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Settling the Score: Portioning with Cardinal Preferences

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Abstract. We study a portioning setting in which a public resource such as time or money is to be divided among a given set of candidates, and each agent proposes a division of the resource. We consider two families of aggregation rules for this setting—those based on coordinate-wise aggregation and those that optimize some notion of welfare—as well as the recently proposed Independent Markets mechanism. We provide a detailed analysis of these rules from an axiomatic perspective, both for classic axioms, such as strategyproofness and Pareto optimality, and for novel axioms, which aim to capture proportionality in this setting. Our results indicate that a simple rule that computes the average of all proposals satisfies many of our axioms, including some that are violated by more sophisticated rules.

1 Introduction

A town council has just received its annual funding from the government, and it needs to determine how to split the budget among constructing new facilities, keeping the streets clean, and ensuring safety in public places. The mayor is in favor of making decisions democratically, so she asks each resident of the town to propose a division of the budget. After having collected the proposals, how should the council aggregate them into an actual allocation?

In the problem of *portioning*, the aim is to divide a resource among a given set of candidates. As illustrated in the example above, a prominent application of portioning is *participatory budgeting*, a democratic framework initiated in Porto Alegre, Brazil in 1989 and used in over 7,000 cities around the world [26].[1] Besides allotting a budget, portioning can also be used to share *time*, for example, when deciding upon the proportion of time to spend on various activities at a conference (e.g., research talks, panels, social gatherings) or different types of music at a graduation party (e.g., classical, rock, jazz).

Most prior works on portioning assumed that each voter's preferences can be represented by a ranking [1], or by an approval ballot [3, 8, 12]. However, in some portioning scenarios these preference formats cannot fully describe the agents' desires. For instance, if a student wants both jazz and rock music to be played at the graduation party, but with more time devoted to rock, her preference is not captured by a ranking or a list of approvals. Likewise, a conference attendee who would like 75% of the time to be spent on research talks, 15% on panels, and 10% on social gatherings ranks these activities in the same way as another attendee who prefers a 40%–35%–25% split, but the actual preferences of these two attendees are quite different.

Recently, Freeman et al. [17] studied portioning with cardinal preferences, wherein each voter is allowed to propose a division of the resource. They observed that, even though the rule that maximizes the utilitarian social welfare is known to be strategyproof (for a specific tie-breaking convention) [20, 25], it tends to put too much weight on majority preferences. In light of this observation, they introduced the *independent markets (IM)* mechanism, which is strategyproof and, in some sense, more proportional. However, while strategyproofness is an important consideration, there may be scenarios where other features of aggregation rules are just as—if not more—desirable. Thus, to identify a suitable aggregation rule for a specific application, it would be useful to (1) build a catalogue of axioms for the portioning setting, and (2) determine which of these axioms are satisfied by popular aggregation rules.

Our Contributions We consider a diverse set of axioms for portioning with cardinal preferences. Besides classic axioms, such as strategyproofness and Pareto optimality, we put forward two novel proportionality axioms, namely, score-unanimity and score-representation (see Section 2 for definitions). We then conduct a systematic study of aggregation rules with respect to these axioms. We focus on two families of portioning rules—those that are based on coordinate-wise aggregation and those that optimize some notion of welfare—as well as the recently proposed Independent Markets mechanism [17]. We also include observations regarding relationships between the axioms. Table 1 summarizes our results. An overview and discussion of our results can be found in Section 3.

Related Work While there is a large body of work on participatory budgeting, most of it focuses on the discrete setting, where each project is either implemented in full or not implemented at all [5]. Our model, where there is a unit of budget to be split arbitrarily among the projects, is usually referred to as portioning [1, 3, 8, 12]. Within the portioning literature, only a few papers consider cardinal preferences. The ground-breaking work of Freeman et al. [17] focuses on designing strategyproof mechanisms for this setting, and proposes the Independent Market mechanism. The follow-up work of Caragiannis et al. [9] studies proportionality guarantees that can be obtained by truthful mechanisms, and provides approximation bounds. In a very recent paper, Goyal et al. [21] study approximation guarantees offered by mechanisms with low sample complexity.

Other related lines of work include probabilistic models in social choice [16, 23, 28], in which the output is a probability distribution

[1] Participatory budgeting is a subject of much recent interest in computational social choice [4, 6, 7, 10, 15, 21, 22, 24, 29, 30]; see the recent survey by Rey and Maly [27]. However, as we discuss in the section on related work, most of the participatory budgeting literature focuses on the discrete setting.

over candidates. There, however, a single candidate is to be chosen according to the probability distribution, in contrast to our setting where we portion among several alternatives. Another relevant topic is probabilistic opinion pooling [11, 19], where the goal is to aggregate probabilistic beliefs which may represent, e.g., forecasts.

2 Preliminaries

We first present the model of portioning with cardinal preferences, and then introduce the axioms and rules that we will study.

2.1 Model

We are given a set $N = \{1, \dots, n\}$ of n *agents* (or *voters*), and a set $C = \{p_1, \dots, p_m\}$ of m *candidates*. Each agent $i \in N$ has a normalized preference score vector $\mathbf{s}_i \in (\mathbb{R}_{\geq 0})^m$ over the candidates, where for each $i \in N$ we have $\mathbf{s}_i = (s_{i1}, \dots, s_{im})$ and $\sum_{j \in [m]} s_{ij} = 1$. An *instance* of our problem is the vector $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$, also referred to as a *preference profile*. For any score vector $\mathbf{x} = (x_1, \dots, x_m)$, agent i's *disutility function* is defined as $d_i(\mathbf{x}) = \sum_{j \in [m]} |s_{ij} - x_j|$, which is the ℓ_1 distance between the agent's score vector $\mathbf{s}_i$ and $\mathbf{x}$. Given an instance $\mathcal{I}$, we aim to find a vector $\mathbf{x}$ with $\sum_{j \in [m]} x_j = 1$ that reflects the agents' collective preferences. To do so, we use *aggregation rules*, which are defined as follows.

Definition 1 (Aggregation rule)**.** *An* aggregation rule F *is a function* $F : (\mathbb{R}_{\geq 0})^{m \times n} \to (\mathbb{R}_{\geq 0})^m$ *that maps a preference profile* $\mathcal{I} \in (\mathbb{R}_{\geq 0})^{m \times n}$ *to an outcome vector* $\mathbf{x} \in (\mathbb{R}_{\geq 0})^m$.

2.2 Axioms

We begin by introducing a basic property, which states that if all agents unanimously agree on a score for a particular candidate, then, in the outcome, this candidate should get exactly that score.

Definition 2 (Score-unanimity)**.** *An aggregation rule F is* score-unanimous *if, for every instance $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ such that for some $j \in [m]$, $\gamma \in [0, 1]$ it holds that $s_{ij} = \gamma$ for all $i \in N$, the score vector $\mathbf{x} = F(\mathcal{I})$ satisfies $x_j = \gamma$.*

Next, we consider another intuitive property, which states that if a single agent increases the score allocated to a particular candidate (while decreasing her scores for other candidates), then this candidate's score in the outcome cannot decrease.

Definition 3 (Score-monotonicity)**.** *An aggregation rule F is* score-monotone *if the following holds: for any two instances $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ and $\mathcal{I}' = (\mathbf{s}'_1, \dots, \mathbf{s}'_n)$ with $F(\mathcal{I}) = \mathbf{x}$ and $F(\mathcal{I}') = \mathbf{x}'$ such that for some $i \in N$, $j \in [m]$ we have (1) $s_{ij} < s'_{ij}$; (2) $s_{ij'} \geq s'_{ij'}$ for all $j' \in [m] \setminus \{j\}$; and (3) $s_{i'j'} = s'_{i'j'}$ for all $i' \in N \setminus \{i\}$, $j' \in [m]$, it holds that $x_j \leq x'_j$.*

Another notion that has recently been studied in the context of portioning is *proportionality* [17]. This property requires that when agents are *single-minded*, i.e., each agent places all of her score on a single candidate, the outcome score for each candidate equals the proportion of agents that favor this candidate. However, agents are rarely single-minded in several applications of portioning, so an appropriate notion of proportionality (or, more broadly, representation) for general preferences is needed. We formulate one such notion for the cardinal preference setting, and refer to it as *score-representation*.

Definition 4 (Score-representation)**.** *An aggregation rule F satisfies* score-representation *if for every instance $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ such that for some $V \subseteq N$, $j \in [m]$, $\gamma \in [0, 1]$ it holds that $s_{ij} \geq \gamma$ for all $i \in V$, the score vector $\mathbf{x} = F(\mathcal{I})$ satisfies $x_j \geq \gamma \cdot \frac{|V|}{n}$.*

Another important property of aggregation rules is strategyproofness (see, e.g., [17]): agents should not be able to lower their disutility by misreporting their score vector.

Definition 5 (Strategyproofness)**.** *An aggregation rule F is* strategyproof *if, for any two instances $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}'$ is obtained from $\mathcal{I}$ by replacing agent i's score vector $\mathbf{s}_i$ with another score vector $\mathbf{s}'$, if $F(\mathcal{I}) = \mathbf{x}$ and $F(\mathcal{I}') = \mathbf{x}'$, then $d_i(\mathbf{x}) \leq d_i(\mathbf{x}')$.*

A related notion is participation. This property states that each agent weakly prefers voting truthfully to withdrawing from the aggregation process. In numerous contexts (particularly for elections), this property incentivizes higher voter turnout.

Definition 6 (Participation)**.** *An aggregation rule F satisfies* participation *if, for any two instances $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}'$ is obtained from $\mathcal{I}$ by adding an additional agent i, if $F(\mathcal{I}) = \mathbf{x}$ and $F(\mathcal{I}') = \mathbf{x}'$, then $d_i(\mathbf{x}) \geq d_i(\mathbf{x}')$.*

We also consider Pareto optimality, which is a basic notion of efficiency.

Definition 7 (Pareto optimality)**.** *An aggregation rule F is* Pareto optimal *(PO) if, for every instance $\mathcal{I}$ and the outcome $\mathbf{x} = F(\mathcal{I})$, there does not exist another outcome $\mathbf{x}'$ such that (1) $d_i(\mathbf{x}') \leq d_i(\mathbf{x})$ for all $i \in N$ and (2) $d_i(\mathbf{x}') < d_i(\mathbf{x})$ for some $i \in N$.*

The last two axioms we consider were studied by Freeman et al. [17].

Definition 8 (Range-respect)**.** *An aggregation rule F is* range-respecting *(RR) if for every instance $\mathcal{I}$, the outcome $\mathbf{x} = F(\mathcal{I})$, and for all $j \in [m]$ it holds that* $\min_{i \in N} s_{ij} \leq x_j \leq \max_{i \in N} s_{ij}$.

Definition 9 (Reinforcement)**.** *An aggregation rule F satisfies* reinforcement *if, for any two instances $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ and $\mathcal{I}' = (\mathbf{s}'_1, \dots, \mathbf{s}'_{n'})$ such that $F(\mathcal{I}) = F(\mathcal{I}') = \mathbf{x}$, for the instance $\mathcal{I}^* = (\mathbf{s}_1, \dots, \mathbf{s}_n, \mathbf{s}'_1, \dots, \mathbf{s}'_{n'})$ we have $F(\mathcal{I}^*) = \mathbf{x}$.*

Remark 10. *The axioms of score-unanimity, score-representation, Pareto optimality, and range-respect can be defined for outcomes rather than voting rules: e.g., we say than an outcome $\mathbf{x}$ for an instance $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ is range-respecting if $\min_{i \in N} s_{ij} \leq x_j \leq \max_{i \in N} s_{ij}$ for all $j \in [m]$ (and similarly for other axioms).*

2.3 Aggregation Rules

We focus on two classes of rules, namely, (1) rules that are based on *coordinate-wise aggregation* and (2) rules that are based on *welfare optimization*. In addition, we will also consider the independent markets mechanism of Freeman et al. [17].

Coordinate-wise Aggregation Rules We start by defining the class of *coordinate-wise* aggregation rules.

Definition 11. *We say that an aggregation rule F is* coordinate-wise *if for each $n \geq 1$ there is a* coordinate-aggregation function *$f_n : (\mathbb{R}_{\geq 0})^n \to \mathbb{R}_{\geq 0}$ such that, given an instance $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$, the function F outputs a vector $\mathbf{x}$ that satisfies $x_j = \frac{f_n(s_{1j}, \dots, s_{nj})}{\sum_{j' \in [m]} f_n(s_{1j'}, \dots, s_{nj'})}$ for each $j \in [m]$.*

	Coordinate-wise					Welfare-based		Other
F	Sum	Max	Min	Med	Prod	UTIL	EGAL	IM
Score-unanimity	✓	X†	X†	X‡	X	✓	✓	X†
Score-monotonicity	✓	✓	✓	✓	✓	✓	?‡	✓
Proportionality	✓	X*	X	X*	X	X*	X*	✓
Score-representation	✓	X	X	X*	X	X	X*	X
Strategyproofness	X	X	X	X	X	✓	X	✓
Participation	✓	✓	✓	?‡	X	✓	✓	✓
Pareto optimality	X‡	X†	X†	X‡	X	✓	✓	X†
Range-respect	✓	X†	X†	X‡	X	✓	✓	X†
Reinforcement	✓	✓	✓	?‡	X	✓	✓	✓

Table 1. Summary of our results. For most of the negative results, we provide a counterexample for the simplest case, but they can be easily extended by cloning agents or adding dummy candidates. The asterisk symbol (*) indicates that the axiom is satisfied for $n = 2$, but may fail when $n \geq 3$ (even if $m = 2$). The dagger symbol (†) indicates that the axiom is satisfied for $m = 2$, but may fail when $m \geq 3$ (even if $n = 2$). The double dagger symbol (‡) indicates that the axiom is satisfied when $\min\{n, m\} = 2$; when used with X, it may fail in more general cases; when used with ?, it indicates that the case $\min\{n, m\} \geq 3$ remains open. Some of the results on UTIL and IM were obtained by Freeman et al. [17].

For each $j \in [m]$, we refer to $y_j = f_n(s_{1j}, \ldots, s_{nj})$ as the *pre-normalization value* for x_j. The vector $\mathbf{y} = (y_1, \ldots, y_m)$ is called the *pre-normalization vector*. In what follows, we will omit the subscript n and write f instead of f_n.

We study five natural coordinate-wise aggregation rules, where f is, respectively, the median (if the number of agents is even, we take the average of the two middle scores), sum, maximum, minimum, or product function (for the last two rules, if $y_j = 0$ for all $j \in [m]$, we set $x_j = 1/m$ for all $j \in [m]$). For brevity, we refer to these rules as med, sum, max, min, and prod rules, respectively. These rules are attractive, because they are intuitive and efficiently computable.

Welfare-based Aggregation Rules We also consider rules that are based on welfare optimization. In particular, we focus on two popular welfare criteria: (1) maximizing the utilitarian welfare (UTIL), i.e., minimizing $\sum_{i \in N} d_i(\mathbf{x})$, and (2) maximizing the egalitarian welfare (EGAL), i.e., minimizing $\max_{i \in N} d_i(\mathbf{x})$. Note that Nash welfare is not well-defined in this setting, as we are considering disutilities.[2]

For UTIL, tie-breaking is important. Following Freeman et al. [17], we break ties in favor of the *maximum entropy division*. Specifically, we assume that UTIL outputs the utilitarian welfare-maximizing outcome $\mathbf{x}$ that minimizes the quantity $\sum_{j \in [m]} (x_j - 1/m)^2$, i.e., the ℓ_2 distance to the uniform distribution $\mathbf{x}_u = (1/m, \ldots, 1/m)$. Importantly, a UTIL outcome is PO, and can be computed in polynomial time. However, the UTIL rule fails proportionality [17].

For EGAL, if there are multiple outcomes that maximize the egalitarian welfare, then we break ties in a "leximin" manner. That is, we minimize the largest disutility, then subject to that, minimize the second-largest disutility, and so on. From this definition, it is clear that EGAL satisfies PO. We make two further observations.

Theorem 2.1. EGAL *can be computed in polynomial time.*

Lemma 2.2. *When $n = 2$, the output of the sum and med rules is an* EGAL *outcome.*

Since there may be multiple EGAL outcomes even after the leximin tie-breaking, we assume for convenience (in light of Lemma 2.2) that for $n = 2$, the output of EGAL is the same as that of the sum rule. Our results for $n \geq 3$ will not depend on this choice, and we allow EGAL to break ties in any consistent manner.

[2] For example, it has been observed that there is no natural equivalent of Nash welfare in the fair allocation of chores [13, 18].

Independent Markets (IM) mechanism Freeman et al. [17] put forward two aggregation mechanisms that both rely on introducing *phantoms*. The first of these mechanisms is called the Independent Markets mechanism, and is described below. The second one is equivalent to UTIL with the maximum entropy division tie-breaking rule (defined earlier).

Definition 12. *For each $c \in \mathbb{R}_{\geq 0}$, the c-coordinate-wise median of candidate p_j is defined as the median of the $2n + 1$ values $\{0, \frac{c}{n}, \frac{2c}{n}, \ldots, \frac{(n-1)c}{n}, c, s_{1j}, \ldots, s_{nj}\}$ (i.e., the agents' scores for p_j and $n + 1$ "phantoms" that are uniformly distributed on $[0, c]$). The mechanism starts with $c = 0$ and continuously increases c, stopping when the sum of all candidates' c-coordinate-wise medians is 1 (if a phantom score is higher than 1, it is taken to be 1). It then outputs the vector of c-coordinate-wise medians.*

All omitted proofs can be found in the full version of our paper [14].

3 Overview and Discussion

Our results offer several insights on portioning rules. As shown in Table 1, the most promising rules with respect to the axioms that we study are the sum rule and UTIL, with the trade-off being that the sum rule fails strategyproofness and Pareto optimality whereas UTIL fails proportionality and score-representation. While the IM mechanism satisfies both strategyproofness and proportionality, it fails other intuitive properties such as score-unanimity, range-respect, and Pareto-optimality; these failures can lead to highly counterintuitive outcomes for the voters and cast doubt on the IM mechanism as an aggregation method.

These trade-offs between various rules may be used to inform decision-making in a wide range of settings. For instance, consider again the scenario where a conference organizer needs to divide time among different activities at a conference. In this case, it is likely difficult for an attendee to accurately predict what other attendees' preferences are, making strategyproofness arguably less relevant as a consideration. On the other hand, strategyproofness could be more important in smaller-scale settings where voters know each other well, e.g., portioning within a family or a small organization. In such a setting, it may not be crucial that the outcome is exactly proportional in the way that the proportionality axiom requires. Additionally, intuitive properties such as score-unanimity and range-respect may be essential in settings where votes are revealed: for example, if all voters vote 0.8 on a certain activity but the rule outputs 0.6 on this activity, this may well lead to dissatisfaction among voters regarding the use of that voting rule.

4 Score-Unanimity and Range-Respect

In this section, we focus on two basic properties defined in Section 2: score-unanimity and range-respect. The former states that if all agents assign a particular candidate p_j the same score, then in the outcome p_j should get exactly that score. The latter mandates that, in the outcome, every candidate's score should lie between the highest and the lowest score that this candidate obtains from the agents.

It is easy to see that if all agents give the exact same score γ to a candidate p_j, then in the outcome of any RR rule F the score of p_j has to be γ, and hence F is also score-unanimous. In fact, as we will show later in Section 8, any PO outcome is also RR. We will now give a direct proof of a more general result.

Theorem 4.1. *Any aggregation rule that is RR or PO is also score-unanimous.*

Proof. The case of RR rules has been considered above. Now, let $\mathbf{x}$ be an outcome returned by a PO rule F on an instance $\mathcal{I}$. Suppose for a contradiction that F is not score-unanimous on $\mathcal{I}$. This means that for some $j \in [m]$ and $\gamma \in [0,1]$ we have $s_{ij} = \gamma$ for all $i \in N$, but $x_j \neq \gamma$. We can assume without loss of generality that $x_j > \gamma$; a similar argument works for the case $x_j < \gamma$. Note that there must exist some $j' \in [m]$ such that $x_{j'} < s_{i'j'}$ for some $i' \in N$. Now, consider the outcome vector $\mathbf{x}'$ where $\mathbf{x}'$ is identical to $\mathbf{x}$, except $x'_j = \gamma$ and $x'_{j'} = x_{j'} + (x_j - \gamma)$. Then, at least one agent's (i') disutility will decrease, and all other agents' disutility does not increase, a contradiction with $\mathbf{x}$ being Pareto optimal. □

Next, we show that of the five coordinate-wise aggregation rules we consider, the sum rule is the only one satisfying score-unanimity.

Theorem 4.2. *The sum rule is score-unanimous.*

Proof. Suppose for some $j \in [m]$ and $\gamma \in [0,1]$ we have $s_{ij} = \gamma$ for all $i \in N$. Then, $x_j = \frac{1}{n}\sum_{i\in N}\gamma = \frac{\gamma\cdot n}{n} = \gamma$. □

The next three results show that the max, min, and med rules satisfy score-unanimity in some special cases, whereas the prod rule fails it even in the simplest setting.

Theorem 4.3. *The max and min rules are score-unanimous when $m = 2$, but may fail to be so when $m \geq 3$ (even when $n = 2$).*

Proof. Since the max and min rules are PO when $m = 2$ (by Proposition 8.6), the property follows by Theorem 4.1.

Next, we show a counterexample for $n = 2$ and $m = 3$. Suppose we have two agents with score vectors $\mathbf{s}_1 = (0, 0.2, 0.8)$ and $\mathbf{s}_2 = (0.8, 0.2, 0)$. Then, the max rule will return $\mathbf{x}_{\text{max}} = (\frac{4}{9}, \frac{1}{9}, \frac{4}{9})$ and the min rule will return $\mathbf{x}_{\text{min}} = (0, 1, 0)$. It is easy to see that score-unanimity is violated for $j = 2$. □

Theorem 4.4. *The med rule is score-unanimous when $n = 2$ or $m \leq 3$, but may fail to be so when $n \geq 3$ and $m \geq 4$.*

Proposition 4.5. *The prod rule may fail score-unanimity for all $n \geq 2$ and $m \geq 2$.*

Proof. Let $n \geq 2$ and $\mathbf{s}_i = (0.8, 0.2, 0, \ldots, 0)$ for all $i \in N$. The prod rule returns $\mathbf{x} = \left(\frac{0.8^n}{0.8^n+0.2^n}, \frac{0.2^n}{0.8^n+0.2^n}, 0, \ldots, 0\right)$. It is easy to verify that $\frac{0.8^n}{0.8^n+0.2^n} > 0.8$ for all $n \geq 2$. □

As for the welfare-based rules, since both UTIL and EGAL are PO (by definition), we get the following as a corollary of Theorem 4.1.

Corollary 4.6. UTIL *and* EGAL *are score-unanimous.*

Finally, we show that the IM mechanism also fails to be score-unanimous in general.

Theorem 4.7. *The IM mechanism is score-unanimous when $m = 2$, but may fail to be so when $m \geq 3$ (even when $n = 2$).*

We have argued that score-unanimity is a special case of RR. Unsurprisingly, it is known that the IM mechanism is not RR in general (one can use the same counterexample as in the proof of Theorem 4.7) [17]. We complement this result by showing that IM satisfies RR for two candidates.

Theorem 4.8. *The IM mechanism is RR when $m = 2$, but may fail to be so when $m \geq 3$ (even when $n = 2$).*

Proof. By the comment before the theorem, it suffices to prove this when $m = 2$. Let $\mathbf{x}$ be the output vector of the IM mechanism. Recall that the stopping condition from the definition of the mechanism mandates that the entries of the output vector $\mathbf{x}$ sum to 1.

For each $j \in \{1, 2\}$, denote z_j^{max} and z_j^{min} as the maximum and minimum values that the n voters have for candidate j, respectively. Suppose for a contradiction that the IM mechanism is not RR when $m = 2$. There are two cases: either all the $n+1$ phantoms are strictly less than z_j^{min} for some $j \in \{1, 2\}$, or all the $n + 1$ phantoms are strictly more than z_j^{max} for some $j \in \{1, 2\}$. The latter case cannot happen since one of the phantoms is always at 0. Thus, we focus on the former case.

Suppose without loss of generality that all $n + 1$ phantoms are strictly less than z_1^{min}, so we have that $x_1 < z_1^{\text{min}}$. Then, since $z_2^{\text{max}} = 1 - z_1^{\text{min}}$, in order for $x_1 + x_2 = 1$, we must have that $x_2 > z_2^{\text{max}}$. This means that all $n + 1$ phantoms must be strictly more than z_2^{max}, which is a contradiction. □

In fact, out of the five coordinate-wise aggregation rules we study, the sum rule is the only one that is RR for all n and m.

Theorem 4.9. *The sum rule is RR.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be the output and pre-normalization vector of the sum rule, respectively. Then $x_j = \frac{y_j}{n}$ for all $j \in [m]$. For each $j \in [m]$, let z_j^{max} and z_j^{min} be the maximum and minimum values that the n voters have for candidate j, respectively. Then, since $y_j = \sum_{i\in N} s_{ij}$, we obtain $n \cdot z_j^{\text{min}} \leq y_j \leq n \cdot z_j^{\text{max}}$. Dividing by n throughout, we get $z_j^{\text{min}} \leq x_j = \frac{y_j}{n} \leq z_j^{\text{max}}$. □

Theorem 4.10. *The max and min rules are RR when $m = 2$, but may fail to be so when $m \geq 3$ (even when $n = 2$).*

Proof. We prove the result for the max rule; the proof for the min rule can be found in the full version [14]. For $m = 2$, let $\mathbf{x}$ be the output vector of the max rule. For each $j \in \{1, 2\}$, denote z_j^{max} and z_j^{min} as the maximum and minimum values that the n voters have for candidate j, respectively. Then $x_j = z_j^{\text{max}}/(z_1^{\text{max}} + z_2^{\text{max}})$ for $j = 1, 2$.

We will show that the max rule is RR for p_1; the same analysis applies to p_2. We have $z_2^{\text{max}} = 1 - z_1^{\text{min}}$. Given that $z_1^{\text{min}} \leq z_1^{\text{max}}$, multiplying both sides by $1-z_1^{\min}$, we get $z_1^{\min}(1-z_1^{\min}) \le z_1^{\max}(1-z_1^{\min})$. Algebraic manipulation gives us $z_1^{\min} \le \frac{z_1^{\max}}{z_1^{\max}+(1-z_1^{\min})} = x_1$, as desired (since $z_2^{\max} = 1 - z_1^{\min}$).

For the upper bound, we know that $z_1^{\min} \le z_1^{\max}$. Adding $1 - z_1^{\min}$ to both sides, we get $1 \le z_1^{\max} + (1 - z_1^{\min})$. Multiplying both sides by $\frac{z_1^{\max}}{z_1^{\max}+(1-z_1^{\min})}$, we get $x_1 = \frac{z_1^{\max}}{z_1^{\max}+(1-z_1^{\min})} \le z_1^{\max}$, as desired (since $z_2^{\max} = 1 - z_1^{\min}$).

A counterexample for the max rule when $n = 2$ and $m = 3$ is as follows. Consider a profile with two agents and three candidates, where the score vectors are $\mathbf{s}_1 = (\frac{2}{3}, 0, \frac{1}{3})$ and $\mathbf{s}_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then, the max rule will return $\mathbf{x} = (0.5, 0.25, 0.25)$, which is not RR. □

Theorem 4.11. *The med rule is RR when $n = 2$ or $m = 2$ or $n = m = 3$, but may fail to be RR when $n \ge 3$ and $m \ge 4$.*

Theorem 4.1 and Proposition 4.5 imply the following:

Corollary 4.12. *The prod rule may fail to be RR for all $n \ge 2$ and $m \ge 2$.*

As for the welfare-based rules, from Theorems 8.2–8.4, we have that both satisfy RR.

Corollary 4.13. UTIL *and* EGAL *are RR.*

5 Score-Monotonicity

We begin this section with a lemma that provides a more general condition for coordinate-wise aggregation rules to satisfy score-monotonicity. This result will be useful for subsequent proofs.

Lemma 5.1. *Suppose that a coordinate-wise aggregation rule F with coordinate-wise aggregation function f is such that for every $i \in N$, $f(z_1, \ldots, z_i, \ldots, z_n) \ge f(z_1, \ldots, z_i', \ldots, z_n)$ whenever $z_i \ge z_i'$. Then F is score-monotone.*

For all our five coordinate-wise aggregation rules, the aggregation function clearly satisfies the condition in Lemma 5.1. Hence, we immediately have the following result.

Theorem 5.2. *All five coordinate-wise aggregation rules satisfy score-monotonicity.*

As for the remaining rules, Freeman et al. [17, Thm. 3] showed that UTIL and the IM mechanism are score-monotone. We show that EGAL is score-monotone in the special cases where $n = 2$ or $m = 2$. The setting where $n, m \ge 3$ is left as an open question.

Theorem 5.3. EGAL *is score-monotone when $n = 2$ or $m = 2$.*

6 Proportionality and Score-Representation

In this section, we focus on the notions of proportionality [17] and score-representation (which is a generalization of proportionality). Freeman et al. [17] proved that the IM mechanism satisfies proportionality. However, we show that their guarantee does not carry over for score-representation in general.

Theorem 6.1. *The IM mechanism may fail score-representation, even when $n = 2$ and $m = 3$.*

Proof. Consider the case of two agents and three candidates, where the score vectors are $\mathbf{s}_1 = (0.8, 0.2, 0)$ and $\mathbf{s}_2 = (0.8, 0, 0.2)$. Then, score-representation mandates that the outcome $\mathbf{x}$ should satisfy $x_1 \ge 0.8$. However, the outcome vector returned by the IM mechanism is $\mathbf{x}' = (0.6, 0.2, 0.2)$, where $x_1 = 0.6 < 0.8$. □

The question of whether the IM mechanism satisfies the score-representation axiom when $m = 2$ remains open.

Now, we show that the sum rule satisfies score-representation (and hence proportionality as well) in general. In fact, among the coordinate-wise aggregation rules we consider, it is the only rule with this property.

Theorem 6.2. *The sum rule satisfies score-representation.*

Proof. Let $S \subseteq N$ be the set of agents whose score for a candidate p_j is at least γ, for some $\gamma \in (0, 1]$. Then $x_j = \frac{1}{n}\left(\sum_{i \in S} s_{ij} + \sum_{i' \in N \setminus S} s_{i'j}\right) \ge \frac{1}{n}\left(\sum_{i \in S} \gamma + 0\right) = \frac{\gamma \cdot |S|}{n}$. □

While the general positive result only holds for the sum rule, we show that the max and med rules satisfy score-representation when $n = m = 2$, whereas the min and prod rules may fail to do so even in the simplest setting.

Theorem 6.3. *The max rule satisfies score-representation when $n = m = 2$, but may fail to do so when $m \ge 3$ (even for $n = 2$) or $n \ge 3$ (even for $m = 2$). It also satisfies proportionality when $n = 2$, but may fail to do so when $n \ge 3$ (even for $m = 2$).*

Proof. We first prove that the max rule satisfies score-representation when $n = m = 2$. Assume without loss of generality that $s_{11} \ge s_{21}$ and $s_{22} \ge s_{12}$. Then, score-representation demands that (i) $x_1 \ge \frac{s_{11}}{2}$ and $x_2 \ge \frac{s_{22}}{2}$, and (ii) $x_1 \ge s_{21}$ and $x_2 \ge s_{12}$. Property (ii) follows from the fact that the max rule is RR when $m = 2$ (Theorem 4.10). We will show the max rule satisfies property (i).

Now, $x_1 = \frac{s_{11}}{s_{11}+s_{22}}$ and $x_2 = \frac{s_{22}}{s_{11}+s_{22}}$. Since $s_{11} + s_{22} \le 2$, $x_1 = \frac{s_{11}}{s_{11}+s_{22}} \ge \frac{s_{11}}{2}$ and $x_2 = \frac{s_{22}}{s_{11}+s_{22}} \ge \frac{s_{22}}{2}$. Thus, property (i) is satisfied.

We now show that the max rule may fail score-representation when $n = 2$ and $m = 3$. Consider a profile with two agents and three candidates, where the score vectors are $\mathbf{s}_1 = (\frac{2}{3}, 0, \frac{1}{3})$ and $\mathbf{s}_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then, since $s_{13} = s_{23} = \frac{1}{3}$, score-representation mandates that $x_3 \ge \frac{1}{3}$. However, the max rule returns $\mathbf{x} = (0.5, 0.25, 0.25)$ with $x_3 = 0.25 < \frac{1}{3}$, failing this condition.

For $m = 2$ and $n \ge 3$, we will show later that the max rule fails the weaker proportionality property.

Next, we show that the max rule satisfies proportionality for $n = 2$ and $m \ge 3$ (it already satisfies the stronger score-representation property when $n = m = 2$). Recall that agents are single-minded when reasoning about proportionality. If both agents give a score of 1 to the same candidate (let it be p_1), then the max rule will return $\mathbf{x} = (1, 0, \ldots, 0)$, satisfying proportionality. Suppose both agents give a score of 1 to different candidates. Without loss of generality, let $s_{11} = 1$ and $s_{22} = 1$, with all other scores being 0. Then, the max rule will return $\mathbf{x} = (0.5, 0.5, 0, \ldots, 0)$, which also satisfies proportionality.

Finally, we show that the max rule may fail proportionality for $n \ge 3$, even for $m = 2$. Suppose we have $n - 1$ agents with score vector $(1, 0)$ and one agent with score vector $(0, 1)$. Then, proportionality states that the score of candidate p_1 should be $\frac{n-1}{n}$. However, the max rule will return the vector $\mathbf{x} = (0.5, 0.5)$. For any $n \ge 3$, we have $x_1 = 0.5 < \frac{n-1}{n}$. The case of more candidates can easily be handled by adding dummy candidates for which every agent has a score of 0. □

Theorem 6.4. *The med rule satisfies score-representation when $n = 2$, but may fail to be proportional when $n \ge 3$ (even for $m = 2$).*

Theorem 6.5. *The min and prod rules satisfy proportionality when $n = m = 2$, but may fail to be so when $n \geq 3$ or $m \geq 3$. Both rules may fail score-representation even when $n = m = 2$.*

Freeman et al. [17] showed that UTIL may fail to satisfy proportionality in general. We show that it satisfies the stronger score-representation property when $n = m = 2$, and proportionality when $n = 2$, but fails in all other cases.

Theorem 6.6. UTIL *satisfies score-representation when $n = m = 2$, but may fail to be so when $m \geq 3$. It is also proportional when $n = 2$, but may fail to be so when $n \geq 3$ (even for $m = 2$).*

We now show that EGAL provides slightly better guarantees.

Theorem 6.7. EGAL *satisfies score-representation when $n = 2$, but may fail to be proportional for any $n \geq 3$ (even for $m = 2$).*

7 Strategyproofness and Participation

Both of the phantom-based mechanisms proposed by Freeman et al. [17] that we study in this work, i.e., the IM mechanism and UTIL, have been proven to satisfy strategyproofness. Unfortunately, none of the five coordinate-wise aggregation rules enjoys this property.

Proposition 7.1. *Each of the sum, med, max, min, and prod rules may fail to be strategyproof for all $n \geq 2$ and $m \geq 2$.*

Next, we show that this negative result extends to EGAL as well.

Proposition 7.2. EGAL *may fail to be strategyproof for all $n \geq 2$ and $m \geq 2$.*

As seen above, the rules we consider may fail strategyproofness even in the simplest case. We will now focus on participation, which is generally viewed as a less demanding property. IM and UTIL are known to satisfy participation [17]. We will now show that three of the five coordinate-wise aggregation rules and EGAL also satisfy this property.

Theorem 7.3. *The sum rule satisfies participation.*

Theorem 7.4. *The max rule satisfies participation.*

Proof. Let $\mathbf{x}$ and $\mathbf{x}'$ be the outcome vector with i not participating and participating, respectively. Let $\mathbf{y}$ and $\mathbf{y}'$ be the corresponding pre-normalization vectors. If $\mathbf{y} = \mathbf{y}'$, then the property is trivially satisfied; thus we assume $\mathbf{y} \neq \mathbf{y}'$.

We state two lemmas whose proofs are in the full version [14].

Lemma 7.5. *For any $j \in [m]$, if $x'_j > x_j$, then $y'_j > y_j$.*

Lemma 7.6. $\sum_{j\in[m]} y'_j \geq 1$.

Let $[m] = S \cup T$, where $S \subseteq [m]$ is the set of indices where for each $\alpha \in S$, $x'_\alpha > x_\alpha$, and $T \subseteq [m]$ is the set of indices where for each $\beta \in T$, $x'_\beta \leq x_\beta$. Now, we have that

$$\sum_{\alpha\in S}(x'_\alpha - x_\alpha) = \sum_{\beta\in T}(x_\beta - x'_\beta). \quad (1)$$

We claim that for all indices $\alpha \in S$, $s_{i\alpha} \geq x'_\alpha$. Suppose for a contradiction that $s_{i\alpha} < x'_\alpha$. Then, $s_{i\alpha} < x'_\alpha = \frac{y'_\alpha}{\sum_{k\in[m]} y'_k} \leq y'_\alpha$, where the rightmost inequality follows from Lemma 7.6. It follows that $y'_\alpha = y_\alpha$. However, by the definition of S and Lemma 7.5, we arrive at a contradiction. Hence, $s_{i\alpha} \geq x'_\alpha > x_\alpha$.

This property shows that for indices in S, agent i's participation results in a decrease in her disutility by exactly $\sum_{\alpha\in S}(x'_\alpha - x_\alpha)$. Together with (1), her net disutility (across $[m]$) from participating is nonnegative, and we obtain the desired result. □

Theorem 7.7. *The min rule satisfies participation.*

Theorem 7.8. *The med rule satisfies participation when $n \leq 2$ or $m = 2$.*

Proposition 7.9. *The prod rule may fail participation for all $n \geq 2$ and $m \geq 2$.*

Theorem 7.10. EGAL *satisfies participation.*

Proof. Let $\mathcal{I}$ and $\mathcal{I}'$ be the instances where i does not participate and participates, respectively. Also let $\mathbf{x}$ and $\mathbf{x}'$ be the corresponding outcome vectors returned by EGAL. Consider the case where $\mathbf{x} \neq \mathbf{x}'$; otherwise participation is trivially satisfied.

Suppose for a contradiction that $d_i(\mathbf{x}) < d_i(\mathbf{x}')$. Let U be the function that takes in a multiset of nonnegative real numbers and outputs a vector containing the elements in the set, sorted in non-increasing order.

Then, define $\mathbf{z} := U(\{d_k(\mathbf{x}) : k \in N \setminus \{i\}\})$ and $\mathbf{z}' := U(\{d_k(\mathbf{x}') : k \in N \setminus \{i\}\})$.

Consider two vectors $\mathbf{v}, \mathbf{v}'$ of the same length V. We say that $\mathbf{v} = \mathbf{v}'$ if $v_k = v'_k$ for all $k \in [V]$. Furthermore, we say that $\mathbf{v} \succ \mathbf{v}'$ if $v_k > v'_k$ for some $k \in [V]$ and $v_{k'} = v'_{k'}$ for all $k' < k$.

Now, since $\mathbf{x}'$ is chosen over $\mathbf{x}$ under instance $\mathcal{I}'$, together with the fact that $d_i(\mathbf{x}) < d_i(\mathbf{x}')$, we have that $z_k > z'_k$ for some $k \in N \setminus \{i\}$ and $z_{k'} = z'_{k'}$ for all $k' < k$, i.e., $\mathbf{z} \succ \mathbf{z}'$. However, since $\mathbf{x}$ is chosen over $\mathbf{x}'$ under instance $\mathcal{I}$, $\mathbf{z}' \succeq \mathbf{z}$, giving us a contradiction. Therefore, the claim is proven. □

8 Pareto Optimality

Next, we turn our attention to Pareto optimality. We first establish the relationships between PO and RR.

Lemma 8.1. *Every PO outcome is RR.*

We now show that PO and RR are equivalent in the special cases of two agents or two candidates. We start by considering the case $n = 2$.

Theorem 8.2. *For $n = 2$, an outcome is PO if and only if it is RR.*

Proof. The forward direction has been established in Lemma 8.1. We prove the other direction. Let the score vectors of the two agents be $\mathbf{s}_1 = (s_{11}, \ldots, s_{1m})$ and $\mathbf{s}_2 = (s_{21}, \ldots, s_{2m})$. Then, for any outcome $\mathbf{x}$ we have that $d_1(\mathbf{x}) + d_2(\mathbf{x}) \geq \sum_{j\in[m]} |s_{1j} - s_{2j}|$. Also, if $\mathbf{x}$ is RR, it holds that $d_1(\mathbf{x}) + d_2(\mathbf{x}) \leq \sum_{j\in[m]} |s_{1j} - s_{2j}|$. Combining the two inequalities, for any RR outcome $\mathbf{x}$, we have that $d_1(\mathbf{x}) + d_2(\mathbf{x}) = \sum_{j\in[m]} |s_{1j} - s_{2j}|$. Thus, any other RR outcome will have the same sum of disutilities, and if one agent were to have a strict decrease in disutility, the other agent must have a strict increase in disutility. This shows that $\mathbf{x}$ is PO. □

The same property is observed in the case of two candidates.

Theorem 8.3. *For $m = 2$, an outcome is PO if and only if it is RR.*

Proof. The forward direction has been established in Lemma 8.1. We prove the other direction. Let the outcome $\mathbf{x}$ be RR. Consider any other outcome $\mathbf{x}'$. Without loss of generality, suppose that $x_1 > x'_1$.

Then, it must be that $x_2 < x_2'$. Since $\mathbf{x}$ is RR, there exists some agent $i \in N$ such that $s_{i1} \geq x_1$. Correspondingly, $s_{i2} \leq x_2$. Note that

$$d_i(\mathbf{x}) = (s_{i1}-x_1)+(x_2-s_{i2}) < (s_{i1}-x_1')+(x_2'-s_{i2}) = d_i(\mathbf{x}').$$

This means that the disutility of agent i increases when going from $\mathbf{x}$ to $\mathbf{x}'$. Thus $\mathbf{x}$ is PO. □

However, if the numbers of agents and candidates exceed two, the relationship between PO and RR becomes one-sided.

Theorem 8.4. *For $n \geq 3$ and $m \geq 3$, every PO outcome is RR. However, the converse may not hold even when $n = m = 3$.*

Building on the above observations, we obtain the following results.

Proposition 8.5. *The sum rule is PO when $n = 2$ or $m = 2$, but may fail to be so when both $n, m \geq 3$.*

Proposition 8.6. *The max and min rules are PO when $m = 2$, but may fail to be so when $m \geq 3$ (even when $n = 2$).*

The following results are corollaries of our results on RR (Section 4) and the relationships between RR and PO established earlier in this section.

Corollary 8.7. *The med rule is PO when $n = 2$ or $m = 2$, but may fail to be so when $n \geq 3$ and $m \geq 4$.*

Corollary 8.8. *The prod rule may fail to be PO for all $n \geq 2$ and $m \geq 2$.*

Corollary 8.9. *The IM mechanism is PO when $m = 2$, but may fail to be so when $m \geq 3$ (even when $n = 2$).*

As for the welfare-based rules, UTIL and EGAL are both PO by definition.

We will now consider PO outcomes from an algorithmic perspective. Finding a PO outcome is easy: we can simply define the vector $\mathbf{x}$ to be exactly $\mathbf{s}_i$ for some agent $i \in N$. Then, $d_i(\mathbf{x}) = 0$, whereas $d_i(\mathbf{x}') > 0$ for any other outcome vector $\mathbf{x}'$. In some problem domains, determining whether an outcome is PO can be computationally difficult [2]. In contrast, our next result shows that in our setting checking the PO property is computationally tractable.

Theorem 8.10. *The problem of determining whether an outcome $\mathbf{x}'$ is PO is polynomial-time solvable.*

9 Reinforcement

The last property we consider is reinforcement. We show that three of the five coordinate-wise aggregation rules satisfy this property. In contrast, the med rule satisfies it in the special case of two agents or two candidates, while the prod rule may fail it even in the simplest setting.

Theorem 9.1. *The sum, max, and min rules satisfy reinforcement.*

Proof. We prove this for the sum rule; the proofs for the max and min rules can be found in the full version [14].

Suppose we have two instances $\mathcal{I} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ and $\mathcal{I}' = (\mathbf{s}_1', \ldots, \mathbf{s}_{n'}')$. Let N and N' be the corresponding sets of agents.

Consider the sum rule. For each $j \in [m]$, if $\frac{1}{n}\sum_{i \in N} s_{ij} = x_j$ and $\frac{1}{n'}\sum_{i \in N'} s_{ij}' = x_j$, then $\sum_{i \in N} s_{ij} = n \cdot x_j$ and $\sum_{i \in N'} s_{ij}' = n' \cdot x_j$. Combining the two, we have that $\sum_{i \in N} s_{ij} + \sum_{i \in N'} s_{ij}' = (n+n') \cdot x_j$. This gives us $\frac{1}{n+n'} \cdot \left(\sum_{i \in N} s_{ij} + \sum_{i \in N'} s_{ij}'\right) = x_j$, as desired. □

Theorem 9.2. *The med rule satisfies reinforcement when $n = 2$ or $m = 2$.*

Proposition 9.3. *The prod rule may fail reinforcement for all $n \geq 2$ and $m \geq 2$.*

As for the rest of the rules, UTIL and the IM mechanism have been shown to satisfy reinforcement [17]. We show that EGAL also satisfies this axiom.

Theorem 9.4. EGAL *satisfies reinforcement.*

Proof. Let $\mathcal{I}$ and $\mathcal{I}'$ be two instances, and let $\mathbf{x}$ be the outcome vector returned by EGAL in both instances. Let N and N' be the sets of agents in those instances, respectively. Consider a third instance $\mathcal{I}^*$ derived by combining the two instances, with the set of agents $N^* := N \cup N'$, and let the outcome returned by EGAL in this combined instance be $\mathbf{x}^*$.

Let U be the function that takes in a multiset of nonnegative real numbers and outputs a vector containing the elements in the set, sorted in non-increasing order. Then, define

$$\mathbf{z} := U(\{d_k(\mathbf{x}) : k \in N\}),\ \mathbf{z}^* := U(\{d_k(\mathbf{x}^*) : k \in N\}),$$
$$\mathbf{y} := U(\{d_k(\mathbf{x}) : k \in N'\}),\ \mathbf{y}^* := U(\{d_k(\mathbf{x}^*) : k \in N'\}),$$
$$\mathbf{w} := U(\{d_k(\mathbf{x}) : k \in N^*\}),\ \mathbf{w}^* := U(\{d_k(\mathbf{x}^*) : k \in N^*\}).$$

Consider two vectors $\mathbf{v}, \mathbf{v}'$. of the same length V. We say that $\mathbf{v} = \mathbf{v}'$ if $v_k = v_k'$ for all $k \in [V]$. Furthermore, we say that $\mathbf{v} \succ \mathbf{v}'$ if $v_k > v_k'$ for some $k \in [V]$ and $v_{k'} = v_{k'}'$ for all $k' < k$.

Additionally, for any two vectors $\mathbf{u}$ and $\mathbf{v}$ of possibly different length, let $\mathbf{u}||\mathbf{v}$ be the vector containing all the elements in both $\mathbf{u}$ and $\mathbf{v}$, sorted in non-increasing order. Note that if we have four vectors $\mathbf{u}, \mathbf{u}', \mathbf{v}, \mathbf{v}'$ such that $\mathbf{u} \succeq \mathbf{u}'$ and $\mathbf{v} \succeq \mathbf{v}'$, then $\mathbf{u}||\mathbf{v} \succeq \mathbf{u}'||\mathbf{v}'$.

Since the outcome returned by EGAL under the combined instance $\mathcal{I}^*$ is $\mathbf{x}^*$, it must be that $\mathbf{w} \succeq \mathbf{w}^*$. If $\mathbf{w} = \mathbf{w}^*$, then $\mathbf{x} = \mathbf{x}^*$ (since we assume that a consistent tie-breaking rule is used for EGAL). Suppose for a contradiction that EGAL does not satisfy reinforcement, i.e., $\mathbf{w} \succ \mathbf{w}^*$.

Now, since $\mathbf{x}$ is chosen over $\mathbf{x}^*$ in instance $\mathcal{I}$, $\mathbf{z}^* \succeq \mathbf{z}$. Also, since $\mathbf{x}$ is chosen over $\mathbf{x}^*$ in instance $\mathcal{I}'$, $\mathbf{y}^* \succeq \mathbf{y}$. Then, we have that $\mathbf{w}^* = \mathbf{z}^*||\mathbf{y}^* \succeq \mathbf{z}||\mathbf{y} = \mathbf{w}$, which contradicts our assumption. Hence, the result follows. □

10 Conclusion

In this work, we analyzed two natural classes of aggregation rules for portioning with cardinal preferences (namely, those based on coordinate-wise aggregation and welfare optimization) as well as the IM mechanism with respect to a number of appealing axiomatic properties. Some of these axioms were proposed in prior work, while others, such as score-representation and score-unanimity, are new. We show that a simple rule that takes the average of the proposals satisfies most of our properties. In contrast, while the IM mechanism possesses the desirable strategyproofness property, it violates some of the other axioms. Thus, in settings where strategyproofness is not a major concern, IM is not necessarily the optimal aggregation rule.

Besides resolving the open questions that remain, avenues for future research include the following: (1) considering other coordinate-wise aggregation rules or characterizing certain rules within this class, (2) studying other classes of aggregation rules, (3) investigating other disutility models (e.g., ℓ_2 or ℓ_∞ norms), and (4) finding a suitable analog of Nash welfare in this setting and exploring its axiomatic properties.

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References

[1] Stéphane Airiau, Haris Aziz, Ioannis Caragiannis, Justin Kruger, Jérôme Lang, and Dominik Peters, 'Portioning using ordinal preferences: Fairness and efficiency', *Artificial Intelligence*, **314**, 103809, (2023).

[2] Haris Aziz, Péter Biró, Jérôme Lang, Julien Lesca, and Jérôme Monnot, 'Efficient reallocation under additive and responsive preferences', *Theoretical Computer Science*, **790**, 1–15, (2019).

[3] Haris Aziz, Anna Bogomolnaia, and Hervé Moulin, 'Fair mixing: the case of dichotomous preferences', *ACM Transactions on Economics and Computation*, **8**(4), 18:1–18:27, (2020).

[4] Haris Aziz, Barton E. Lee, and Nimrod Talmon, 'Proportionally representative participatory budgeting: Axioms and algorithms', in *Proceedings of the 17th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pp. 23–31, (2018).

[5] Haris Aziz and Nisarg Shah, 'Participatory budgeting: Models and approaches', in *Pathways Between Social Science and Computational Social Science: Theories, Methods, and Interpretations*, eds., Tamás Rudas and Gábor Péli, 215–236, Springer, (2021).

[6] Dorothea Baumeister, Linus Boes, and Christian Laußmann, 'Time-constrained participatory budgeting under uncertain project costs', in *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI)*, pp. 74–80, (2022).

[7] Gerdus Benade, Swaprava Nath, Ariel D. Procaccia, and Nisarg Shah, 'Preference elicitation for participatory budgeting', in *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, pp. 376–382, (2017).

[8] Anna Bogomolnaia, Hervé Moulin, and Richard Stong, 'Collective choice under dichotomous preferences', *Journal of Economic Theory*, **122**(2), 165–184, (2005).

[9] Ioannis Caragiannis, George Christodoulou, and Nicos Protopapas, 'Truthful aggregation of budget proposals with proportionality guarantees', in *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*, pp. 4917–4924, (2022).

[10] Jiehua Chen, Martin Lackner, and Jan Maly, 'Participatory budgeting with donations and diversity constraints', in *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*, pp. 9323–9330, (2022).

[11] Robert T. Clemen, 'Combining forecasts: A review and annotated bibliography', *International Journal of Forecasting*, **5**(4), 559–583, (1989).

[12] Conal Duddy, 'Fair sharing under dichotomous preferences', *Mathematical Social Sciences*, **73**, 1–5, (2015).

[13] Soroush Ebadian, Dominik Peters, and Nisarg Shah, 'How to fairly allocate easy and difficult chores', in *Proceedings of the 21st International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pp. 372–380, (2022).

[14] Edith Elkind, Warut Suksompong, and Nicholas Teh, 'Settling the score: Portioning with cardinal preferences', *arXiv*, **abs/2307.15586**, (2023).

[15] Roy Fairstein, Dan Vilenchik, Reshef Meir, and Kobi Gal, 'Welfare vs. representation in participatory budgeting', in *Proceedings of the 21st International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pp. 409–417, (2022).

[16] Peter C. Fishburn, 'A probabilistic model of social choice: Comment', *The Review of Economic Studies*, **42**(2), 297–301, (1975).

[17] Rupert Freeman, David Pennock, Dominik Peters, and Jennifer Wortman Vaughan, 'Truthful aggregation of budget proposals', *Journal of Economic Theory*, **193**, 105234, (2021).

[18] Rupert Freeman, Sujoy Sikdar, Rohit Vaish, and Lirong Xia, 'Equitable allocations of indivisible chores', in *Proceedings of the 19th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pp. 384–392, (2020).

[19] Christian Genest and James V. Zidek, 'Combining probability distributions: A critique and an annotated bibliography', *Statistical Science*, **1**(1), 114–135, (1986).

[20] Ashish Goel, Anilesh K. Krishnaswamy, Sukolsak Sakshuwong, and Tanja Aitamurto, 'Knapsack voting for participatory budgeting', *ACM Transactions on Economics and Computation*, **7**(2), 8:1–8:27, (2019).

[21] Mohak Goyal, Sukolsak Sakshuwong, Sahasrajit Sarmasarkar, and Ashish Goel, 'Low sample complexity participatory budgeting', in *Proceedings of the 50th International Colloquium on Automata, Languages and Programming (ICALP)*, pp. 70:1–70:20, (2023).

[22] D. Ellis Hershkowitz, Anson Kahng, Dominik Peters, and Ariel D. Procaccia, 'District-fair participatory budgeting', in *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI)*, pp. 5464–5471, (2021).

[23] Michael D. Intriligator, 'A probabilistic model of social choice', *The Review of Economic Studies*, **40**(4), 553–560, (1973).

[24] Pallavi Jain, Nimrod Talmon, and Laurent Bulteau, 'Partition aggregation for participatory budgeting', in *Proceedings of the 20th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pp. 665–673, (2021).

[25] Tobias Lindner, Klaus Nehring, and Clemens Puppe, 'Allocating public goods via the midpoint rule', in *Proceedings of the 9th International Meeting of the Society of Social Choice and Welfare*, (2008).

[26] Participatory Budgeting Project. What is PB? http://www.participatorybudgeting.org/what-is-pb/, 2023. Accessed July 19, 2023.

[27] Simon Rey and Jan Maly, 'The (computational) social choice take on indivisible participatory budgeting', *arXiv*, **abs/2303.00621**, (2023).

[28] P. M. Rice, 'Comments on a probabilistic model of social choice', *The Review of Economic Studies*, **44**(1), 187–188, (1977).

[29] Gogulapati Sreedurga, Mayank Ratan Bhardwaj, and Yadati Narahari, 'Maxmin participatory budgeting', in *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI)*, pp. 489–495, (2022).

[30] Nimrod Talmon and Piotr Faliszewski, 'A framework for approval-based budgeting methods', in *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI)*, pp. 2181–2188, (2019).