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Complexity of Control by Adding or Deleting Edges in Graph-Restricted Weighted Voting Games

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Abstract. Graph-restricted weighted voting games generalize weighted voting games, a well-studied class of succinct simple games, by embedding them into a communication structure: a graph whose vertices are the players some of which are connected by edges. In such games, only connected coalitions are taken into consideration for calculating the players' power indices. We focus on the probabilistic Penrose–Banzhaf index [5] and the Shapley–Shubik index [18] and study the computational complexity of manipulating these games by an external agent who can add edges to or delete edges from the graph. For the problems modeling such scenarios, we raise some of the lower bounds obtained by Kaczmarek and Rothe [9] from NP- or DP-hardness to PP-hardness, where PP is probabilistic polynomial time. We also solve one of their open problems by showing that it is a coNP-hard problem to maintain the Shapley–Shubik index of a given player in a graph-restricted weighted voting game when edges are deleted.

1 Introduction

Weighted voting games naturally model settings in which, e.g., the members of some committee have certain voting weights, and for a decision to pass some quota needs to be reached. To measure the influence of players in weighted voting games, power indices like the probabilistic Penrose–Banzhaf index [5] and the Shapley–Shubik index [18] are used. Following the observation that voting weight does not equal political influence, these power indices formalize the notion of "king makers" by identifying influence with the frequency by which a committee member may be the decisive vote in a coalition. A quite recent example for the difference between voting weight and political power is in the 36th Israeli government that was formed in 2021: Out of 61 members of parliament, coming from seven parties, Naftali Bennett, the leader of a party with only seven members, succeeded in being the prime minister showing that his political power was exceeding that of his relative voting weight.

Graph-restricted weighted voting games generalize weighted voting games by assuming some communication structure among the players. Myerson [11] introduced *graph-restricted games*, i.e., cooperative games with undirected graphs that describe which sets of players, together, can form a coalition. Using the model of Myerson, Napel et al. [12] defined graph-restricted weighted voting games and Skibski et al. [19] studied the players' power in these games in terms of their computational complexity.

Israeli politics may serve as an example of the relevance of topological structures in the context of weighted voting games. For example, the 36th Israeli government that was mentioned above contained quite a diverse set of political parties, spanning from the political left through the political center and to the political right. It is quite unreasonable that a coalition of only political left and right could have been formed, due to the political divide between the left and right; however, the inclusion of the political center may have helped in bridging the ideological gaps between them, thus making the coalition structurally connected and therefore viable. Generally speaking, one may imagine either a graph containing a vertex for each party and a missing edge between pairs of parties that may not sit together in the same coalition (due to ideological gaps and/or personal disputes) or perhaps a graph with three cliques: one for the left, one for the center, one for the right parties; and with some edges between the cliques. This naturally raises the question of how the influence (or power) of given player may change, depending on which topological structure is used.

For both the probabilistic Penrose–Banzhaf and the Shapley–Shubik index, we study the complexity of manipulating the power of a given player in a graph-restricted weighted voting game via an alteration of the underlying graph. Kaczmarek and Rothe [9] defined control by adding edges and by deleting edges with the goal of increasing, decreasing, or maintaining a given player's power in such games, and they established the first hardness results. We improve some of their lower bounds for these control problems by raising their NP- or DP-hardness results to PP-hardness for control by adding and by deleting edges when the goal is to increase or to decrease a given player's power.[1] Note that a PP-hardness lower bound was also achieved by Rey and Rothe [16] for the different scenario of merging and splitting players in weighted voting games (the latter is a.k.a. false-name manipulation), as introduced and first studied by Aziz et al. [1]. Indeed, we will reduce from problems that Rey and Rothe [16] showed to be PP-complete and will apply their proof technique to our setting. This proof technique has also been applied

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[1] Gill [8] introduced PP as the class of problems solvable by probabilistic Turing machines. The complexity class DP, introduced by Papadimitriou and Yannakakis [13], is the class of problems that can be written as the difference of two NP sets; this class is is also known as the second level of the boolean hierarchy over NP, first studied by Cai et al. [3].

in the context of control by adding players to or deleting players from weighted voting games (which are not graph-restricted) by Rey and Rothe [17] and Kaczmarek and Rothe [10].

For the goal of maintaining a given player's power in graph-restricted weighted voting games, Kaczmarek and Rothe [9] showed that control by adding edges is coNP-hard for both power indices and control by deleting edges is coNP-hard for the Penrose–Banzhaf index, leaving this question open for the Shapley–Shubik index. We solve this open question by showing that this problem is coNP-hard as well. Our complexity results for lower bounds are summarized in Table 1. As also shown by Kaczmarek and Rothe [9], the best known upper bound for all our problems is $\mathrm{NP}^{\mathrm{PP}}$, the class of problems solvable by an NP oracle Turing machine accessing a PP oracle.

Table 1. Overview of complexity results for control problems in graph-restricted weighted voting games with respect to the Shapley–Shubik (φ) and the probabilistic Penrose–Banzhaf index (β).

Goal		**Control by adding edges**	**Control by deleting edges**
Decrease	β	PP-hard (Theorem 5)	PP-hard (Theorem 6)
	φ	PP-hard (Theorem 5)	PP-hard (Theorem 6)
Increase	β	PP-hard (Theorem 3)	PP-hard (Theorem 6)
	φ	PP-hard (Theorem 4)	PP-hard (Theorem 6)
Maintain	β	coNP-hard [9]	coNP-hard [9]
	φ	coNP-hard [9]	coNP-hard (Theorem 7)

In Section 2, we provide the needed definitions and notation. In Sections 3 and 4, we present our results regarding control by adding edges and by deleting edges, respectively, with the goal to increase, decrease, or maintain the Shapley–Shubik or the Penrose–Banzhaf power index of a distinguished player. We conclude in Section 5.

2 Preliminaries

We discuss the needed notions from cooperative game theory and computational complexity. We assume the reader to be familiar with the basic concepts of graph theory. For an undirected graph $G = (V, E)$, we denote the set of nonedges of G by $\overline{E} = \{\{x, y\} \mid x, y \in V \wedge x \neq y \wedge \{x, y\} \notin E\}$. Further, for a subset $E' \subseteq \overline{E}$ of nonedges of G, we denote by $G_{\cup E'} = (V, E \cup E')$ the graph that results from G by adding the elements of E' as new edges to it. Similarly, for a subset $E'' \subseteq E$ of edges of G, we denote by $G_{\setminus E''} = (V, E \setminus E'')$ the graph that results from G by deleting the edges of E'' from it.

Let $N = \{1, \dots, n\}$ denote a set of players. A *coalitional game* is a pair (N, v), where $v : 2^N \to \mathbb{R}_{\geq 0}$ assigns a nonnegative real value to each coalition (i.e., subset) of players; it is said to be *simple* if it is *monotonic* (i.e., $v(A) \leq v(B)$ whenever $A \subseteq B$) and $v(C) \in \{0, 1\}$ for each $C \subseteq N$ (where $v(C) = 1$ means that coalition C *wins*, and $v(C) = 0$ means that C *loses*). A *weighted voting game* $\mathcal{G} = (w_1, \dots, w_n; q)$ is a simple coalitional game with players N that consists of a quota $q \in \mathbb{N}$ (i.e., a given threshold) and nonnegative integer weights, where w_i is the weight of player $i \in N$. For each coalition $S \subseteq N$, letting $w_S = \sum_{i \in S} w_i$, S *wins* if $w_S \geq q$, and otherwise it *loses*.

The players' significance in a given game is usually measured by so-called *power indices*, which take into consideration how many coalitions a player is pivotal for (i.e., how many coalitions $S \subseteq N \setminus \{i\}$ a player i can make win: $v(S \cup \{i\}) - v(S) = 1$). We study two of the most popular power indices:

- the *probabilistic Penrose–Banzhaf power index*, defined by

$$\beta(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))}{2^{n-1}},$$

which Dubey and Shapley [5] introduced as an alternative to the original *normalized Penrose–Banzhaf index* [14, 2], and
- the *Shapley–Shubik power index*, introduced by Shapley and Shubik [18] as

$$\varphi(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} |S|!(n - 1 - |S|)!(v(S \cup \{i\}) - v(S))}{n!}.$$

A *graph-restricted weighted voting game* is a weighted voting game $\mathcal{G} = (w_1, \dots, w_n; q)$ with players $N = \{1, \dots, n\}$ together with a graph $G = (N, E)$, where

$$v(S) = \begin{cases} 1 & \text{if } S \text{ has a connected part } S' \text{ with } w_{S'} \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

Graph-restricted weighted voting games are a generalization of weighted voting games, where weighted voting games are the special cases with a complete graph as their communication structure. In this situation, only the total weight of a coalition determines whether it wins or loses. However, if we limit the possibilities in communication among players, a coalition's weight alone is not enough.

Before we present the way, due to Skibski et al. [19], for how to define appropriate power indices in graph-restricted weighted voting games (also used mainly in our proofs), let us first define a few useful notions referring to coalitions in the sense of graph restrictions. For $S \subseteq N$, we denote a *maximal connected subset of S in G* as S/G. The *set of all winning connected coalitions* is defined as $\mathcal{WC} = \{S \subseteq N \mid w_S \geq q \text{ and } S \text{ is connected}\}$ and the *set of winning connected coalitions with player i* is denoted by $\mathcal{WC}_i$. The *set of all pivotal winning connected coalitions of player i* is defined as $\mathcal{PWC}_i = \{S \in \mathcal{WC}_i \mid ((S \setminus \{i\})/G) \cap \mathcal{WC} = \emptyset\}$. Further, let $\mathcal{N}(i) = \{j \in N \mid \{i, j\} \in E\}$ denote the neighborhood of vertex i in graph $G = (N, E)$, and let $\mathcal{N}(S) = \left(\bigcup_{i \in S} \mathcal{N}(i)\right) \setminus S$ be the set of neighbors of a subset $S \subseteq N$ of vertices.

Skibski et al. [19] presented the following formulas for computing the Penrose–Banzhaf power index and the Shapley–Shubik power index in graph-restricted weighted voting games using only the set of all pivotal winning connected coalitions of a player.

Theorem 1 (Skibski et al. [19]) *Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with players N. The Penrose–Banzhaf index of player $i \in N$ in $(\mathcal{G}, G)$ satisfies*

$$\beta((\mathcal{G}, G), i) = \sum_{S \in \mathcal{PWC}_i} \frac{1}{2^{|S| + |\mathcal{N}(S)| - 1}}.$$

The Shapley–Shubik index of player i in $(\mathcal{G}, G)$ satisfies

$$\varphi((\mathcal{G}, G), i) = \sum_{S \in \mathcal{PWC}_i} \frac{(|S| - 1)!|\mathcal{N}(S)|!}{(|S| + |\mathcal{N}(S)|)!}.$$

We assume familiarity with the basic concepts of computational complexity theory, such as the well-known complexity classes P (*deterministic polynomial time*), NP (*nondeterministic polynomial time*), and coNP (the class of problems that are complements of NP sets); also, recall PP (*probabilistic polynomial time* [8]) and the class DP that contains the differences of two NP sets [13] from Footnote 1. Note that P ⊆ NP ⊆ DP ⊆ PP and P ⊆ coNP ⊆ DP.

We will use the notions of completeness and hardness for a complexity class based on the polynomial-time many-one reducibility: A problem X (polynomial-time many-one) reduces to a problem Y (for short, $X \leq_{\mathrm{m}}^{\mathrm{p}} Y$) if there exists a polynomial-time computable function ρ such that for each input x, $x \in X$ if and only if $\rho(x) \in Y$; Y is hard for a complexity class $\mathcal{C}$ if $C \leq_{\mathrm{m}}^{\mathrm{p}} Y$ for each $C \in \mathcal{C}$; and Y is hard for $\mathcal{C}$ if Y is $\mathcal{C}$-hard and $Y \in \mathcal{C}$.

We recall some well-known NP-complete problems [7] the counting versions of which will be used later on to provide reductions in our hardness proofs:

- SUBSETSUM: Given a sequence $(a_1, \dots, a_n)$ of positive integers and an integer q, does there exist a subset $A \subseteq \{1, \dots, n\}$ such that $\sum_{i \in A} a_i = q$?
- X3C: Given a set B, with $|B| = 3k$ for some $k \in \mathbb{N}$, and a family $\mathcal{S}$ of three-element subsets of B, does there exist an exact cover, i.e., a subfamily $\mathcal{S}^* \subseteq \mathcal{S}$ such that each element from B is contained in exactly one set in $\mathcal{S}^*$?

Valiant [21] introduced $\#\mathrm{P}$ as the class of functions that give the number of solutions of NP problems. Deng and Papadimitriou [4] showed that computing the Shapley–Shubik index of a player in a given weighted voting game is complete for $\#\mathrm{P}$ via *functional* polynomial-time many-one reductions. Prasad and Kelly [15] proved that computing the probabilistic Penrose–Banzhaf index is *parsimoniously* complete for $\#\mathrm{P}$, i.e., $\#\mathrm{P}$-hardness is shown by a functional polynomial-time many-one reduction that preserves the number of solutions. Skibski et. al. [19] observed that computing the Shapley–Shubik index and the probabilistic Penrose–Banzhaf index in *graph-restricted* weighted voting games is $\#\mathrm{P}$-complete as well.

$\#\mathrm{P}$ and PP, even though the former is a class of functions and the latter a class of decision problems, are closely related by the well-known result that $\mathrm{P}^{\mathrm{PP}} = \mathrm{P}^{\#\mathrm{P}}$; note that P^{PP} contains the entire polynomial hierarchy by Toda's celebrated result [20]. We also use in our PP-hardness proofs the following two problems that were shown to be PP-complete by Rey and Rothe [16]:

COMPARE-#SUBSETSUM-RR	
Given:	A sequence $(a_1, \dots, a_n)$ of positive integers, where $\alpha = \sum_{i=1}^{n} a_i$.
Question:	Is it true that $\#\text{SUBSETSUM}((a_1, \dots, a_n), \frac{\alpha}{2} - 2) > \#\text{SUBSETSUM}((a_1, \dots, a_n), \frac{\alpha}{2} - 1)$?

The problem COMPARE-#SUBSETSUM-ЯЯ is defined analogously, except that the inequality is inverted in the question, i.e., "$>$" is replaced by "$<$."

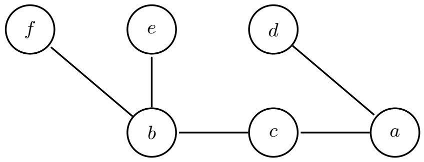
Finally, we will use the following lemma in some of our proofs. The result regarding X3C instances in it is due to Faliszewski and Hemaspaandra [6] and Rey and Rothe [16] observed that it can also be transferred to instances of SUBSETSUM since #X3C parsimoniously reduces to #SUBSETSUM.

Lemma 2 *Every* X3C *instance* $(B', \mathcal{S}')$ *can be transformed into an* X3C *instance* $(B, \mathcal{S})$*, where* $|B| = 3k$ *and* $|\mathcal{S}| = n$*, such that* $\frac{k}{n} = \frac{2}{3}$ *without changing the number of solutions. Consequently, we can assume that the size of each solution in a* SUBSETSUM *instance is* $\frac{2n}{3}$*, that is, each subsequence summing up to the given quota contains the same number of elements.*

3 Adding Edges to a Communication Structure

Before we delve into the computational complexity of control by adding edges in graph-restricted weighted voting games, we consider an example that shows that if we add new edges to the communication structure, connecting players other than the distinguished player i with each other, all situations are possible to happen: i's power can increase, decrease, or be maintained by such a control action.

Example 1 *Consider the graph-restricted weighted voting game* $(\mathcal{G}, G)$ *with players* $N = \{a, b, \dots, f\}$, $\mathcal{G} = (1, 2, 4, 2, 2, 2; 7)$ *(i.e., player* a *has weight* 1*, player* c *has weight* 4*, all other players have weight* 2*, and the quota is* 7*), and the following communication structure* G*:*



For player a, $\mathcal{PWC}_a = \{\{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$. *After adding an edge between players* e *and* f*, the set* $\mathcal{PWC}_a$ *does not change. If we add an edge between the players* e *and* d*, player* a *will become pivotal for the coalitions* $\{a, b, d, e\}$ *and* $\{a, c, d, e\}$*. Finally, if an edge between* c *and* d *is added, player* a *will not be pivotal for the coalition* $\{a, b, c, d\}$ *anymore. Indeed, it is even possible that adding edges among other players can reduce the distinguished player's power indices even to* 0.

We begin our technical treatment by considering the impact of adding new edges to the communication structure of a given graph-restricted weighted voting game. By this structural change to the game, we allow some players to communicate with each other for whom this was impossible before.

Let us define the decision problem in which we ask whether some power index PI can be *increased*:

CONTROL BY ADDING EDGES TO INCREASE PI	
Given:	A graph-restricted weighted voting game $(\mathcal{G}, G)$ with players $N = \{1, \dots, n\}$, a communication structure $G = (N, E)$, $\lvert E\rvert < \binom{n}{2}$, a distinguished player $p \in N$, and a nonnegative integer k.
Question:	Is it sufficient to add k or fewer edges $E' \subseteq \overline{E}$, $E' \neq \emptyset$, to G to obtain a new graph-restricted weighted voting game $(\mathcal{G}, G_{\cup E'})$ for which it holds that $\mathrm{PI}((\mathcal{G}, G_{\cup E'}), p) > \mathrm{PI}((\mathcal{G}, G), p)$?

Analogously, we define the decision problems for *decreasing* and *maintaining* a distinguished player's power (by replacing "$>$" in the question by "$<$" or "$=$").

Kaczmarek and Rothe [9] showed that control by adding edges to increase a given player's Penrose–Banzhaf index is DP-hard and that this problem is NP-hard for the Shapley–Shubik index. We improve these lower bounds to PP-hardness for both power indices, starting with the Penrose–Banzhaf index.

Theorem 3 *Control by adding edges between players to increase a distinguished player's Penrose–Banzhaf index in a graph-restricted weighted voting game is* PP*-hard.*

Proof. To prove PP-hardness, we reduce from the PP-hard problem COMPARE-#SUBSETSUM-RR. Given an instance

$A = (a_1, \ldots, a_n)$ of COMPARE-#SUBSETSUM-RR, where $\alpha = \sum_{i=1}^{n} a_i$, we define

$$\xi_1 = \#\text{SUBSETSUM}\Big(A, \frac{\alpha}{2} - 1\Big),$$
$$\xi_2 = \#\text{SUBSETSUM}\Big(A, \frac{\alpha}{2} - 2\Big)$$

and construct an instance of our control problem as follows. Let $z_1 = \cdots = z_n = 1$ and choose r_1, r_2, q, and ℓ such that

$$\begin{aligned}10^{r_1} &> n + \sum_{j=1}^{n} (a_j \cdot 10^{r_2}),\\ 10^{r_2} &> n,\\ q &= 2\sum_{j=1}^{n} a_j \cdot 10^{r_1} + 1, \text{ and}\\ 10^{\ell} &> 2q + n + \sum_{j=1}^{n} (a_j \cdot 10^{r_1} + a_j \cdot 10^{r_2}) + 1.\end{aligned}$$

Construct the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\begin{aligned}\mathcal{G} = \Big(&1, 10^{\ell}, q - \Big(\frac{\alpha}{2} - 1\Big) \cdot 10^{r_1}, a_1 \cdot 10^{r_1}, \ldots, a_n \cdot 10^{r_1},\\ &q - \Big(\frac{\alpha}{2} - 2\Big) \cdot 10^{r_2}, a_1 \cdot 10^{r_2}, \ldots, a_n \cdot 10^{r_2},\\ &z_1, \ldots, z_n; 10^{\ell} + q + 1\Big)\end{aligned}$$

with $3n + 4$ players in N, distinguished player 1, and the following communication graph $G = (N, E)$, displayed in Figure 1: The players 1, 2, 3, and all players with weights $z_1, \ldots, z_n$ form a complete subgraph, the players $4, \ldots, n+3$ are connected with each other and with all previous players except the players 1 and 2, and the players $n + 4, \ldots, 2n + 4$ form another complete component without any edges to the remaining players. Call the first component X and the other component Y. So, the only edges that can possibly be added are edges between the components X and Y, and within X, between the players from $\{1, 2\}$ and those from $\{4, \ldots, n + 3\}$.

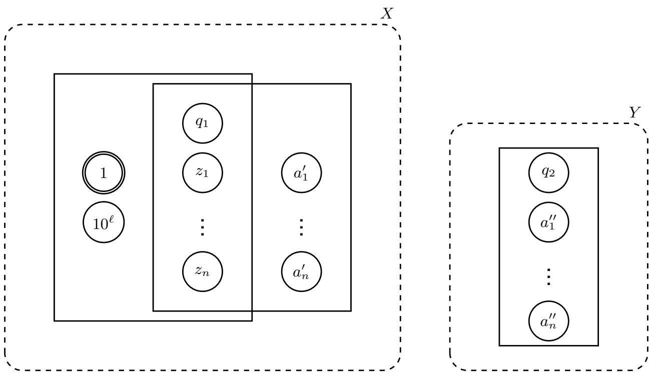


Figure 1. Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 3. The weights of the players are used instead of the player names, where $q_1 = q - \left(\frac{\alpha}{2} - 1\right) \cdot 10^{r_1}$, $q_2 = q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^{r_2}$, $a'_i = a_i \cdot 10^{r_1}$, and $a''_i = a_i \cdot 10^{r_2}$ for $i \in \{1, \ldots, n\}$. The double circle indicates the distinguished player. The dashed rectangles represent connected components and regular rectangles represent complete subgraphs of G.

Player 1's Penrose–Banzhaf index in this game is

$$\beta((\mathcal{G}, G), 1) = \frac{\xi_1}{2^{2n+2}}.$$

Let the addition limit be $k = 1$. We will show that

$$(\exists e \in \overline{E})\ [\beta((\mathcal{G}, G_{\cup\{e\}}), 1) - \beta((\mathcal{G}, G), 1) > 0] \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. Add an edge e between the distinguished player 1 and, for example, player $n+4$ with weight $q - (\frac{\alpha}{2} - 2) \cdot 10^{r_2}$. Then 1's Penrose–Banzhaf index will increase to

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_1}{2^{2n+3}} + \frac{\xi_2}{2^{2n+3}}\\ &> \frac{\xi_1 + \xi_1}{2^{2n+3}} = \frac{\xi_1}{2^{2n+2}} = \beta((\mathcal{G}, G), 1).\end{aligned}$$

From left to right, assume now that $\xi_1 \geq \xi_2$. Adding any of the edges inside of the component X will not change the distinguished player's Penrose–Banzhaf index. Let us focus on edges connecting the components X and Y with each other. Let e be an added edge, and consider the following possibilities which players are connected by e:

Case 1: e connects either player 1 or player 2 with player $n + 4$. Then

$$\beta((\mathcal{G}, G_{\cup\{e\}}), 1) = \frac{\xi_1}{2^{2n+3}} + \frac{\xi_2}{2^{2n+3}} \leq \frac{\xi_1 + \xi_1}{2^{2n+3}} = \beta((\mathcal{G}, G), 1).$$

Case 2: e connects either player 1 or 2 with one of the players from $Y \setminus \{n + 4\}$, say i, and let $\xi'_2 \leq \xi_2$ be the number of coalitions containing i for which player 1 is pivotal. Then

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_1}{2^{2n+3}} + \frac{\xi'_2}{2^{2n+3}} \leq \frac{\xi_1 + \xi_2}{2^{2n+3}}\\ &\leq \frac{\xi_1 + \xi_1}{2^{2n+3}} = \beta((\mathcal{G}, G), 1).\end{aligned}$$

Case 3: e connects any player from Y with one of the players from $\{3, \ldots, n + 3\}$. This will only add a neighbor to those coalitions counted in the old Penrose–Banzhaf index of player 1 and will not make player 1 pivotal for any other coalition, so 1's new Penrose–Banzhaf index will not be greater.

Case 4: e connects the component Y with any player from $\{2n + 5, \ldots, 3n + 4\}$. In this case, the Penrose–Banzhaf index of player 1 will not change, since the latter players are only neighbors of the set counted in player 1's old Penrose–Banzhaf index and they cannot be contained in any coalition for which 1 could be pivotal in a game with the given set of players and weights and the given quota.

Therefore, it is impossible for the distinguished player's Penrose–Banzhaf index to increase by adding any edge. ❑

By slightly modifying the reduction provided in the previous proof, we obtain the same result for the Shapley–Shubik index. The proof of Theorem 4 is omitted due to space limitations.

Theorem 4 *Control by adding edges between players to increase a distinguished player's Shapley–Shubik index in a graph-restricted weighted voting game is* PP*-hard.*

For the goal of decreasing a given player's power, Kaczmarek and Rothe [9] established NP-hardness for control by adding edges with respect to both power indices. We again improve their results by showing PP-hardness, using the same reduction for both problems.

Theorem 5 *Control by adding edges between players to decrease a distinguished player's Penrose–Banzhaf or Shapley–Shubik index in a graph-restricted weighted voting game is* PP*-hard.*

Proof. Again, we prove PP-hardness of our two control problems by reducing from the PP-hard problem COMPARE-#SUBSETSUM-RR. Let $A = (a_1, \dots, a_n)$ with $\alpha = \sum_{i=1}^{n} a_i$ be a given instance of COMPARE-#SUBSETSUM-RR. Define

$$\xi_1 = \#\text{SUBSETSUM}\Big(A, \frac{\alpha}{2} - 1\Big) \quad \text{and}$$
$$\xi_2 = \#\text{SUBSETSUM}\Big(A, \frac{\alpha}{2} - 2\Big)$$

and construct an instance of our control problem (for either the Penrose–Banzhaf or the Shapley–Shubik index) as follows.

Let $z = 1$ and $r_2 = 1$ and choose r_1, q, and ℓ such that

$$\begin{aligned} 10^{r_1} &> 2z + \sum_{j=1}^{n} a_j \cdot 10^{r_2}, \\ q &= 2\sum_{j=1}^{n} a_j \cdot 10^{r_1} + 1, \text{ and} \\ 10^{\ell} &> 2q + 2z + 2\sum_{j=1}^{n} (a_j \cdot 10^{r_1} + a_j \cdot 10^{r_2}) + 1. \end{aligned}$$

Consider the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\begin{aligned} \mathcal{G} = \Big(&1, 10^l, z, z, \\ &q - \Big(\frac{\alpha}{2} - 2\Big) \cdot 10^{r_1} - z, a_1 \cdot 10^{r_1}, \dots, a_n \cdot 10^{r_1}, \\ &q - \Big(\frac{\alpha}{2} - 1\Big) \cdot 10^{r_2} - 2z, a_1 \cdot 10^{r_2}, \dots, a_n \cdot 10^{r_2}; \\ &10^l + q + 1\Big) \end{aligned}$$

with $2n + 6$ players, distinguished player 1 and the following communication graph $G = (N, E)$: All players but player 4 form a complete subgraph, whereas player 4 is an isolated vertex. Thus we can only add edges between player 4 and the large complete subgraph.

Due to space limitations, the proof of correctness of our reduction is omitted for the Penrose–Banzhaf index. Turning now to the Shapley–Shubik power index, consider the above reduction between the problem COMPARE-#SUBSETSUM-RR and our control problem. By Lemma 2, we may assume that every set counted in ξ_1 and in ξ_2 has the same size t. Our distinguished player's Shapley–Shubik index in the constructed game $(\mathcal{G}, G)$ is

$$\varphi((\mathcal{G}, G), 1) = \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!}.$$

Let the addition limit be $k = 1$. We will show that

$$(\exists e \in \overline{E})\ [\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) < 0] \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. After adding an edge e between players 3 and 4, the Shapley–Shubik index of our distinguished player will decrease to

$$\begin{aligned} &\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) \\ &\quad = \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad < \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad = \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\quad = \varphi((\mathcal{G}, G), 1). \end{aligned}$$

From left to right, assume now that $\xi_1 \geq \xi_2$. Let e be an added edge (which, recall, can only connect player 4 with any of the other players). Consider the following cases.

Case 1: e connects player 4 with either player 1 or player 2. Then

$$\begin{aligned} &\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) \\ &\quad = 2\xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad \geq 2\xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad \geq \varphi((\mathcal{G}, G), 1). \end{aligned}$$

Case 2: e connects player 4 with player 3. Then

$$\begin{aligned} &\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) \\ &\quad = \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad \geq \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad = \varphi((\mathcal{G}, G), 1). \end{aligned}$$

Case 3: e connects player 4 with player 5. Then

$$\begin{aligned} \varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= 2\xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} \\ &= \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} \\ &\quad + \frac{2n-t+2}{t+4} \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \varphi((\mathcal{G}, G), 1), \end{aligned}$$

since $\frac{2n-t+2}{t+4} \geq 1$ due to the fact that $t < n$.

Case 4: e connects player 4 with one of the players from $\{6, \dots, n+5\}$, say with player j. Let $\xi_2' \leq \xi_2$ be the number of coalitions containing j for which our distinguished player 1 is pivotal. Then

$$\begin{aligned} &\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) \\ &= (\xi_2 - \xi_2') \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + 2\xi_2' \frac{(t+3)!(2n-t+2)!}{(2n+6)!} \\ &= (\xi_2 - \xi_2') \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\quad + 2\frac{2n-t+2}{2n+6} \xi_2' \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &= (\xi_2 - \xi_2') \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\quad + 2\left(1 - \frac{t+4}{2n+6}\right) \xi_2' \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\geq (\xi_2 - \xi_2') \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + \xi_2' \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &= \varphi((\mathcal{G}, G), 1). \end{aligned}$$

Case 5: e connects player 4 with player $n+6$. Then

$$\begin{aligned} &\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) \\ &\quad = \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad = \varphi((\mathcal{G}, G), 1) + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\quad \geq \varphi((\mathcal{G}, G), 1). \end{aligned}$$

Case 6: Finally, e connects player 4 with one of the players from $\{n+7,\ldots,2n+6\}$, say with player j. Let $\xi_1' \leq \xi_1$ be the number of coalitions containing j for which our distinguished player 1 is pivotal. Then

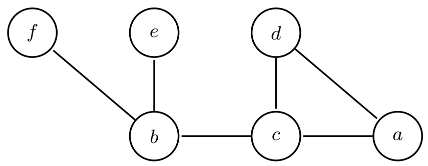
$$\begin{aligned}\varphi((\mathcal{G},G_{\cup\{e\}}),1) \\ &= \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + \xi_1' \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &= \varphi((\mathcal{G},G),1) + \xi_1' \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \varphi((\mathcal{G},G),1).\end{aligned}$$

Therefore, in each case, also the distinguished player's Shapley–Shubik index does not decrease. ❑

4 Deleting Edges from a Communication Structure

We now turn to control by deleting edges and study how limiting the communication among players can change their power in a graph-restricted weighted voting game. Again, we start with an example.

Example 2 *Let us consider again the graph-restricted weighted voting game $(\mathcal{G},G)$ from Example 1 and let us modify this game's communication graph creating a new game $(\mathcal{G},H)$ with the following graph H:*



Let us consider player a again. In the new game, $\mathcal{PWC}_a = \{\{a,b,c\},\{a,c,d\}\}$. If we delete the edge between a and c, player a will stop being pivotal for the coalition $\{a,b,c\}$. If we remove the edge between the players a and d, a will stay pivotal for the same coalitions as before. If we delete the edge between c and d, we will get the game $(\mathcal{G},G)$, so player a will be pivotal for one coalition more than before our manipulation.

As in the previous section, we define the problem of control by deleting edges to increase a power index PI; the other two definitions (where the goal is to decrease or to maintain an index) are again analogous.

CONTROL BY DELETING EDGES TO INCREASE PI	
Given:	A graph-restricted weighted voting game $(\mathcal{G},G)$ with players $N = \{1,\ldots,n\}$, a communication structure $G = (N,E)$, a distinguished player $p \in N$, and a positive integer $k \leq \|E\|$.
Question:	Can at most k edges $E' \subseteq E$, $E' \neq \emptyset$, be deleted from G such that for the new game $(\mathcal{G},G_{\setminus E'})$, it holds that $\mathrm{PI}((\mathcal{G},G_{\setminus E'}),p) > \mathrm{PI}((\mathcal{G},G),p)$?

For control by deleting edges to increase or to decrease a given player's power, Kaczmarek and Rothe [9] established DP-hardness for the Penrose–Banzhaf index and NP-hardness for the Shapley–Shubik index. Our results in Theorem 6 again improve all four lower bounds to PP-hardness (as summarized in Table 1 in Section 1). Note that these problems are PP-hard to solve even if we delete only a single edge from the given communication graph.

Theorem 6 *Control by deleting edges between players to increase or to decrease a distinguished player's Penrose–Banzhaf or Shapley–Shubik index in a graph-restricted weighted voting game is* PP*-hard.*

Proof. We show PP-hardness of control by deleting edges to decrease the distinguished player's Penrose–Banzhaf or Shapley–Shubik index by means of a reduction from the COMPARE-#SUBSETSUM-RR problem. PP-hardness of control by deleting edges to *increase* the distinguished player's Penrose–Banzhaf or Shapley–Shubik index can be proven analogously with the same reduction when starting from the COMPARE-#SUBSETSUM-ЯЯ problem.

Let $A = (a_1,\ldots,a_n)$ be a given instance of COMPARE-#SUBSETSUM-RR with $\alpha = \sum_{i=1}^{n} a_i$. Let

$$\xi_1 = \#\text{SUBSETSUM}\Big(A, \frac{\alpha}{2}-1\Big),$$

$$\xi_2 = \#\text{SUBSETSUM}\Big(A, \frac{\alpha}{2}-2\Big)$$

and construct the control problem instance consisting of a graph-restricted weighted voting game

$$\mathcal{G} = \Big(1, a_1, \ldots, a_n, 2\alpha, 1; \frac{5\alpha}{2}\Big)$$

with $n+3$ players, distinguished player 1, and the following communication graph $G = (N,E)$: All players except $n+3$ form a complete subgraph and player $n+3$ is connected with the distinguished player 1 by an edge called x.

Let t be the size of each solution according to Lemma 2. Player 1's Penrose–Banzhaf index in this game is

$$\beta((\mathcal{G},G),1) = \frac{\xi_1+\xi_2}{2^{n+2}},$$

and player 1's Shapley–Shubik index in it is

$$\varphi((\mathcal{G},G),1) = \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} + \xi_2 \frac{(t+2)!(n-t)!}{(n+3)!}.$$

Let the deletion limit be $k = 1$. We will first show that for the distinguished player's Penrose–Banzhaf index, we have

$$(\exists e \in E)\ [\beta((\mathcal{G},G_{\setminus\{e\}}),1) - \beta((\mathcal{G},G),1) < 0] \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. Then, after deleting the edge x, the Penrose–Banzhaf index of player 1 will decrease to

$$\beta((\mathcal{G},G_{\setminus\{x\}}),1) = \frac{\xi_1}{2^{n+1}} = \frac{2\xi_1}{2^{n+2}} < \frac{\xi_1+\xi_2}{2^{n+2}} = \beta(\mathcal{G},1).$$

From left to right, assume that $\xi_1 \geq \xi_2$. We have

$$\beta((\mathcal{G},G_{\setminus\{x\}}),1) = \frac{\xi_1}{2^{n+1}} = \frac{2\xi_1}{2^{n+2}} \geq \frac{\xi_1+\xi_2}{2^{n+2}} = \beta(\mathcal{G},1),$$

so the Penrose–Banzhaf index of player 1 does not decrease by deleting x. If we remove any other edge, 1's Penrose–Banzhaf index will not change at all.

Similarly, we now show that for the distinguished player's Shapley–Shubik index, we have

$$(\exists e \in E)\ \varphi((\mathcal{G},G),1) - \varphi((\mathcal{G},G_{\setminus\{e\}}),1) > 0 \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. After deleting the edge x, the Shapley–Shubik index of the distinguished player will decrease to

$$\begin{aligned}\varphi((\mathcal{G}, G_{\setminus\{x\}}), 1) &= \xi_1 \frac{(t+1)!(n-t)!}{(n+2)!} \\ &= \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} \frac{n+3}{n+1-t} \\ &= \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} + \xi_1 \frac{(t+2)!(n-t)!}{(n+3)!} \\ &< \varphi((\mathcal{G}, G), 1).\end{aligned}$$

From left to right, assume that $\xi_1 \geq \xi_2$. We have

$$\begin{aligned}\varphi((\mathcal{G}, G_{\setminus\{x\}}), 1) &= \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} + \xi_1 \frac{(t+2)!(n-t)!}{(n+3)!} \\ &\geq \varphi((\mathcal{G}, G), 1),\end{aligned}$$

so the distinguished player's Shapley–Shubik index does not decrease. If we remove any other edge, the distinguished player's Shapley–Shubik index will not change at all. ❑

Finally, we consider the goal of maintaining a player's Shapley–Shubik index in a graph-restricted weighted voting game when edges are deleted from the communication graph. The complexity of this problem was left open by Kaczmarek and Rothe [9] (who showed coNP-hardness only for the Penrose–Banzhaf index).

We solve this open question by showing coNP-hardness also for the Shapley–Shubik index—the same lower bound, but obtained via a completely different construction the proof of correctness of which will be rather involved and is omitted due to space limitations.

Theorem 7 *Control by deleting edges to maintain a distinguished player's Shapley–Shubik index in a graph-restricted weighted voting game is* coNP-*hard.*

Proof Sketch. We can show coNP-hardness of our control problem by providing a reduction from the complement of the SUBSETSUM problem.

Let $((a_1, \ldots, a_n), q)$ be an instance of SUBSETSUM and let $\alpha = \sum_{i=1}^{n} a_i$. From this instance, we construct our control problem instance as follows.

Set the values y_1, y_2, and z as follows:

$$\begin{aligned} y_1 &= n+1, \\ y_2 &= 2n+2, \text{ and} \\ z &= 2 \cdot (\alpha \cdot 10^{\ell} + n + y_1 + y_2) + 1, \end{aligned}$$

where the positive integer ℓ is chosen such that

$$10^{\ell} > n + y_1 + y_2 = 4n + 3.$$

Further, let $a'_i = a_i \cdot 10^{\ell}$ and $a''_i = z - a_i \cdot 10^{\ell} - \lfloor \frac{n}{2} \rfloor$ for $i \in \{1, \ldots, n\}$, and let $b_1 = z - q \cdot 10^{\ell} - y_1$ and $b_2 = z - q \cdot 10^{\ell} - y_2$. Define the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\begin{aligned}\mathcal{G} = \Big(&1, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n, \\ &\underbrace{1, \ldots, 1}_{n}, b_1, y_1, b_2, y_2, 10^t; 10^t + z + 1\Big)\end{aligned}$$

with $3n+6$ players the last of which is heavier than the total weight of all other players (i.e., if w_i is the weight of player i, then t is

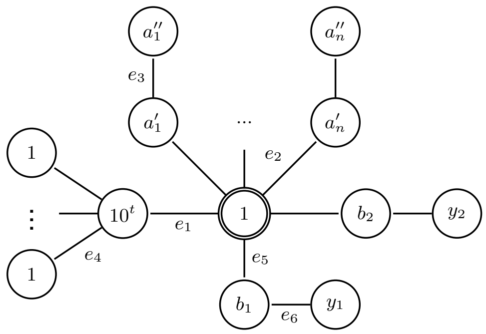


Figure 2. Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 7. The double circle indicates the distinguished player.

chosen such that $w_{3n+6} = 10^t > \sum_{i=1}^{3n+5} w_i$). Our distinguished player is 1, and we define the communication structure $G = (N, E)$ as shown in Figure 2 (again labeling the vertices with the weights of the players instead of their names to enhance readability).

Set the deletion limit $k = 1$ and let ξ be the number of SUBSETSUM $((a_1, \ldots, a_n), q)$'s solutions. We can show that

$$(\exists e \in E)\, [\varphi((\mathcal{G}, G_{\setminus\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) = 0] \iff \xi = 0,$$

again omitting the full proof due to space limitations. Specifically, if $\xi = 0$, we can show that after removing an edge between the players of weights b_1 and y_1 or between the players of weights b_2 and y_2, the Shapley–Shubik index of player 1 remains unchanged. Otherwise, if $\xi > 0$, removing any edge will cause either increasing or decreasing the index. Therefore, the control problem of deleting edges from a communication graph to maintain a given player's Shapley–Shubik power index is coNP-hard. ❑

5 Conclusions

We have analyzed structural manipulation of communication structures in graph-restricted weighted voting games by adding or deleting edges in the graphs. We have studied the computational complexity of the problems of whether such a manipulation can increase, decrease, or maintain the (probabilistic) Penrose–Banzhaf or the Shapley–Shubik power index of a given player. Specifically, we have improved the known lower bounds for these problems due to Kaczmarek and Rothe [9] for the goals of increasing and decreasing these power indices from NP- or DP-hardness to PP-hardness. Further, we have shown coNP-hardness for control by deleting edges to maintain a given player's Shapley–Shubik index, thus providing a first lower bound for this problem and solving an open question raised by Kaczmarek and Rothe [9].

Interesting tasks for future research include the question of whether our complexity lower bounds can be raised even further and whether we can pinpoint the complexity of these problems exactly, as well as considering further power indices such as the normalized Penrose–Banzhaf index [2, 14].

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