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Mechanism Design for Ad Auctions with Display Prices

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Abstract. In various applications, ads are displayed together with prices, so as to provide a direct comparison among similar products or services. The price-displaying feature not only influences the consumers' decision, but also affects the bidding behavior of advertisers. In this paper, we study ad auctions with display prices from the perspective of mechanism design, in which advertisers are asked to submit both the product costs and the display prices of their commodities. We first provide a characterization for all individually rational and incentive-compatible mechanisms in the presence of display prices, then use it to design ad auctions in two scenarios. In the former scenario, the display prices are assumed to be exogenously determined. For this scenario, we derive the welfare-maximizing and revenue-maximizing auctions for any given display price profile. In the latter, advertisers are allowed to strategize their display prices freely. We investigate two families of allocation policies within the scenario and identify the equilibrium display prices accordingly. Our findings demonstrate the impact of display prices on the design of ad auctions, and highlight how platforms can utilize display price information to optimize the performance of ad delivery.

1 Introduction

Online advertising has been an indispensable part of modern advertising market. According to the newly released report of IAB [12], full-year online advertising revenue has reached $209.7 billion in 2022. An important reason for wide-spread adoption of online advertisement comes from its high return on investment for advertisers, compared to other traditional marketing methods [20]. As an efficient tool of deriving revenue, auctions are commonly used to allocate the display opportunities. Every day, millions of ad auctions are conducted in real time to decide which advertisers' ads are shown, how these ads are arranged, and what the advertisers are charged. To date, online advertising platforms [9, 19, 7] have developed various types of products for different types of advertisers, such as pay-per-mille or pay-per-impression (PPM), pay-per-click (PPC), and pay-per-action (PPA). In the classic ad auction setting, such as sponsored search, ads are presented in the form of hyper-links together with relevant keywords or well-designed creatives, serving as portals of advertisers' websites or products. Which ads are displayed depends on advertisers' bids and the relevance of their ads to the context [24]. However, in numerous real applications, like Temu or Ttrip, ads (or products) are displayed also together with the prices, e.g., it can be the per-night price of a room or the group purchase price of a commodity. The price-displaying feature brings two significant changes for the advertising system. On the one hand, the prices provide a direct comparison among similar products or services, which can easily influence the consumers' decision. One the other hand, the price information also affects the advertisers' bidding behavior and the efficiency of the deployed ad auctions [2]. As the display price has changed the bidding language and the way advertisers participate in the ad auction, fundamental investigation into mechanism design for auctions with display prices should be made.

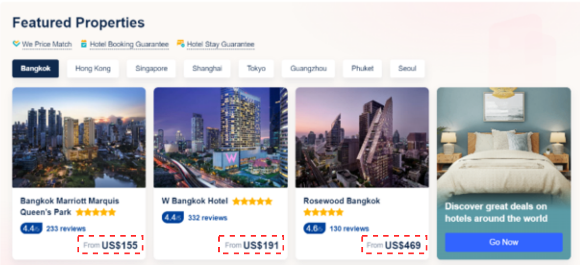

Figure 1: Ads with display prices on Ttrip.

In this paper, we study how the presence of display prices affects ad auction design. In our model, advertisers are asked to submit both the product costs and display prices of their commodities and the advertising platform allocates the display opportunities and decides the charges based on the submitted information. Our model differs from the classic one in two ways. Firstly, rather than submitting a single bid for the display opportunities, we ask advertisers to submit both the costs and prices of their products. Secondly, in classic ad auctions, a conversion of an ad, like a purchase, is exogenously determined and is independent of the submitted information. However, in our model conversions are essentially determined by the submitted display prices. Based on the framework of mechanism design, we carry out a systemic investigation on ad auctions with display prices. Our contributions advance the state of the art in the following ways:

- We propose the formal model of ad auctions with display prices, where advertisers are asked to submit not only the product costs of their commodities but also the display prices.
- We characterize all incentive-compatible and individuality rational auctions in the presence of display prices. (Theorems 1, 2)
- In non-strategic price settings, we derive the welfare-maximizing and revenue-maximizing auctions for any given display price profile. (Theorems 3, 5)
- In strategic price settings, we characterize the equilibrium price report for two families of allocation policies, namely the price-independent allocation policy and the affine maximizer allocation policy. (Propositions 1, 2)

Our results show that the display prices do affect the design of ad

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auctions, and the advertising platforms can leverage such information to further optimize the performance of ad delivery.

1.1 Related Work

In addition to the great industrial success, ad auctions have attracted a lot of attention from the research community. Since Overture, for the first time, adopted the generalized first price auction mechanism in its sponsored search system in 1997 [5, 13], many researchers from economics, computer sciences, management science, etc. have been working on different aspects of ad auctions for decades. Some of them focus on studying the theoretical properties of the deployed ad auctions [6, 24, 26], while some others are devoted to design new auction mechanisms towards different scenarios and objectives [15, 16, 8]. With the development of AI, it is even possible to design ad auctions automately [1, 23, 27], based on advanced machine learning techniques and massive transaction data. To the best of our knowledge, Castiglioni et al. [2] is the very first to study ad auctions with display prices. The authors studied the allocation efficiency of two widely used ad auctions, namely VCG [25, 3, 10] and GSP [6], in the presence of display prices, and analyzed the Price of Anarchy (PoA) and the Price of Stability (PoS) in the direct and indirect realizations of these two auctions, respectively. In contrast, we focus on the counterpart and study new ad auctions involving display prices from the perspective of mechanism design.

The reminder of this paper is organized as follows. Section 2 presents the formal model of auction with display prices and defines several general concepts of an auction mechanism. Section 3 characterizes all truthful auctions with display prices. Following the characterization, Section 4 investigates the welfare-maximizing and revenue-maximizing auctions, under the assumption of non-strategic display prices. Section 5 studies two families of auction mechanisms in strategic price settings and Section 6 summarizes this work.

2 Preliminaries

Suppose there is a set N of advertisers and K available ad slots, i.e., display opportunities. For each advertiser $i \in N$, let $c_i \in [\underline{c}_i, \overline{c}_i]$ denote the product cost of her commodity, which is private information and is derived from a distribution $\mathcal{C}_i$. In addition, let p_i denote the display price that i sets for her commodity. The advertising platform runs an auction to allocate the display opportunities. Besides the product cost, each advertiser is additionally asked to report her display price to the auction. Since product costs are private information, advertisers can game the auction mechanism to benefit themselves via strategic actions. Accordingly, let (c_i', p_i) denote i's report, where c_i' is the reported product cost and p_i is the reported display price. For convenience, we use $\mathbf{c}'$ and $\mathbf{p}$ to denote the reported costs and display prices of all advertisers, respectively. In addition, let $\mathbf{c}'_{-i}$ and $\mathbf{p}_{-i}$ be the reported costs and display prices of all advertisers except i, i.e., $\mathbf{c}' = (c_i', \mathbf{c}'_{-i})$ and $\mathbf{p} = (p_i, \mathbf{p}_{-i})$. The formal definition of auction mechanisms with display prices is given below.

Definition 1. *An* auction mechanism $\mathcal{M} = (\pi, x)$ *consists of an allocation policy* $\pi = \{\pi_i\}_{i\in N}$ *and a payment policy* $x = \{x_i\}_{i\in N}$*, where* $\pi_i : \mathcal{R}_+^{|N|} \times \mathcal{R}_+^{|N|} \to \{0, 1\}$ *and* $x_i : \mathcal{R}_+^{|N|} \times \mathcal{R}_+^{|N|} \to \mathcal{R}$ *are the allocation and payment policies for i, respectively.*

Given all advertisers' reports $(\mathbf{c}', \mathbf{p})$, $\pi_i(\mathbf{c}', \mathbf{p})$ indicates whether or not advertiser i wins a slot and $x_i(\mathbf{c}', \mathbf{p})$ denotes the amount each advertiser i pays to the advertising platform. The revenue generated by an auction mechanism is defined by the sum of all advertisers' payments, denoted by $R(\mathbf{c}', \mathbf{p}, \mathcal{M}) = \sum_{i\in N} x_i(\mathbf{c}', \mathbf{p})$.

Due to the limited number of display opportunities, the mechanism cannot over-allocate the slots.

Definition 2. *An allocation policy is feasible if for all reports* $(\mathbf{c}', \mathbf{p})$*, we have that* $\sum_{i\in N} \pi_i(\mathbf{c}', \mathbf{p}) \le K$*.*

In the following contents, we only consider feasible allocation policies. The display prices provide a direct comparison among similar commodities or services, which would further influence or determine the purchasing behavior of consumers. Given a set Z of winners and their display prices $\mathbf{p}^Z = \{p_i\}_{i\in Z}$, we use $\lambda_i(\mathbf{p}^Z)$ to denote the probability that a conversion of i's commodity, like a purchase, is acquired under display price vector $\mathbf{p}^Z$, aka *the conversion rate*. In this paper, we focus on separable conversion rate functions, in which each consumer is only influenced by the commodity and the display price associated with the commodity, i.e., $\lambda_i(\mathbf{p}^Z) = \lambda_i(p_i)$. With the assumption of separable conversion rate functions, the expected value advertiser i achieves when her commodity is displayed on the advertising platform can be formulated as $v_i(c_i, p_i) = (p_i - c_i)\lambda_i(p_i)$. Given an auction mechanism $\mathcal{M} = (\pi, x)$ and reports $(\mathbf{c}', \mathbf{p})$, advertiser i's utility function is quasi-linear and is defined as follows:

$$u_i\big(c_i, \mathbf{c}', \mathbf{p}, (\pi, x)\big) = v_i(c_i, p_i)\pi_i(\mathbf{c}', \mathbf{p}) - x_i(\mathbf{c}', \mathbf{p}). \quad (1)$$

We next present several properties that an auction mechanism should satisfy. Given an auction mechanism $\mathcal{M}$, the social welfare obtained in $(\mathbf{c}', \mathbf{p})$, denoted by $W(\mathbf{c}', \mathbf{p}, \mathcal{M})$, is defined as the total utilities of all participants (including the advertising platform), which can be formulated as $W(\mathbf{c}', \mathbf{p}, \mathcal{M}) = \sum_{i\in N} v_i(c_i, p_i)\pi_i(\mathbf{c}', \mathbf{p})$. We say an auction mechanism is efficient with reported prices (EF-RP) if for all equilibrium reports $(\mathbf{c}', \mathbf{p})$ it maximizes $W(\mathbf{c}', \mathbf{p}, \mathcal{M})$.

Definition 3. *An auction mechanism* $\mathcal{M}$ *is* efficient with reported prices (EF-RP) *if for all* $\mathbf{c}$ *and all equilibrium reports* $(\mathbf{c}', \mathbf{p})$

$$\mathcal{M} \in \arg\max_{\mathcal{M}'} W(\mathbf{c}', \mathbf{p}, \mathcal{M}'). \quad (2)$$

Let $\Pi_i(\mathbf{c}) = \arg_{\mathbf{p}'} \max_{\mathbf{p}', \pi'} \sum_{i\in N} v_i(c_i, p_i')\pi_i'(\mathbf{c}, \mathbf{p}')$ denote the space of the optimal price profiles that maximize the social welfare for a given $\mathbf{c}$. We say an auction mechanism $\mathcal{M}$ is efficient (EF) if it maximizes $W(\mathbf{c}', \mathbf{p}, \mathcal{M})$ and the reported prices $\mathbf{p} \in \Pi_i(\mathbf{c})$.

Definition 4. *An auction mechanism* $\mathcal{M}$ *is* efficient (EF) *if for all* $\mathbf{c}$ *and all equilibrium reports* $(\mathbf{c}', \mathbf{p})$*, we have that* $\mathbf{p} \in \Pi_i(\mathbf{c})$ *and*

$$\mathcal{M} \in \arg\max_{\mathcal{M}'} W(\mathbf{c}', \mathbf{p}, \mathcal{M}'). \quad (3)$$

In other words, the EF-RP property only asks the mechanism to maximize the social welfare with respect to the reported display prices, while the EF property requires the mechanism to maximize the social welfare at the optimal display prices $\mathbf{p} \in \Pi_i(\mathbf{c})$. Clearly, if an auction mechanism is efficient, it is also efficient with reported prices, but the reverse is not true.

Definition 5. *An auction mechanism* $\mathcal{M}$ *is* incentive-compatible *(IC) if for all i, all* c_i*, all* $\mathbf{p}$*, and all* $\mathbf{c}'$*,*

$$u_i\big(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}, \mathcal{M}\big) \ge u_i\big(c_i, (c_i', \mathbf{c}'_{-i}), \mathbf{p}, \mathcal{M}\big). \quad (4)$$

Incentive compatibility requires that submitting true product costs is a dominant strategy for all advertisers, i.e., each advertiser's utility is maximized by acting truthfully, no matter what the others do. Another important concept is called individual rationality which guarantees that each advertiser will not receive a negative utility when revealing her product cost truthfully.

Definition 6. *An auction mechanism $\mathcal{M}$ is* individually rational *(IR) if for all i, all c_i, all $\mathbf{p}$, and all $\mathbf{c}'_{-i}$,*

$$u_i\big(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}, \mathcal{M}\big) \geq 0. \tag{5}$$

If an auction mechanism violates the IR property, some advertisers may obtain negative utility when revealing their true costs, in which case quitting the auction system is the best reply. Therefore, individual rationality is also known as the participation constraint.

In the following contents, we study auction mechanisms that satisfy EF(-RP), IC, IR and other desired properties in the presence of display prices. We first characterize a set of conditions for an auction mechanism to be IC and IR, then using these conditions to design ad auctions towards different scenarios and objectives.

3 Characterizations of IC and IR Auctions

In this section, we characterize all auction mechanisms involving display prices that are incentive-compatible and individually rational. We first present two conditions that an incentive-compatible auction mechanism should hold, then show that the two conditions are also sufficient for an auction mechanism to be incentive-compatible.

Lemma 1. *If an auction mechanism $\mathcal{M}$ is incentive-compatible, then π_i is non-increasing with c_i for all i, all $\mathbf{p}$ and all $\mathbf{c}'_{-i}$.*

Proof. Consider two possible costs c_i^1 and c_i^2 of advertiser i with $c_i^1 > c_i^2$. Incentive compatibility requires that for all $\mathbf{p}$ and $\mathbf{c}'_{-i}$,

$$\begin{aligned} &v_i(c_i^1, p_i)\pi(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) \geq \\ &v_i(c_i^1, p_i)\pi(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}), \end{aligned} \tag{6}$$

and

$$\begin{aligned} &v_i(c_i^2, p_i)\pi(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}) \geq \\ &v_i(c_i^2, p_i)\pi(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}). \end{aligned} \tag{7}$$

Adding above two inequalities, we obtain that

$$(v_i(c_i^1, p_i) - v_i(c_i^2, p_i))(\pi_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) - \pi_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p})) \geq 0.$$

Recall that $v_i(c_i, p_i)$ is decreasing with c_i, therefore the above inequality leads to the fact that

$$\pi_i(c_i^1, \mathbf{c}'_{-i}, \mathbf{p}) \leq \pi_i(c_i^2, \mathbf{c}'_{-i}, \mathbf{p}). \tag{8}$$

That is, π_i should be non-increasing with c_i in any IC auction. □

Lemma 2 unfolds the interconnections of the payment policy and the allocation policy. It shows that in any IC auction, the allocation policy essentially pins the payment policy.

Lemma 2. *If an auction mechanism $\mathcal{M}$ is incentive-compatible, then for all i, all $\mathbf{p}$ and all $\mathbf{c}'_{-i}$, $x_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ can be formulated as*

$$\begin{aligned} &v_i(c_i, p_i)\pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i)\int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz \\ &- U_i(\mathbf{c}'_{-i}, \mathbf{p}), \end{aligned} \tag{9}$$

where $U_i(\mathbf{c}'_{-i}, \mathbf{p})$ is independent of i's cost report.

Proof. Given any IC auction $\mathcal{M} = (\pi, x)$, based on Definition 4 we have that for all i, all $\mathbf{p}$ and all $\mathbf{c}'_{-i}$ the following equation must hold:

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) = \max_{c'_i} v_i(c_i, p_i)\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}).$$

By the envelope theorem, the above equation is equivalent to the condition of

$$\begin{aligned} &\frac{\partial u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p})}{\partial c_i} \\ &= \frac{\partial(v_i(c_i, p_i)\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}) - x_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}))}{\partial c_i}\Big|_{c'_i = c_i} \end{aligned} \tag{10}$$

which can be simplified as

$$\frac{\partial u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p})}{\partial c_i} = -\lambda_i(p_i)\pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}). \tag{11}$$

Integrating both sides of formula (11) over $[c_i, \bar{c}_i]$ on c_i, we can get that in any IC auction, advertiser i's utility can be denoted by

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) = U_i(\mathbf{c}'_{-i}, \mathbf{p}) + \lambda_i(p_i)\int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz,$$

where $U_i(\mathbf{c}'_{-i}, \mathbf{p}) = u_i(\bar{c}_i, (\bar{c}_i, \mathbf{c}'_{-i}), \mathbf{p})$ is independent of advertiser i's cost report. Based on formula (1), it is clear that advertiser i's payment $x_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ can be formulated as (9). □

Since both $\lambda_i(p_i)$ and $\pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ are non-negative, Lemma 2 also suggests that each advertiser's utility is non-increasing with her true product cost in any IC auction. We next show that the above two conditions are also sufficient for an auction to be IC.

Theorem 1. *An auction mechanism $\mathcal{M}$ is incentive-compatible if and only if for all i, all $\mathbf{p}$ and all $\mathbf{c}'_{-i}$,*

1. *π_i is non-increasing with c_i,*
2. *x_i can be formulated as*

$$\begin{aligned} &v_i(c_i, p_i)\pi_i(c_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i)\int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz \\ &- U_i(\mathbf{c}'_{-i}, \mathbf{p}). \end{aligned}$$

Proof. To prove the theorem, it suffices to prove that if an auction mechanism $\mathcal{M}$ satisfies Condition 1 and Condition 2, then it is IC. Given an advertiser i with true cost c_i, to prove IC, we need to show that the inequality

$$u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) \geq u_i(c_i, (c'_i, \mathbf{c}'_{-i}), \mathbf{p}) \tag{12}$$

holds for all c'_i, all $\mathbf{p}$ and all $\mathbf{c}'_{-i}$. Plugging in the formula of $x_i(c_i, \mathbf{c}'_{-i}, \mathbf{p})$ and making simplification accordingly, inequality (12) can be reformulated as

$$\begin{aligned} \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz \geq\ &\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p})(c'_i - c_i) \\ &+ \int_{c'_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz. \end{aligned} \tag{13}$$

Case 1: If $c'_i \geq c_i$, then inequality (13) is equivalent to

$$\int_{c_i}^{c'_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz \geq \pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p})(c'_i - c_i), \tag{14}$$

which is true under Condition 1.

Case 2: If $c'_i < c_i$, then inequality (13) is equivalent to

$$\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p})(c_i - c'_i) \geq \int_{c'_i}^{c_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})dz, \tag{15}$$

which is also true under Condition 1. □

Besides the IC property, another desired property is individual rationality, which requires the advertiser's utility to be non-negative when acting truthfully.

Theorem 2. *An incentive-compatible auction mechanism $\mathcal{M}$ is individually rational if and only if for all i, all $\mathbf{p}$ and all $\mathbf{c}'_{-i}$,*

$$U_i(\mathbf{c}'_{-i}, \mathbf{p}) \geq 0. \tag{16}$$

Proof. ("$\Rightarrow$") For any IC auction $\mathcal{M} = (\pi, x)$, we have that advertiser i's utility $u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p})$ is identical to

$$U_i(\mathbf{c}'_{-i}, \mathbf{p}) + \lambda_i(p_i) \int_{c_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz.$$

Since $\lambda_i(p_i) \geq 0$ and $\pi_i(z, \mathbf{c}'_{-i}, \mathbf{p})$ is non-negative, we know that if $U_i(\mathbf{c}'_{-i}, \mathbf{p}) \geq 0$ then $u_i(c_i, (c_i, \mathbf{c}'_{-i}), \mathbf{p}) \geq 0$.

("$\Leftarrow$") If $U_i(\mathbf{c}'_{-i}, \mathbf{p}) < 0$ for some $\mathbf{p}$ and $\mathbf{c}'_{-i}$, then advertiser i with $(\bar{c}_i, p_i)$ will obtain a negative utility, which violates IR. □

Theorems 1 and 2 show the importance of the price information in ad auction design. On the one hand, the platform can leverage the information of display prices to allocate the ad slots. On the other hand, the display price is also a key component in determining the chargers of each advertiser. In order to elicit advertisers' true costs and optimize the overall performance of the adverting platform, the display prices need be taken into consideration when designing ad auctions. Nevertheless, evaluating the performance of a given auction mechanism is non-trivial as we need to figure out how each advertiser submits her display price in the auction. If the display prices are exogenously determined, then techniques developed for traditional auctions without display prices can apply here with slight modification. Otherwise, we need to identify the reported display prices in equilibrium at first, which presents inherent challenges.

In the following contents, we first study auctions with non-strategic display prices, then turn to the general scenario where advertisers can strategize their display prices in their favor. As the advertising platform acts as a profit-maximizing agent, for convenience's sake the term $U_i(\mathbf{c}_{-i}, \mathbf{p})$ is treated as zero hereafter for all i, all $\mathbf{c}_{-i}$ and all $\mathbf{p}$, without violating IC and IR.

4 Auction Design with Non-Strategic Prices

In many real-world scenarios, advertisers rarely adjust, or are sometimes prohibited from adjusting, the display prices of their commodities. For example, brand advertisers aim to build brand awareness and long-term relationships with customers, and the display prices of their products rarely changes. Moreover, many sales platforms, such as Tmall and Meituan, stipulate that the listing price of a product cannot be changed at will it is released, or can only be changed periodically. The display prices in above situations can be considered exogenously determined and fixed. Motivated by these scenarios, this section focuses on designing welfare-maximizing and revenue-maximizing ad auctions in non-strategic display price settings.

4.1 Welfare-Maximizing Auction Design

Social welfare reflects the overall efficiency of an auction mechanism, which is defined by the summation of all participants' utilities. Based on Theorems 1 and 2, we next propose an auction mechanism, called welfare maximizer with reported prices (abbreviated as WM-RP), to maximize the social welfare in non-strategic price settings. As the reported display prices are assumed to be fixed, here the pursued property is EF-RP (see Definition 3).

Welfare Maximizer with Reported Prices

- **Allocation Policy:** Given reports $(\mathbf{c}', \mathbf{p})$, allocate the ad slots to maximize $\sum_{i \in N} v_i(c'_i, p_i)\pi_i(\mathbf{c}', \mathbf{p})$, break tie arbitrarily.
- **Payment Policy:** For each advertiser $i \in N$, her payment $x_i(\mathbf{c}', \mathbf{p})$ is defined below:
  - if $\pi_i(\mathbf{c}', \mathbf{p}) = 0$, then $x_i(\mathbf{c}', \mathbf{p}) = 0$;
  - if $\pi_i(\mathbf{c}', \mathbf{p}) = 1$, then $x_i(\mathbf{c}', \mathbf{p})$ is defined as

    $$v_i(v_i^{-1}(v^{(K+1)}(\mathbf{c}', \mathbf{p}), p_i), p_i),$$

  where v_i^{-1} is the inverse function of v_i w.r.t. c_i and $v^{(K+1)}(\mathbf{c}', \mathbf{p})$ denotes the $K+1$ highest reported value.

In other words, in WM-RP the slots are allocated to the advertisers with the top K highest reported values, and the winning advertisers pay the $K+1$ highest reported value to the advertising platform and the losers pay zero. Next, we prove that WM-RP maximizes the social welfare for any given display price profile.

Theorem 3. *WM-RP is IC, IR and EF-RP.*

Proof. According to the definitions of WM-RP and EF-RP, to prove this theorem it is sufficient to show that WM-RP is IC and IR. Firstly, it is straightforward that the allocation policy is non-increasing with c_i for all i, all $\mathbf{p}$ and all $\mathbf{c}_{-i}$, so the first condition of Theorem 1 is satisfied. Secondly, we show that the payment policy of WM-RP is identical to (9). According to the allocation policy, the slots will be allocated to the advertisers with the top K highest reported values. Given reports $(\mathbf{c}', \mathbf{p})$, let $v^{(K+1)}(\mathbf{c}', \mathbf{p})$ be the $K+1$ highest value under $(\mathbf{c}', \mathbf{p})$. For all losers i, $\pi_i(\mathbf{c}', \mathbf{p}) = 0$ for all $c''_i \geq c'_i$ and therefore i's payment $x_i(\mathbf{c}', \mathbf{p})$ is zero according to (9). For a winner i, we know that her allocation $\pi_i(\mathbf{c}', \mathbf{p}) = 1$ as long as

$$v_i(c'_i, p_i) \geq v^{(K+1)}(\mathbf{c}', \mathbf{p}), \tag{17}$$

which is equivalent to the condition of

$$c'_i \leq v_i^{-1}(v^{(K+1)}(\mathbf{c}', \mathbf{p}), p_i), \tag{18}$$

where v_i^{-1} is the inverse function of v_i w.r.t c_i (recall that v_i is decreasing with c_i, so the reverse function v_i^{-1} is existing). Let $U_i(\mathbf{c}'_{-i}, \mathbf{p}) = 0$ for all i, all $\mathbf{c}'_{-i}$ and all $\mathbf{p}$. Then according to (9), advertiser i's payment $x_i(\mathbf{c}', \mathbf{p})$ can be expressed as

$$v_i(c'_i, p_i)\pi_i(c'_i, \mathbf{c}'_{-i}, \mathbf{p}) - \lambda_i(p_i) \int_{c'_i}^{\bar{c}_i} \pi_i(z, \mathbf{c}'_{-i}, \mathbf{p}) dz \tag{19}$$

$$= v_i(c'_i, p_i) - \lambda_i(p_i)(v_i^{-1}(v^{(K+1)}(\mathbf{c}', \mathbf{p}), p_i) - c'_i) \tag{20}$$

$$= v_i(v_i^{-1}(v^{(K+1)}(\mathbf{c}', \mathbf{p}), p_i), p_i). \tag{21}$$

□

Note that v_i^{-1} is the inverse function of v_i, therefore the winner's payment $v_i(v_i^{-1}(v^{(K+1)}(\mathbf{c}', \mathbf{p}), p_i), p_i)$ is exactly $v^{(K+1)}(\mathbf{c}', \mathbf{p})$–the $K+1$ highest reported value. In addition, given any allocation policy, the only freedom for an IC auction is $U_i(\mathbf{c}'_{-i}, \mathbf{p})$. Since IR requires $U_i(\mathbf{c}'_{-i}, \mathbf{p}) \geq 0$ and we set $U_i(\mathbf{c}'_{-i}, \mathbf{p})$ to be zero in WM-RP, the following result is straightforward.

Corollary 1. *Among all IR, IC and EF-RP auctions, WM-RP maximizes the platform's revenue.*

Besides the allocation efficiency, another desiderata of the platform is revenue. We next investigate ad auctions that maximize the platform's revenue in non-strategic display price settings.

4.2 Revenue-Maximizing Auction Design

The following lemma gives a succinct description of advertiser's expected payment, which is key in characterizing revenue-maximizing auctions, and its proof follows the proof of Lemma 3 in [21].

Lemma 3. *Given any IC and IR auction $\mathcal{M}$, any reported price profile $\mathbf{p}$ and any cost profile of others $\mathbf{c}_{-i}$, the expected payment of advertiser i can be represented by*

$$\mathrm{E}_{c_i\sim\mathcal{C}_i}[x_i(\mathbf{c},\mathbf{p})] = \mathrm{E}_{c_i\sim\mathcal{C}_i}[\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)], \tag{22}$$

where $\mathcal{C}_i'$ is the p.d.f of $\mathcal{C}_i$ and $\phi_i(c_i,p_i) = v_i(c_i,p_i) - \lambda_i(p_i)\frac{\mathcal{C}_i(c_i)}{\mathcal{C}_i'(c_i)}$ is defined as the virtual value of advertiser i with respect to (c_i,p_i).

Proof. Given an IC and IR auction $\mathcal{M}$, reported prices $\mathbf{p}$ and $\mathbf{c}_{-i}$, according to the proof of Theorem 1, we know that advertiser i's expected payment $\mathrm{E}_{c_i\sim\mathcal{C}_i}[x_i(\mathbf{c},\mathbf{p})]$ is identical to

$$\int_{\underline{c}_i}^{\bar{c}_i}[v_i(y,p_i)\pi_i(y,\mathbf{c}_{-i},\mathbf{p})]\mathcal{C}_i'(y)dy$$
$$-\int_{\underline{c}_i}^{\bar{c}_i}\mathcal{C}_i'(y)\int_y^{\bar{c}_i}\lambda_i(p_i)\pi_i(z,\mathbf{c}_{-i},\mathbf{p})dzdy. \tag{23}$$

Since p_i is independent of advertiser i's cost c_i, we can change the order of integration for the latter term of formula (23):

$$\int_{\underline{c}_i}^{\bar{c}_i}\mathcal{C}_i'(y)\int_y^{\bar{c}_i}\lambda_i(p_i)\pi_i(z,\mathbf{c}_{-i},\mathbf{p})dzdy$$
$$=\int_{\underline{c}_i}^{\bar{c}_i}\lambda_i(p_i)\pi_i(z,\mathbf{c}_{-i},\mathbf{p})\int_{\underline{c}_i}^{z}\mathcal{C}_i'(y)dydz \tag{24}$$
$$=\int_{\underline{c}_i}^{\bar{c}_i}\lambda_i(p_i)\pi_i(z,\mathbf{c}_{-i},\mathbf{p})\mathcal{C}_i(z)dz. \tag{25}$$

Now, $\mathrm{E}_{c_i\sim\mathcal{C}_i}[x_i(\mathbf{c},\mathbf{p})]$ can be formulated as

$$\int_{\underline{c}_i}^{\bar{c}_i}[v_i(y,p_i)\pi_i(y,\mathbf{c}_{-i},\mathbf{p}) - \lambda_i(p_i)\pi_i(y,\mathbf{c}_{-i},\mathbf{p})\frac{\mathcal{C}_i(y)}{\mathcal{C}_i'(y)}]\mathcal{C}_i'(y)dy$$
$$=\mathrm{E}_{c_i\sim\mathcal{C}_i}[\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)], \tag{26}$$

where $\phi_i(c_i,p_i) = v_i(c_i,p_i) - \lambda_i(p_i)\frac{\mathcal{C}_i(c_i)}{\mathcal{C}_i'(c_i)}$. □

Based on Lemma 3, we can characterize the seller's expected revenue for any IC and IR auction and any given display price profile.

Theorem 4. *Given a display price profile $\mathbf{p}$ and any IC and IR auction $\mathcal{M}$, the expected revenue of the platform can be denoted by*

$$\mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[\sum_{i\in N}x_i(\mathbf{c},\mathbf{p})] = \mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[\sum_{i\in N}\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)]. \tag{27}$$

Proof. Take the expectation, with respect to $\mathbf{c}_{-i}$, of both sides of

$$\mathrm{E}_{c_i\sim\mathcal{C}_i}[x_i(\mathbf{c},\mathbf{p})] = \mathrm{E}_{c_i\sim\mathcal{C}_i}[\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)], \tag{28}$$

we obtain that

$$\mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[x_i(\mathbf{c},\mathbf{p})] = \mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)]. \tag{29}$$

Applying linearity of expectations, we can obtain that

$$\mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[\sum_{i\in N}x_i(\mathbf{c},\mathbf{p})] = \sum_{i\in N}\mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[x_i(\mathbf{c},\mathbf{p})] \tag{30}$$
$$=\sum_{i\in N}\mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)] \tag{31}$$
$$=\mathrm{E}_{\mathbf{c}\sim\mathcal{C}}[\sum_{i\in N}\pi_i(\mathbf{c},\mathbf{p})\phi_i(c_i,p_i)]. \tag{32}$$

□

Theorem 4 shows that the platform's expected revenue is identical to the expectation of the virtual social welfare. To maximize the revenue, we can maximize the virtual social welfare pointwisely for any $(\mathbf{c},\mathbf{p})$. Based on this observation, we now propose the virtual welfare maximizer with reported prices (abbreviated as VWM-RP).

Virtual Welfare Maximizer with Reported Prices

- **Allocation Policy:** Given reports $(\mathbf{c}',\mathbf{p})$, allocate the ad slots to maximize $\sum_{i\in N}\phi_i(c_i',p_i)\pi_i(\mathbf{c}',\mathbf{p})$, break tie arbitrarily.
- **Payment Policy:** For each advertiser $i\in N$, her payment $x_i(\mathbf{c}',\mathbf{p})$ is defined below:
  - if $\pi_i(\mathbf{c}',\mathbf{p}) = 0$, then $x_i(\mathbf{c}',\mathbf{p}) = 0$;
  - if $\pi_i(\mathbf{c}',\mathbf{p}) = 1$, then $x_i(\mathbf{c}',\mathbf{p})$ is defined as

  $$v_i(\phi_i^{-1}(\max\{\phi^{(K+1)}(\mathbf{c}',\mathbf{p}),0\},p_i),p_i),$$

  where ϕ_i^{-1} is the inverse function of ϕ_i w.r.t. c_i and $\phi^{(K+1)}(\mathbf{c}',\mathbf{p})$ denotes the $K+1$ highest virtual value.

Different from WM-RP, in VWM-RP the allocation policy maximizes the virtual social welfare. Note that if all advertisers' virtual values are negative, the platform will not allocate the slots according to the allocation policy, i.e., VWM-RP does not satisfy EF-RP. To implement the allocation policy of VWM-RP, we need a regular condition on the cost distributions, which is defined below.

Definition 7. *A distribution $\mathcal{C}_i$ is regular if the virtual value function $\phi_i(c_i,p_i)$ is non-increasing with c_i for all p_i.*

Recall that $\phi_i(c_i,p_i)$ can be reformulated as $v_i(c_i+\frac{1}{\sigma_i(c_i)},p_i)$, where $\sigma(c_i) = \mathcal{C}_i'(c_i)/\mathcal{C}_i(c_i)$ is the reverse hazard rate of $\mathcal{C}_i$. Since $v_i(\cdot,\cdot)$ is non-increasing with the first variable, a sufficient condition for regularity is $\sigma(\cdot)$ is non-increasing. We next prove that VWM-RP maximizes the seller's revenue for regular cost distributions.

Theorem 5. *Given a set of regular distributions $\{\mathcal{C}_i\}_{i\in N}$, VWM-RP is IC and IR, and maximizes the platform's expected revenue for any fixed display price profile.*

Proof. If $\{\mathcal{C}_i\}_{i\in N}$ are regular, then the allocation policy of VWM-RP is non-increasing with c_i. Following the proof of Theorem 3, we can verify that the payment policy of VWM-RP is consistent with (9), therefore VWM-RP is IC and IR according to Theorem 1. Since VWM-RP maximizes the virtual welfare $\sum_{i\in N}\pi_i(\mathbf{c}',\mathbf{p})\phi_i(c_i',p_i)$ pointwisely for all $(\mathbf{c}',\mathbf{p})$, then based on Theorem 4, we know that VWM-RP maximizes the seller's expected revenue. □

If $\mathcal{C}_i$ is not regular, we can use the "ironing technique" to obtain a surrogate $\tilde{\mathcal{C}}_i$ [21], which is regular, and replaces $\mathcal{C}_i$ in VWM-PR.

5 Auction Design with Strategic Prices

In this section, we study the general setting where advertisers are allowed to report display prices in their favor. Since product costs are private information for all advertisers, Bayesian Nash Equilibrium (BNE) is a suitable solution concept for advertisers' price report.

Definition 8. *Given an IC and IR auction $\mathcal{M}$ and a set of product costs* $\mathbf{c}$*, a strategy profile $\mathbf{p}^{\mathcal{M}}(\mathbf{c}) = (p_i^{\mathcal{M}}(c_i))_{i \in N}$ forms a Bayesian Nash Equilibrium if for all i and all p_i',*

$$\begin{aligned} &\mathrm{E}_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[u_i(c_i, \mathbf{c}, (p_i^{\mathcal{M}}(c_i), \mathbf{p}_{-i}^{\mathcal{M}}(\mathbf{c}_{-i})), \mathcal{M})] \\ &\geq \mathrm{E}_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[u_i(c_i, \mathbf{c}, (p_i', \mathbf{p}_{-i}^{\mathcal{M}}(\mathbf{c}_{-i})), \mathcal{M})], \end{aligned}$$

where $\mathbf{p}_{-i}^{\mathcal{M}}(\mathbf{c}_{-i}) = (p_j^{\mathcal{M}}(c_j))_{j \in N \setminus \{i\}}$.

In other words, a strategy profile $\mathbf{p}^{\mathcal{M}}(\mathbf{c})$ is a BNE if no one can gain more utilities by unilaterally deviating from $\mathbf{p}^{\mathcal{M}}(\mathbf{c})$. Finding BNE in games is known to be a hard problem both analytically and computationally [22]. Previous works derived the BNE analytically only for the simplest auction settings [14]. Recall that the display price information enters into both the allocation and payment policies according to Theorem 1, hence it is impossible to obtain a closed form of the equilibrium price report for all auctions. For tractability, this section studies two special classes of allocation policies, namely the price-independent allocation policy and the affine maximizer allocation policy, where the equilibrium prices are given analytically.

5.1 Price-Independent Allocation Policy

We first investigate price-independent allocation policies, in which the allocation of ad slots is independent of the display price information. It can apply to applications where the display price information is unavailable before the auction or the advertisers tend to adjust their display prices dynamically after the auction. The formal definition of the price-independent allocation policy is given below.

Definition 9 (Price-Independent Allocation Policy)**.** *We say an allocation policy π is price-independent (PI) if for all $i \in N$, and any two reports $(\mathbf{c}', \mathbf{p}^1)$ and $(\mathbf{c}', \mathbf{p}^2)$,*

$$\pi_i(\mathbf{c}', \mathbf{p}^1) = \pi_i(\mathbf{c}', \mathbf{p}^2).$$

Based on Theorem 1, we can easily derive the equilibrium prices for PI allocation policies.

Proposition 1. *Given any IC and IR auction $\mathcal{M}$ with a PI allocation policy, the following price report forms the unique BNE:*

$$\overline{p}_i^{\mathcal{M}}(c_i) = \arg\max_{p_i'} \{\lambda_i(p_i')\}, \forall i \in N. \tag{33}$$

Proof. Suppose $\mathcal{M}$ is IC and IR, then according to Theorem 1 advertiser i's utility can be simplified as

$$u_i(c_i, (c_i, \mathbf{c}_{-i}), \mathbf{p}, \mathcal{M}) = \lambda_i(p_i) \int_{c_i}^{\overline{c}_i} \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz.$$

If the allocation policy is PI, then the term $\int_{c_i}^{\overline{c}_i} \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz$ is independent of p_i, and the expected utility of i can be denoted by

$$\begin{aligned} &\mathrm{E}_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[u_i(c_i, \mathbf{c}, \mathbf{p}, \mathcal{M})] \\ &= \mathrm{E}_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[\lambda_i(p_i) \int_{c_i}^{\overline{c}_i} \pi_i(z, \mathbf{c}_{-i}, \mathbf{p}) dz] \quad (34) \\ &= \lambda_i(p_i) \mathrm{E}_{\mathbf{c}_{-i} \sim \mathcal{C}_{-i}}[\int_{c_i}^{\overline{c}_i} \pi_i(z, \mathbf{c}_{-i}) dz]. \quad (35) \end{aligned}$$

To maximize the expected utility, i can simply choose the price p_i that maximize $\lambda_i(p_i)$, no matter what the others report. Therefore, $(\overline{p}_i^{\mathcal{M}}(c_i))_{i \in N}$ forms the unique BNE, where the uniqueness is from that submitting $\overline{p}_i^{\mathcal{M}}(c_i)$ is a dominant strategy for all $i \in N$. □

Proposition 1 suggests that $\overline{p}_i^{\mathcal{M}}(c_i)$ is also a dominant strategy for all $i \in N$. If the platform searches ad auctions over PI allocation policies, then the following auction maximizes the expected revenue.

Virtual Welfare Maximizer with Price-Independent Allocations (VWM-PIA)

Given a set of conversion rate functions $\{\lambda_i\}_{i \in N}$, let $(\overline{p}_i)_{i \in N}$ be a price profile, where $\overline{p}_i = \arg\max_{p_i'} \lambda_i(p_i')$.

- **Allocation Policy:** Given reports $(\mathbf{c}', \mathbf{p})$, allocate the ad slots to maximize $\sum_{i \in N} \phi_i(c_i', \overline{p}_i) \pi_i(\mathbf{c}', \mathbf{p})$, break tie arbitrarily.
- **Payment Policy:** For each advertiser $i \in N$, her payment $x_i(\mathbf{c}', \mathbf{p})$ is defined below:
  - if $\pi_i(\mathbf{c}', \mathbf{p}) = 0$, then $x_i(\mathbf{c}', \mathbf{p}) = 0$;
  - if $\pi_i(\mathbf{c}', \mathbf{p}) = 1$, then $x_i(\mathbf{c}', \mathbf{p})$ is defined as

$$v_i(\phi_i^{-1}(\max\{\phi^{(K+1)}(\mathbf{c}', \mathbf{p}), 0\}, p_i), p_i).$$

Theorem 6. *Given a set of regular distributions $\{\mathcal{C}_i\}_{i \in N}$, VWM-PIA is IC and IR, and maximizes the platform's expected revenue over all PI allocation policies.*

Proof. According to Proposition 1, if the allocation policy is PI, then the equilibrium price is cost-independent. Hence, Lemma 3 and Theorem 4 can extend to auctions with PI allocation policies. Since the allocation policy is PI and the cost distributions are regular, one can check that the payment policy of VWM-PIA is consistent with (9), i.e., VWM-PIA is IC and IR. In addition, as the allocation policy is PI, then each advertiser will submit $\overline{p}_i$ according to Proposition 1. Recall that VWM-PIA maximizes the virtual welfare pointwisely for each report, then according to Theorem 4 VWM-PIA maximizes the platform's expected revenue over all PI allocation policies. □

5.2 Affine Maximizer Allocation Policy

This section investigates another family of allocation policy, called affine maximizer allocation policy. In such an allocation policy, ad slots are allocated in such a way as to maximize the weighted and boosted social welfare. The advertising platform can utilize this kind of allocation policy to artificially increase the winning chance of advertisers with low values or outright ban certain outcomes in order to boost the revenue [11]. The formal definition of an affine maximizer allocation policy is given below.

Definition 10. *Given a set of advertiser weights $\mathbf{w} = (w_i \in \mathcal{R}_+)_{i \in N}$ and a set of allocation boosts $\mathbf{b} = (b_o \in \mathcal{R})_{o \in \mathcal{O}}$, an allocation policy π is called an affine maximizer if for all reports $(\mathbf{c}', \mathbf{p})$, it picks the following outcome:*

$$\pi(\mathbf{c}', \mathbf{p}) \in \arg\max_{\pi'(\mathbf{c}', \mathbf{p}) \in \mathcal{O}} W^{\mathbf{w}, \mathbf{b}}(\mathbf{c}', \mathbf{p}),$$

where $\mathcal{O}$ denotes the set of all feasible allocations, and

$$W^{\mathbf{w}, \mathbf{b}}(\mathbf{c}', \mathbf{p}) = b_{\pi'(\mathbf{c}', \mathbf{p})} + \sum_{i \in N} w_i v_i(c_i', p_i) \pi_i'(\mathbf{c}', \mathbf{p})$$

is the weighted and boosted social welfare under $\pi'(\mathbf{c}', \mathbf{p})$.

For convenience, we use $\pi^{\mathbf{w},\mathbf{b}}$ to denote the affine maximizer allocation policy defined by $(\mathbf{w}, \mathbf{b})$ and use $W^{\mathbf{w},\mathbf{b}}$ to denote the weighted and boosted social welfare function under $\pi^{\mathbf{w},\mathbf{b}}$. According to Definition 10, it is clear that $\pi^{\mathbf{w},\mathbf{b}}$ is non-increasing with c_i for all possible $(\mathbf{w}, \mathbf{b})$ and all $i \in N$. Based on Theorems 1 and 2, we can design a payment policy denoted as $x^{\mathbf{w},\mathbf{b}}$ for any $\pi^{\mathbf{w},\mathbf{b}}$, ensuring that $(\pi^{\mathbf{w},\mathbf{b}}, x^{\mathbf{w},\mathbf{b}})$ is individually rational and incentive-compatible. This auction mechanism, commonly referred to as the affine maximizer auction, is formally defined as follows.

Affine Maximizer Auctions (AMA)

Predefine a set of advertiser weights $\mathbf{w}$ and allocation boosts $\mathbf{b}$.

- **Allocation Policy:** Given advertiser' reports $(\mathbf{c}', \mathbf{p})$, choosing an outcome $\pi(\mathbf{c}', \mathbf{p}) \in \mathcal{O}$ that maximizes

$$b_{\pi(\mathbf{c}',\mathbf{p})} + \sum_{i\in N} w_i v_i(c'_i, p_i)\pi_i(\mathbf{c}', \mathbf{p}),$$

i.e., applying the affine maximizer allocation policy $\pi^{\mathbf{w},\mathbf{b}}$.
- **Payment Policy:** For each advertiser $i \in N$, her payment $x_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p})$ is defined as

$$\frac{W^{\mathbf{w},\mathbf{b}}(\mathbf{c}'_{-i}, \mathbf{p}_{-i}) - W^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p})}{w_i} + v_i(c'_i, p_i)\pi_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}),$$

where $W^{\mathbf{w},\mathbf{b}}(\mathbf{c}'_{-i}, \mathbf{p}_{-i})$ denotes the maximum affine social welfare without i's participation.

In the above description, the w_i and the b_o are constant parameters. Therefore, every AMA mechanism is exactly characterized by $|N| + |\mathcal{O}|$ parameters. For any assignments of the parameters, we can prove that the corresponding AMA is IC and IR.

Proposition 2. *AMA is IC and IR, and the following price report forms the unique BNE:*

$$\tilde{p}_i^{\mathcal{M}}(c_i) = \arg\max_{p'_i}\{v_i(c_i, p'_i)\}, \forall i \in N. \tag{36}$$

Proof. Given any instance of AMA, denoted as $\mathcal{M}^{\mathbf{w},\mathbf{b}}$, and reports $(\mathbf{c}', \mathbf{p})$, advertiser i's utility can be formulated as

$$\begin{aligned} &u_i(c_i, \mathbf{c}', \mathbf{p}, \mathcal{M}^{\mathbf{w},\mathbf{b}}) = v_i(c_i, p_i)\pi_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}) - x_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}) \\ &= \frac{1}{w_i}[\sum_{j\in N\setminus\{i\}} w_j v_j(c'_j, p_j)\pi_j^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}) + w_i v_i(c_i, p_i)\pi_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}) \\ &+ b_{\pi^{\mathbf{w},\mathbf{b}}(\mathbf{c}',\mathbf{p})} - W^{\mathbf{w},\mathbf{b}}(\mathbf{c}'_{-i}, \mathbf{p}_{-i})]. \end{aligned} \tag{37}$$

Suppose i reports her true cost, then her utility can be simplified as

$$\frac{1}{w_i}[W^{\mathbf{w},\mathbf{b}}((c_i, \mathbf{c}'_{-i}), \mathbf{p}) - W^{\mathbf{w},\mathbf{b}}(\mathbf{c}'_{-i}, \mathbf{p}_{-i})] \geq 0. \tag{38}$$

Therefore, AMA is IR. Next, we show misreporting is not beneficial for advertiser all $i \in N$. Note that the term $W^{\mathbf{w},\mathbf{b}}(\mathbf{c}'_{-i}, \mathbf{p}_{-i})$ is independent of i's report, and hence i's utility is only determined by the following component:

$$\begin{aligned} &\sum_{j\in N\setminus\{i\}} w_j v_j(c'_j, p_j)\pi_j^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}) + w_i v_i(c_i, p_i)\pi_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p}) \\ &+ b_{\pi^{\mathbf{w},\mathbf{b}}(\mathbf{c}',\mathbf{p})}. \end{aligned} \tag{39}$$

Moreover, note that the coefficient of $\pi_i^{\mathbf{w},\mathbf{b}}(\mathbf{c}', \mathbf{p})$ is $w_i v_i(c_i, p_i)$ other than $w_i v_i(c'_i, p_i)$. Hence, for any price p_i and any others' reports $(\mathbf{c}'_{-i}, \mathbf{p}_{-i})$, the value of (39) is maximized by reporting c'_i truthfully according to the definition of affine maximizer allocation policy, i.e., AMA is IC. In addition, the value of (39) is non-decreasing with $v_i(c_i, p_i)$, implying that advertiser i's utility is also non-decreasing with $v_i(c_i, p_i)$. Therefore, choosing a display price $\tilde{p}_i^{\mathcal{M}}(c_i) = \arg\max_{p'_i} v_i(c_i, p'_i)$ can maximize advertiser i' utility, no matter what the others report. Combined with above analysis, we conclude that AMA is individually rational and incentive-compatible, and the price report profile $(\tilde{p}_i^{\mathcal{M}}(c_i))_{i\in N}$ forms the unique BNE in AMA for all advertisers, where the uniqueness is from the fact that submitting $\tilde{p}_i^{\mathcal{M}}(c_i)$ is a dominant strategy for i. □

Recall that the allocation policy of WM-RP is a special instance of the affine maximizer allocation policy. Based on Proposition 2, we can obtain the following important result.

Proposition 3. *WM-RP is IR, IC and EF in strategic price settings.*

Proof. According to the definition of WM-RP and Definition 10, we know that the allocation policy of WM-RP is an instance of the affine maximizer with $\mathbf{b} = \{b_o = 0\}_{o\in\mathcal{O}}$ and $\mathbf{w} = \{w_i = 1\}_{i\in N}$. Since WM-RP is IC and IR for all reported prices, then each advertiser will report $(c_i, \tilde{p}_i^{\mathcal{M}}(c_i))$ to maximize her own utility. Clearly, for any $\mathbf{c}$, we have that the price vector $(\tilde{p}_i^{\mathcal{M}}(c_i))_{i\in N}$ belongs to $\Pi(\mathbf{c})$, i.e., $(\tilde{p}_i^{\mathcal{M}}(c_i))_{i\in N} \in \arg_{\mathbf{p}'} \max_{\mathbf{p}',\pi'} \sum_{i\in N} v_i(c_i, p'_i)\pi'_i(\mathbf{c}, \mathbf{p}')$. Based on Definition 4, Theorem 3 and Proposition 2, we can conclude that WM-RP is not only EF-RP, it is also EF. □

Proposition 3 has important implications. It demonstrates that the advertising platform can design practical ad auctions, namely WM-RP, to maximize the overall social welfare, regardless of whether or not advertisers strategize the display prices of their commodities. To optimize the platform's revenue over all affine maximizer allocation policies, we only need to adjust the parameters $(\mathbf{w}, \mathbf{b})$ in AMA, which is a typical optimization problem. Since there is no known short-cut for calculating the expected revenue [11], previous studies have developed many techniques to search for the (approximate) optimal parameters, e.g., grid-based gradient descent approach [17, 18], LP-based heuristic [11], neutral networks [4].

6 Conclusion

In this paper, we formulated the problem of ad auction design in the presence of display prices. We characterized all IC, IR auctions with display prices and analyzed the welfare-maximizing and revenue-maximizing auctions under various scenarios. Besides theoretical implications, the study results also provide meaningful insights for practitioners designing ad auctions to optimize the performance of ad delivery. For the objective of maximizing social welfare, our findings (Theorem 3, Corollary 1 and Proposition 3) suggest that the platforms can simply adopt WM-RP. For the profit-maximizing objective, the results (Theorems 5 and 6) indicate that the platforms should set a display-price-based reserve price for each advertiser. As Lemma 3 cannot generally extend to the strategic price settings, designing revenue-maximizing auctions in general settings presents inherent challenges, and we leave the full analysis as future work.

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