq dist(\delta_1, \delta_2)$. We now prove $mindist_k(\phi, \phi') \geq mindist_{k'}(\phi, \phi')$. Suppose $mindist_k(\phi, \phi') = dist(\delta_1, \delta_2)$. By Prop. 13, there are $\delta'_1, \delta'_2 \in S_{k'}$ s.t. δ'_i is an expansion of δ_i, $i = 1, 2$, and $dist(\delta'_1, \delta'_2) = dist(\delta_1, \delta_2)$. ■

The following result shows the mindist between two ACCFs is simply equal to their distance.

Theorem 4 *Let* ϕ *and* ϕ' *be two ACCFs. Then* $dist(\phi, \phi') = mindist(\phi, \phi')$.

Proof: Let $k = md(\phi)$, $k' = md(\phi')$. Suppose $k \leq k'$. We prove by induction on k. Base: $k = 0$. The proof is easy. Induction: $k > 0$. Let $\phi = \phi_0 \wedge \bigwedge_{a\in\mathcal{B}} \nabla_a\Phi_a$, $\phi' = \phi'_0 \wedge \bigwedge_{a\in\mathcal{B}'} \nabla_a\Phi'_a$. We first prove $mindist(\phi, \phi') \leq dist(\phi, \phi')$. Let $\delta = \delta_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Psi_a \in S_{k'}(\phi)$ and $\delta' = \delta'_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Psi'_a \in S_{k'}(\phi')$. Let $\delta_0^* \in S_0(\phi_0)$ and $\delta_0^{*'} \in S_0(\phi'_0)$ s.t. $dist(\delta_0^*, \delta_0^{*'}) = dist(\phi_0, \phi'_0)$. We let δ^* be the same as δ except $w(\delta^*) = \delta_0^*$ and for $a \in \mathcal{B}\cap\mathcal{B}'$, $R_a(\delta^*) = \Psi_a^*$ which we define in the following. Let $\delta^{*'}$ be the same as δ' except $w(\delta^{*'}) = \delta_0^{*'}$, for $a \notin \mathcal{B}\cup\mathcal{B}'$, $R_a(\delta^{*'}) = \Psi_a$, and for $a \in \mathcal{B}\cap\mathcal{B}'$, $R_a(\delta^{*'}) = \Psi_a^{*'}$ which we define now. Let $a \in \mathcal{B}\cap\mathcal{B}'$. For each $\phi_a \in \Phi_a$, let $\phi'_a \in \Phi'_a$ s.t. $dist(\phi_a, \phi'_a) = dist(\phi_a, \Phi'_a)$. By induction, $mindist(\phi_a, \phi'_a) = dist(\phi_a, \phi'_a)$. Thus there exist $\delta_a \in S_{k'-1}(\phi_a)$

and $\delta'_a \in S_{k'-1}(\phi'_a)$ s.t. $dist(\delta_a, \delta'_a) = dist(\phi_a, \phi'_a)$. Remove the ∇_a parts from δ_a and δ'_a, add the results to Ψ^*_a and $\Psi^{*\prime}_a$, respectively. Do the above similar for Φ'_a. Then $\delta^* \in S_{k'}(\phi)$, $\delta^{*\prime} \in S_{k'}(\phi')$, and for $a \in \mathcal{B} \cap \mathcal{B}'$, $dist(\Psi^*_a, \Psi^{*\prime}_a) = dist(\Phi_a, \Phi'_a)$, hence $dist(\delta^*, \delta^{*\prime}) = dist(\phi, \phi')$. Thus $mindist(\phi, \phi') \leq dist(\phi, \phi')$.

We now prove $mindist(\phi, \phi') \geq dist(\phi, \phi')$. Let $\delta = \delta_0 \wedge \bigwedge_{a \in \mathcal{A}} \nabla_a \Psi_a \in S_{k'}(\phi)$ and $\delta' = \delta'_0 \wedge \bigwedge_{a \in \mathcal{A}} \nabla_a \Psi'_a \in S_{k'}(\phi')$. We show that $dist(\delta, \delta') \geq dist(\phi, \phi')$. Obviously, $dist(\delta_0, \delta'_0) \geq dist(\phi_0, \phi'_0)$. So it suffices to show that for each $a \in \mathcal{B} \cap \mathcal{B}'$, $dist(\Psi_a, \Psi'_a) \geq dist(\Phi_a, \Phi'_a)$. Note $dist(\Psi_a, \Psi'_a) = \max(\{dist(\psi_a, \Psi'_a) | \psi_a \in \Psi_a\} \cup \{dist(\psi'_a, \Psi_a) | \psi'_a \in \Psi'_a\})$. To prove $\max(S_1) \geq \max(S_2)$, we prove that for each $s_2 \in S_2$ there is $s_1 \in S_1$ s.t. $s_2 \leq s_1$. For each $\phi_a \in \Phi_a$, there is $\psi_a \in \Psi'_a$ s.t. $\psi_a \models \psi'_a$. We show that $dist(\psi_a, \Psi'_a) \geq dist(\phi_a, \Phi'_a)$. Note $dist(\psi_a, \Psi'_a) = \min(\{dist(\psi_a, \psi'_a) | \psi'_a \in \Psi'_a\})$. To prove $\min(S_1) \geq \min(S_2)$, we prove that for each $s_1 \in S_1$ there is $s_2 \in S_2$ s.t. $s_1 \geq s_2$. For any $\psi'_a \in \Psi'_a$ there is $\phi'_a \in \Phi'_a$ s.t. $\psi'_a \models \phi'_a$. By induction, $dist(\phi_a, \phi'_a) = mindist(\phi_a, \phi'_a)$. Let $\delta_a \in S_{k'-1}(\phi_a)$. We add the ∇_a part of δ_a into ψ_a (resp. ψ'_a) and get ψ^*_a (resp. $\psi^{*\prime}_a$). Then $\psi^*_a \in S_{k'-1}(\phi_a)$ and $\psi^{*\prime}_a \in S_{k'-1}(\phi'_a)$ and $dist(\psi_a, \psi'_a) = dist(\psi^*_a, \psi^{*\prime}_a) \geq mindist(\phi_a, \phi'_a) = dist(\phi_a, \phi'_a)$. So $dist(\psi_a, \Psi'_a) \geq dist(\phi_a, \Phi'_a)$. Similarly, for each $\phi'_a \in \Phi'_a$, there is $\psi'_a \in \Psi'_a$ s.t. $dist(\psi'_a, \Psi_a) \geq dist(\phi'_a, \Phi_a)$. Thus $dist(\Psi_a, \Psi'_a) \geq dist(\Phi_a, \Phi'_a)$. ∎

We now show how to compute the mindist between formulas of special form from the mindist between the subformulas. This result will be used in the syntactic characterizations of our revision for formulas of special form (Prop. 17 and 18).

Proposition 14 *Let* $\mathcal{B} \subseteq \mathcal{A}$. *Let* $\phi = \bigwedge_{a \in \mathcal{B}} B_a \phi_a$ *and* $\phi' = \bigwedge_{a \in \mathcal{B}} B_a \phi'_a$ *be satisfiable alternating formulas. Then* $mindist(\phi, \phi') = \sum_{a \in \mathcal{B}} mindist(\phi_a, \phi'_a)$.

Proof: We first show LHS $\leq$ RHS. For $a \in \mathcal{B}$, let $\delta_a \in S_{k-1}(\phi_a)$ and $\delta'_a \in S_{k-1}(\phi'_a)$ s.t. $dist(\delta_a, \delta'_a) = mindist(\phi_a, \phi'_a)$. We remove the ∇_a part from δ_a (resp. δ'_a) and get δ^-_a (resp. $\delta^{-\prime}_a$). Let $\delta^* \in S_k(\phi)$. Let δ (resp. δ') be the same as δ^* except that for each $a \in \mathcal{B}$, $R_a(\delta) = \{\delta^-_a\}$ (resp. $\{\delta^{-\prime}_a\}$). Then $\delta \in S_k(\phi)$ and $\delta' \in S_k(\phi')$ s.t. $dist(\delta, \delta') = \sum_{a \in \mathcal{B}} dist(\delta^-_a, \delta^{-\prime}_a) = \sum_{a \in \mathcal{B}} dist(\delta_a, \delta'_a) = \sum_{a \in \mathcal{B}} mindist(\phi_a, \phi'_a)$.

We now show LHS $\geq$ RHS. Let $\delta = \delta_0 \wedge \bigwedge_{a \in \mathcal{A}} \nabla_a \Psi_a \in S_k(\phi)$ and $\delta' = \delta'_0 \wedge \bigwedge_{a \in \mathcal{A}} \nabla_a \Psi'_a \in S_k(\phi')$. Then $dist(\delta, \delta') \geq \sum_{a \in \mathcal{B}} dist(\Psi_a, \Psi'_a)$. Also, for each $\psi_a \in \Psi_a$, $\psi_a \models \phi_a$, and for each $\psi'_a \in \Psi'_a$, $\psi'_a \models \phi'_a$. Thus $dist(\Psi_a, \Psi'_a) \geq mindist(\phi_a, \phi'_a)$. So $dist(\delta, \delta') \geq \sum_{a \in \mathcal{B}} mindist(\phi_a, \phi'_a)$. Thus LHS $\geq$ RHS. ∎

4 A model-theoretic definition

In this section, we adapt distance-based belief revision from propositional logic to KD45n, using ACFs as models. Then we show our revision satisfies all of the AGM postulates, and give syntactic characterizations for formulas of special form.

Like propositional belief revision, $\phi \circ \mu$ selects from the models of μ those that are closest to models of ϕ. However, we have to specify the depth k of models we use.

Definition 21 Let ϕ and ϕ' be two satisfiable formulas, and $k \geq \max(md(\phi), md(\phi'))$. We define $S_k(\phi \circ_k \mu) = \{\delta \in S_k(\mu) \mid$ there is $\delta' \in S_k(\phi)$ s.t. $dist(\delta, \delta') = mindist(\phi, \mu)\}$.

We first show that $\phi \circ_k \mu$ is independent of k:

Theorem 5 *Let* $k' \geq k \geq \max\{md(\phi), md(\mu)\}$. *Then* $\phi \circ_k \mu \Leftrightarrow \phi \circ_{k'} \mu$.

Proof: We first show for each $\delta_{k'} \in S_{k'}(\phi \circ_{k'} \mu)$, there is $\delta_k \in S_k(\phi \circ_k \mu)$ s.t. $\delta_{k'} \models \delta_k$. Let $\delta_{k'} \in S_{k'}(\phi \circ_{k'} \mu)$, $\delta'_{k'} \in S_{k'}(\phi)$ s.t. $dist(\delta_{k'}, \delta'_{k'}) = mindist(\phi, \mu)$. By Prop. 10, there exist $\delta_k, \delta'_k \in S_k$, where $\delta_{k'}$ is an expansion of δ_k and $\delta'_{k'}$ is an expansion of δ'_k. Also, $\delta_k \models \mu$ and $\delta'_k \models \phi$. Hence $dist(\delta_k, \delta'_k) \geq dist(\delta_{k'}, \delta'_{k'}) = mindist(\phi, \mu)$. By Prop. 12, $dist(\delta_k, \delta'_k) \leq dist(\delta_{k'}, \delta'_{k'}) = mindist(\phi, \mu)$. So $dist(\delta_k, \delta'_k) = mindist(\phi, \mu)$. Thus $\delta_k \in S_k(\phi \circ_k \mu)$. By Prop. 9, $\delta_{k'} \models \delta_k$. We now show for each $\delta_k \in S_k(\phi \circ_k \mu)$, there is $\Delta_{k'} \subseteq S_{k'}(\phi \circ_{k'} \mu)$ s.t. $\delta_k \models \bigvee \Delta_{k'}$. Let $\delta_k \in S_k(\phi \circ_k \mu)$, $\delta'_k \in S_k(\phi)$ s.t. $dist(\delta_k, \delta'_k) = mindist(\phi, \mu)$. Let $\delta_{k'} \in S_{k'}$ be any expansion of δ_k. By Prop. 13, there is $\delta'_{k'} \in S_{k'}$ s.t. $\delta'_{k'}$ is an expansion of δ'_k and $dist(\delta_{k'}, \delta'_{k'}) = dist(\delta_k, \delta'_k) = mindist(\phi, \mu)$. By Prop. 10, $\delta_{k'} \models \mu$ and $\delta'_{k'} \models \phi$. So $\delta_{k'} \in S_{k'}(\phi \circ_{k'} \mu)$. Let $\Delta_{k'} \subseteq S_{k'}$ be the set of expansions of δ_k. By Prop. 9, $\delta_k \Leftrightarrow \bigvee \Delta_{k'}$. ∎

Based on the above theorem, we can omit the subscript, and have

Definition 22 $\phi \circ \mu$ denotes $\phi \circ_k \mu$ where $k = \max\{md(\phi), md(\mu)\}$.

The following proposition shows that our revision operator coincides with Dalal's in the propositional case.

Proposition 15 *If* ϕ *and* μ *are propositional,* $\phi \circ \mu \Leftrightarrow \phi \circ_d \mu$.

Proof: If ϕ and μ are propositional, then models for them are propositional models, and the distance notion is the Hamming distance between propositional models. ∎

Then we have the result that our revision satisfies all the AGM postulates (R1-6) [17].

Proposition 16 *Let* $\phi, \mu, \psi, \phi_i, \mu_i \in \mathcal{L}_B$. *We have*

(R1) $\phi \circ \mu \models \mu$;
(R2) *If* $\phi \wedge \mu$ *is satisfiable, then* $\phi \circ \mu \Leftrightarrow \phi \wedge \mu$;
(R3) *If* μ *is satisfiable, then* $\phi \circ \mu$ *is satisfiable;*
(R4) *If* $\phi_1 \Leftrightarrow \phi_2$ *and* $\mu_1 \Leftrightarrow \mu_2$, *then* $\phi_1 \circ \mu_1 \Leftrightarrow \phi_2 \circ \mu_2$;
(R5) $(\phi \circ \mu) \wedge \psi \models \phi \circ (\mu \wedge \psi)$;
(R6) *If* $(\phi \circ \mu) \wedge \psi$ *is satisfiable, then* $\phi \circ (\mu \wedge \psi) \models (\phi \circ \mu) \wedge \psi$.

Proof: (R1) Obviously, $S_k(\phi \circ \mu) \subseteq S_k(\mu)$.

(R2) If $\phi \wedge \mu$ is satisfiable, then $S_k(\phi) \cap S_k(\mu)$ is not empty. Hence $S_k(\phi \circ \mu) = S_k(\phi) \cap S_k(\mu)$.

(R3) If μ is satisfiable, then $S_k(\mu)$ is not empty, hence $S_k(\phi \circ \mu)$ is not empty.

(R4) Let $k = \max\{md(\phi_1), md(\phi_2), md(\mu_1), md(\mu_2)\}$. Then $S_k(\phi_1) = S_k(\phi_2)$ and $S_k(\mu_1) = S_k(\mu_2)$. Hence $S_k(\phi_1 \circ \mu_1) = S_k(\phi_2 \circ \mu_2)$.

(R5) Let $\delta \in S_k((\phi \circ \mu) \wedge \psi)$. Then $\delta \in S_k(\mu \wedge \psi)$, and there exists $\delta' \in S_k(\phi)$ s.t. $dist(\delta, \delta') = mindist(\phi, \mu)$. So $dist(\delta, \delta') = mindist(\phi, \mu) \leq mindist(\phi, \mu \wedge \psi) \leq dist(\delta, \delta')$, so $mindist(\phi, \mu) = mindist(\phi, \mu \wedge \psi) = dist(\delta, \delta')$. So $\delta \in S_k(\phi \circ (\mu \wedge \psi))$.

(R6) By the proof of (R5), if $\delta \in S_k((\phi \circ \mu) \wedge \psi) \neq \emptyset$, then $mindist(\phi, \mu) = mindist(\phi, \mu \wedge \psi)$. Let $\delta \in S_k(\phi \circ (\mu \wedge \psi))$. Then there is $\delta' \in S_k(\phi)$ s.t. $dist(\delta, \delta') = mindist(\phi, \mu \wedge \psi) = mindist(\phi, \mu)$. So $\delta \in S_k(\phi \circ \mu)$, hence $\delta \in S_k((\phi \circ \mu) \wedge \psi)$. ∎

The following result shows that $B_a \phi_a \circ B_a \mu_a \Leftrightarrow B_a(\phi_a \circ \mu_a)$ under certain conditions. Its proof uses Proposition 14.

Proposition 17 *Let $B_a\phi_a$ and $B_a\mu_a$ be satisfiable alternating formulas. Then $B_a\phi_a \circ B_a\mu_a \Leftrightarrow B_a(\phi_a \circ \mu_a)$.*

Proof: $\Rightarrow$: Let $\delta = \delta_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Psi_a \in S_k(B_a\mu_a)$ and $\delta' = \delta'_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Psi'_a \in S_k(B_a\phi_a)$ s.t. $dist(\delta,\delta') = mindist(B_a\phi_a, B_a\mu_a) = mindist(\phi_a,\mu_a)$, by Proposition 14. Then $dist(\Psi_a,\Psi'_a) = mindist(\phi_a,\mu_a)$. Thus for each $\psi_a \in \Psi_a$, there exists $\psi'_a \in \Psi'_a$ s.t. $dist(\psi_a,\psi'_a) = dist(\psi_a,\Psi'_a) \leq mindist(\phi_a,\mu_a)$. We show $\psi_a \models \phi_a \circ \mu_a$, hence $\delta \in S_k(B_a(\phi_a \circ \mu_a))$. Let $\delta_a \in S_{k-1}(\psi_a)$. Then $\delta_a = \psi_a \wedge \nabla_a\Phi_a$ for some Φ_a. Let $\delta'_a = \psi'_a \wedge \Phi_a$. Then $\delta_a \in S_{k-1}(\mu_a)$ and $\delta'_a \in S_{k-1}(\phi_a)$. So $dist(\psi_a,\psi'_a) = dist(\delta_a,\delta'_a) \geq mindist(\phi_a,\mu_a)$. Thus $dist(\delta_a,\delta'_a) = mindist(\phi_a,\mu_a)$. So $\delta_a \in S_{k-1}(\phi_a \circ \mu_a)$.

$\Leftarrow$: Let $\delta = \delta_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Psi_a \in S_k(B_a(\phi_a \circ \mu_a))$. Then for each $\psi_a \in \Psi_a$, $\psi_a \models \phi_a \circ \mu_a$. Let $\delta_a \in S_{k-1}(\psi_a)$. Then $\delta_a = \psi_a \wedge \nabla_a\Phi_a$ for some Φ_a, and $\delta_a \in S_{k-1}(\phi_a \circ \mu_a)$. Then there is $\delta'_a \in S_{k-1}(\phi_a)$ s.t. $dist(\delta_a,\delta'_a) = mindist(\phi_a,\mu_a)$. Since both $\phi_a,\mu_a \in \mathcal{L}_B^{-a}$, the ∇_a parts of δ_a and δ'_a must be the same. So $\psi_a \models \mu_a$, hence $\delta \in S_k(B_a\mu_a)$. Let Ψ'_a be the set of formulas obtained from removing the ∇_a part from some δ'_a. Let δ' be the same as δ except that $R_a(\delta') = \Psi'_a$. Then $\delta' \in S_k(B_a\phi_a)$, and $dist(\delta,\delta') = dist(\Psi_a,\Psi'_a) = mindist(\phi_a,\mu_a) = mindist(B_a\phi_a, B_a\mu_a)$. Thus $\delta \in S_k(B_a\phi_a \circ B_a\mu_a)$. ■

Then we show that under certain conditions, revision of conjunctions reduces to conjunctions of revisions.

Proposition 18 *Let $\mathcal{B} \subseteq \mathcal{A}$. Let $\phi = \bigwedge_{a\in\mathcal{B}} B_a\phi_a$ and $\mu = \bigwedge_{a\in\mathcal{B}} B_a\mu_a$ be satisfiable alternating formulas. Then $\phi \circ \mu \Leftrightarrow \bigwedge_{a\in\mathcal{B}} B_a\phi_a \circ B_a\mu_a$.*

Proof: We only prove the case that $\mathcal{B} = \{a,b\}$. The general case can be similarly proved.

$\Rightarrow$: Let $\delta = \delta_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Phi_a \in S_k(B_a\mu_a \wedge B_b\mu_b)$ and $\delta' = \delta'_0 \wedge \bigwedge_{a\in\mathcal{A}} \nabla_a\Phi'_a \in S_k(B_a\phi_a \wedge B_b\phi_b)$ s.t. $dist(\delta,\delta') = mindist(\phi,\mu) = mindist(\phi_a,\mu_a) + mindist(\phi_b,\mu_b)$, by Prop. 14. We must have $dist(\delta,\delta') = dist(\Phi_a,\Phi'_a) + dist(\Phi_b,\Phi'_b)$. Thus $dist(\Phi_a,\Phi'_a) + dist(\Phi_b,\Phi'_b) = mindist(\phi_a,\mu_a) + mindist(\phi_b,\mu_b)$. Since $dist(\Phi_a,\Phi'_a) \geq mindist(\phi_a,\mu_a)$ and $dist(\Phi_b,\Phi'_b) \geq mindist(\phi_b,\mu_b)$, we have $dist(\Phi_a,\Phi'_a) = mindist(\phi_a,\mu_a)$ and $dist(\Phi_b,\Phi'_b) = mindist(\phi_b,\mu_b)$. We now show $\delta \in S_k(B_a\phi_a \circ B_a\mu_a)$. Similarly, $\delta \in S_k(B_b\phi_b \circ B_b\mu_b)$. Obviously, $\delta \in S_k(B_a\mu_a)$. Let δ'' be the same as δ' except that $R_b(\delta'') = \Phi_b$. Then $\delta'' \in S_k(B_a\phi_a)$. We have $dist(\delta,\delta'') = dist(\Phi_a,\Phi'_a) = mindist(\phi_a,\mu_a) = mindist(B_a\phi_a, B_a\mu_a)$. Thus $\delta \in S_k(B_a\phi_a \circ B_a\mu_a)$.

$\Leftarrow$: Let $\delta \in S_k(B_a\mu_a \wedge B_b\mu_b)$, $\delta_1 \in S_k(B_a\phi_a)$, $\delta_2 \in S_k(B_b\phi_b)$, $dist(\delta,\delta_1) = mindist(B_a\phi_a, B_a\mu_a) = mindist(\phi_a,\mu_a)$, $dist(\delta,\delta_2) = mindist(B_b\phi_b, B_b\mu_b) = mindist(\phi_b,\mu_b)$. Let δ' be the same as δ except $R_a(\delta') = R_a(\delta_1)$ and $R_b(\delta') = R_b(\delta_2)$. Then $\delta' \in S_k(B_a\phi_a \wedge B_b\phi_b)$, and $dist(\delta,\delta') = dist(R_a(\delta), R_a(\delta_1)) + dist(R_b(\delta), R_b(\delta_2)) = mindist(\phi_a,\mu_a) + mindist(\phi_b,\mu_b) = mindist(B_a\phi_a \wedge B_b\phi_b, B_a\mu_a \wedge B_b\mu_b)$. Thus $\delta \in S_k(\phi \circ \mu)$. ■

Finally, we give a syntactic characterization of our belief revision operator on proper ACDFs (see Def. 12 and Prop. 5). By Propositions 15, 17 and 18, we get the following result, showing that for proper ACDFs, higher-order belief revision nicely reduces to propositional belief revision along each path.

Theorem 6 *Let $\phi = \bigwedge_{p\in P} B_p\phi_p$ and $\mu = \bigwedge_{p\in P'} B_p\mu_p$ be proper ACDFs. Then*

$$\phi \circ \mu \Leftrightarrow \bigwedge_{p\in P-P'} B_p\phi_p \wedge \bigwedge_{p\in P'-P} B_p\mu_p \wedge \bigwedge_{p\in P\cap P'} B_p[\phi_p \circ_d \mu_p]$$

where $\circ_d$ is Dalal's propositional revise operator.

By this theorem, the plausible result we give for Example 1 is indeed the result of our revision operator.

Finally, we comment that our work takes a perfect external point of view, *i.e.*, considers revision of beliefs of a third person. Our work can handle both a private revision, and a semi-private revision. For example, suppose agent a senses that ϕ is true, agent b is aware of the sensing action, but not the result, and agent c is oblivious. Then the new information will be represented by $B_a\phi \wedge B_b(B_a\phi \vee B_a\neg\phi)$. However, we cannot yet handle a public revision, for which we need common belief.

5 Syntactic characterizations

In this section, we give syntactic characterizations of our revision operator. Recall that in KD45n, each formula is equivalent to an ACDF. We begin with a syntactic characterization for ACCFs, based on which we get one for ACDFs.

As for ACCFs, we first define the notions of partitions and prime partitions, then characterize revision based on prime partitions, and finally show how to compute prime partitions.

Definition 23 A partition of an ACCF ϕ is a set Φ of ACCFs s.t. $\phi \Leftrightarrow \bigvee \Phi$.

Note that we do not require formulas in Φ to be exclusive. For example, $\{q \wedge p, q \wedge \neg p\}$ is a partition of q.

As shown by Theorem 4, for two ACCFs ϕ and ϕ', $dist(\phi,\phi') = mindist(\phi,\phi')$. This result is used in the proofs of this section. It is easy to prove the following:

Proposition 19 *Let ϕ and μ be two ACCFs, Ψ a partition of μ. Let $\psi \in \Psi$. If $dist(\phi,\psi) > dist(\phi,\mu)$, then $\phi \circ \mu \models \neg\psi$; If $dist(\phi,\psi) = dist(\phi,\mu)$ and ψ is an ACF, then $\psi \models \phi \circ \mu$.*

Proof: By Theorem 4, if $dist(\phi,\psi) > dist(\phi,\mu)$, there exist $\delta \in S_k(\phi)$ and $\delta' \in S_k(\mu)$ s.t. $dist(\delta,\delta') = dist(\phi,\mu) = mindist(\phi,\mu)$, and for all $\delta'' \in S_k(\psi)$, $dist(\delta,\delta'') \geq mindist(\phi,\psi) = dist(\phi,\psi) > dist(\phi,\mu)$. So δ'' will not be included in $S_k(\phi \circ \mu)$. If $dist(\phi,\psi) = dist(\phi,\mu)$ and ψ is an ACF, then there is $\delta \in S_k(\phi)$ s.t. $dist(\delta,\psi) = dist(\phi,\mu) = mindist(\phi,\mu)$, so $\psi \models \phi \circ \mu$. ■

This motivates us to define the notion of a prime partition.

Definition 24 Let ϕ and μ be two ACCFs, Ψ a partition of μ. We say Ψ is a *prime partition* of μ w.r.t. ϕ if for all $\psi \in \Psi$, if $dist(\phi,\psi) = dist(\phi,\mu)$, then ψ is an ACF.

As an easy corollary of Prop. 19, we get

Theorem 7 *Let ϕ and μ be two ACCFs. If Ψ is a prime partition of μ w.r.t. ϕ then*

$$\phi \circ \mu \Leftrightarrow \bigvee\{\psi \in \Psi \mid dist(\phi,\psi) = dist(\phi,\mu)\}$$
$$\Leftrightarrow \mu \wedge \bigwedge\{\neg\psi \mid \psi \in \Psi, dist(\phi,\psi) > dist(\phi,\mu)\}.$$

Obviously, $S_k(\mu)$ is always a prime partition of μ. However, each member of $S_k(\mu)$ is an ACF. In the following, we show how to find more compact prime partitions.

The following proposition shows how to get a partition of an ACCF recursively:

Algorithm 1: primepart(ϕ,μ,d)

Input: Two ACCFs ϕ, μ, and $d \geq 0$
Output: A prime partition of μ w.r.t. ϕ

Get a partition Ψ of μ by applying Prop. 20
$S := \emptyset$
foreach $\mu^* \in \Psi$ **do**
  if *μ^* is not an ACF and $dist(\phi, \mu^*) = d$* **then**
    Φ :=primepart(ϕ,μ^*,d); $S := S \cup \Phi$
  else
    $S := S \cup \{\mu^*\}$
return S

Proposition 20 *Let ϕ be an ACCF. We consider three cases:*

1. *ϕ is propositional: ϕ can be partitioned according to the truth values of atoms not appearing in ϕ.*
2. *ϕ is $\phi_1 \wedge \phi_2$: Let Ψ_1 be a partition of ϕ_1. Then $\{\psi_1 \wedge \phi_2 \mid \psi_1 \in \Psi_1\}$ is a partition of ϕ.*
3. *ϕ is $\nabla_a(\{\phi_a\} \cup S)$: Let Ψ_a be a partition of ϕ_a. Then $\{\nabla_a(S' \cup S) \mid S' \subseteq \Psi_a$ and $S' \neq \emptyset\}$ is a partition of ϕ.*

Proof: We prove the third case when $|\Psi_a| = 2$. Let $\phi_a = \phi_1 \vee \phi_2$. suppose $\delta \models \nabla_a(\{\phi_a\} \cup S)$. There are 3 cases: $\delta \models \hat{B}_a\phi_1 \wedge \neg\hat{B}_a\phi_2$, then $\delta \models \nabla_a(\{\phi_1\} \cup S)$; $\delta \models \hat{B}_a\phi_2 \wedge \neg\hat{B}_a\phi_1$, then $\delta \models \nabla_a(\{\phi_2\} \cup S)$; $\delta \models \hat{B}_a\phi_1 \wedge \hat{B}_a\phi_2$, then $\delta \models \nabla_a(\{\phi_1, \phi_2\} \cup S)$. ■

Algorithm 1 gives a nondeterministic recursive algorithm to compute a prime partition. When computing $\phi \circ \mu$, we call primepart($\phi, \mu, dist(\phi, \mu)$).

In the worst case, the algorithm will output a set of ACFs. Since the size and number of CFs are non-elementary, the algorithm has a worst-case non-elementary complexity.

Example 4 Let $\phi = p \wedge \nabla_a\{p, q\}$, and $\mu = \nabla_a\{p\}$. Then $dist(\phi, \mu) = 0$. Figure 5 shows the process of computing a prime partition of μ w.r.t. ϕ. Each node μ^* is also marked with $dist(\phi, \mu^*)$, $\mu_{11} = p \wedge \nabla_a\{p \wedge q\}$, $\mu_{12} = p \wedge \nabla_a\{p \wedge \neg q\}$, and $\mu_{13} = p \wedge \nabla_a\{p \wedge q, p \wedge \neg q\}$. The prime partition is the set of the leaves of the tree. So $\phi \circ \mu \Leftrightarrow q \wedge \mu_{11} \vee \neg q \wedge \mu_{11} \vee q \wedge \mu_{13} \vee \neg q \wedge \mu_{13} \Leftrightarrow \mu_{11} \vee \mu_{13}$. Also, $\phi \circ \mu \Leftrightarrow \mu \wedge \neg\mu_2 \wedge \neg\mu_{12}$.

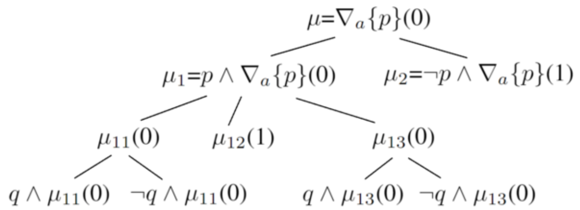

Figure 5. An example of computing a prime partition

We now move to ACDFs.

Proposition 21 *Each ACDF ϕ is equivalent to a disjunction of ACCFs.*

Proof: We prove by induction on ϕ. The cases of propositional terms and disjunctions are easy. Now let $\phi = \phi_0 \wedge \bigwedge_{a \in \mathcal{B}} \nabla_a\Phi_a$. By induction, ϕ_0 and each member of Φ_a is equivalent to a disjunction of ACCFs. By repeatedly applying Prop. 20, ϕ is equivalent to a disjunction of ACCFs. ■

Theorem 8 *Let $\phi = \bigvee \Phi$, $\mu = \bigvee \Phi'$ be two ACDFs, where Φ and Φ' are nonempty sets of ACCFs. Then*

$$mindist(\phi, \mu) = min\{dist(\phi_i, \mu_j) \mid \phi_i \in \Phi, \mu_j \in \Phi'\},$$

$$\phi \circ \mu \Leftrightarrow \bigvee_{\langle \phi_i, \mu_j \rangle \text{ s.t. } dist(\phi_i, \mu_j) = mindist(\phi, \mu)} \phi_i \circ \mu_j.$$

Proof: We know $S_k(\phi \circ \mu) = \{\delta \in S_k(\mu) \mid$ there is $\delta' \in S_k(\phi)$ s.t. $dist(\delta, \delta') = mindist(\phi, \mu)\}$. So it is equal to the union of $\{\delta \in S_k(\mu_j) \mid$ there is $\delta' \in S_k(\phi_i)$ s.t. $dist(\delta, \delta') = mindist(\phi_i, \mu_j)\}$, *i.e.*, $S_k(\phi_i \circ \mu_j)$, where the union is taking over all $\langle \phi_i, \mu_j \rangle$ s.t. $dist(\phi_i, \mu_j) = mindist(\phi, \mu)$. ■

So the above syntactic characterization takes the same form as that for Dalal's revision in Theorem 1. Since in KD45n, each formula is equivalent to an ACDF, the above theorem gives a syntactic characterization of our revision for the whole language.

6 Related work and conclusions

Now we would like to compare our work with those of Aucher [3] and Caridroit *et al.* [7]. First of all, all three works give distance-based model-theoretic definitions of multi-agent belief revision. Second, [7] and ours take a perfect external point of view, *i.e.*, consider revision of beliefs of a third person, while [3] assumes an internal point of view, *i.e.*, consider revision of beliefs of one of the agents. Third, [3] and [7] use minimal Kripke models under bisimulation in their model-theoretic definitions, while we use ACFs as models of formulas. In addition, we give syntactic characterizations of our revision, while [3] and [7] do not. Finally, their definitions of distances between Kripke models are based on weakenings of standard bisimulation or based on tuples or aggregations of distances between sets of valuations for different depths in the two models, while our definition of distances between ACFs is an inductive one where the distance between ACFs of depth $k + 1$ is defined using the distance between ACFs of depth k.

Another related work is [4] where Aucher explored the progression of a KB ϕ w.r.t. an action represented by a formula ϕ'. He represented both ϕ and ϕ' as disjunctions of CFs (canonical formulas), and defined the progression of ϕ w.r.t. ϕ' as the disjunction of progression of a CF δ of ϕ w.r.t. a CF δ' of ϕ' applicable in δ, which is defined by induction on the depth of δ and δ'. Our work goes beyond [4] in there ways. First, we propose alternating CFs. Second, our definition of revision has to consider the issue of minimal change. Finally, our syntactic characterizations give more compact representation of the revision results than just taking the disjunction of the minimal ACFs.

To conclude, in this paper, we gave a model-theoretic definition of belief revision in KD45n and its syntactic characterizations. To give a model-theoretic definition, we propose alternating canonical formulas (ACFs) and treat them as models for formulas. To give a syntactic characterization for ACDFs, we propose a new fragment of ACDFs called ACCFs which subsumes ACFs. We define distances between ACCFs based on the notion of Hausdorff distance between two sets. To give a syntactic characterization for ACCFs, we propose the concept of prime partitions. Finally, we show that each ACDF is equivalent to a disjunction of ACCFs, and give a syntactic characterization for ACDFs, in the same form as that for Dalal's propositional revision. In the future, we would like to extend the work of this paper to incorporate common belief and address belief update in KD45n.

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