Proceedings of CECNet 2022
A.J. Tallón-Ballesteros (Ed.)

doi:10.3233/FAIA220574

Uncertainty Measurement of Variable Precision Fuzzy Soft Rough Set Model

Xinyi WANG [a,b] and Qinghai WANG [a,b,1]
[a] School of Computing, Qinghai Normal University, Xining, China
[b] Academy of Plateau Science and Sustainability, Xining, China

Abstract. Aiming at solving the uncertainty of knowledge in variable precision fuzzy soft rough set model. Firstly, from the perspectives of the approximated set and fuzzy equivalent soft approximation space, the uncertainty measurement methods of variable precision fuzzy soft rough set model are defined, such as information granularity, information entropy, rough entropy, etc; Secondly, many related theorems of the measurement methods are given and proved, the monotonicity of the measurement methods are obtained when the fuzzy equivalent soft relation satisfies the inclusion condition; Finally, the feasibility and validity of the definition are verified by an example.

Keywords. variable precision, soft rough set, measurement of uncertainty

1. Introduction

Because of the complexity, diversity and motion change of knowledge in practical problems, people are uncertainty about the expression of knowledge. There are two main reasons for the emergence of uncertainty: on the one hand, people's cognition of knowledge is limited; On the other hand, knowledge has a certain degree of randomness, fuzziness, instability and so on. To deal with the uncertainty problems, we should express and measure the uncertainty of knowledge through proper methods. Entropy is first used to describe the state of matter in a thermodynamic system, later it is used to describe the degree of irregular motion in a thermodynamic system. With the development of information theory, people found that entropy can be used to describe the uncertainty degree of knowledge, entropy combined with probability theory can also measure the uncertainty of knowledge. Inspired by this, Shannon proposed the concept of information entropy to measure the uncertainty of knowledge[1]. Many subsequent researchers defined different entropies to measure uncertainty problems according to different situations. For example, rough entropy is used to measure the roughness of rough set models, fuzzy entropy is used to measure the fuzziness of rough set models. Entropy is widely used in data processing[2], attribute reduction[5] and other fields.

Entropy is one of the methods to measure uncertainty. Compared with other uncertainty measurement methods, entropy has the advantages of simple definition, simple calculation and intuitive reflection of uncertainty. Combining probability theory with entropy, Liang et al. defined rough entropy to measure the roughness of rough set

[1] Corresponding Author, Qinghai WANG, School of Computing, Qinghai Normal University, Xining, China; E-mail: wangqinghai@qhnu.edu.cn.

models. They explored the relationship between different measurement methods, and verified that this measurement method based on entropy could also be used to measure the fuzziness of rough set models[8]. Liang et al. proved the relationship between rough entropy, information entropy and information granularity. They discussed the monotonicity of uncertainty problems in approximate space in this paper[9]. Qian et al. based on the relationship between two kinds of knowledge in the approximation space, proposed to measure the uncertainty in incomplete information systems using combinatorial entropy[10]. Yang et al. discussed the related properties of information entropy and the relationship between information entropy and other uncertainty measurement methods in intuitionistic fuzzy information systems[11]. Mi et al. discussed uncertainty measurement from the perspective of fuzziness. In the partition based on fuzzy rough set model, fuzzy entropy was defined to measure the fuzziness of knowledge, which reflected the degree of ambiguity of knowledge itself[12]. Chen et al. designed the rough entropy thresholding segmentation algorithm, constructed upper and lower approximate sets' rough uncertainty, and verified the effectiveness of the algorithm through experiments[13]. It can be seen that entropy is an effective mathematical method to measure the uncertainty of rough set model and its related models. The use of entropy measure is more suitable for solving the uncertainty problem of mathematical model of practical problems, which can help people to understand the nature and characteristics of data model objectively, also help to apply the model to solve practical problems.

Soft set theory is different from the existing parametric tools used to deal with uncertain data[14]. It uses the mapping between objects and attributes to express the relationship between them. In order to further combine soft set with rough set and fuzzy set to solve mathematical problems in real life, fuzzy soft set, soft rough set and other mathematical models have been proposed and popularized. Fuzzy soft set uses the parameterized feature of soft set to describe the relationship between objects and attributes in fuzzy environment[15]. Khan mentioned in the paper that fuzzy soft set can solve the problem of actual data decision-making in an imprecise environment[16]. Feng discusses the related properties and theorems of fuzzy sets in soft rough approximation space and verifies that the combination of soft sets and different mathematical models can be applied to solve more complex mathematical problems[17]. The advantage and limitation of fuzzy soft set model and fuzzy soft rough set model are shown in Table 1:

Table 1. The advantage and limitation of fuzzy soft set model and fuzzy soft rough set model

Set	Advantage	Limitation
Fuzzy soft set	It can express the relationship between objects and attributes in fuzzy environment.	It is not conducive to the processing of fuzzy environment data.
Fuzzy soft rough set	It is good for processing the data in fuzzy environment.	It is strict with the classification requirements for the processed data.

Variable precision rough set[18], as an extended rough set model using fault-tolerant mechanism, solves the problem of excessive classification requirements in classical rough set models in accurate environments. Subsequently, Katezberg and Ziarko[19] proposed a variable precision rough set model based on asymmetric boundaries u and l ($0 \leq l < u \leq 1$), which improved the original variable precision rough set model by using the upper and lower approximations of the asymmetric boundary limiting set. Xing et al. applied the variable precision rough set model combined with Bayesian method to the diagnosis of tumor cases[20]. Wu et al. proposed the image multi-threshold segmentation algorithm using variable precision rough set to solve the classification error problem of background subgraph and target subgraph[21]. In recent years, variable precision fuzzy rough set model has become a mathematical tool to solve the problem of strict classification requirements in fuzzy environment. The model is beneficial to reduce the influence of noise data. A Mieszkowicz-Rolka and L Rolka used the research method of excluding objects that may affect the quality of the upper approximation and lower approximation results through cut-set when constructing the variable precision fuzzy rough set mode[22]. In the membership function approximated under the definition, the "better" objects are used instead of all objects; in the membership function of the upper approximation, the identification range of the upper approximation is expanded by some "good" objects after intercepting. Zhang et al. proposed a variable precision fuzzy rough set model based on coverage, extended the variable precision rough set model to cover fuzzy environment, discussed the properties of the model and cited examples to prove[23]. Gong et al. defined implicit operators and membership functions, explained the principle and application of operators, cited examples to prove the feasibility of the model[24]. Combining the knowledge expression characteristics of soft set theory with the robustness of variable precision fuzzy rough set model is helpful to analyze the mathematical model that built on the premise of fuzzy soft relation, solving the related mathematical problems of constructing fuzzy knowledge from actual fuzzy data.

Based on the variable precision fuzzy soft rough set model, this paper proposes many uncertainty measurement methods based on the idea of entropy, deeply studies the mathematical characteristics of the variable precision fuzzy soft rough set model, which is helpful to analyze and measure the uncertainty in the process of constructing fuzzy soft knowledge from actual fuzzy data. Firstly, from the perspectives of the approximated set and fuzzy equivalent soft approximation space, the uncertainty measurement methods of variable precision fuzzy soft rough set model are defined; Secondly, many related theorems of the measurement methods are given and proved, the monotonicity of the measurement methods are obtained when the fuzzy equivalent soft relation satisfies the inclusion condition; Finally, the feasibility and validity of the definition are verified by an example.

2. Preliminaries

Define 2.1[25] Let U be a finite universe of discourse. A fuzzy set A on U is described as: $A = \{(u, \mu_A(u)) | u \in U\}$, in which $\mu_A(u)$ denotes the possible membership degrees of the elements $u \in U$ to the set A, with the conditions: for $\forall u \in U$, $0 \le \mu_A(u) \le 1$. All fuzzy sets on U are represented as $F(U)$. $\forall A, B \in F(U)$ $\forall u \in U$, there has $A \cap B = \frac{\mu_A(u) \wedge \mu_B(u)}{u}$. The cardinality of a fuzzy set A is defined as: $|A| = \sum_{u \in U} \mu_A(u)$. The α −level cut set of fuzzy set A, denoted by A_α, is defined as : $A_\alpha = \{u \in U | \mu_A(u) \ge \alpha\}$, for $\alpha \in [0,1]$. Meanwhile, we call the set supp $A = \{u \in U | \mu_A(u) > 0\}$ is the support of set A.

Define 2.2[14] Let U be a finite universe of discourse, E be a set of parameters of U, $A \subseteq E$, $F(U)$ be all fuzzy sets on U. When the mapping f satisfies the condition $f: E \to F(U)$, the pair $\varepsilon = (f, E)$ is a fuzzy soft set over universe U. $R: U \times E \to [0,1]$ is called a fuzzy soft relation from U to E, which is defined as $R = \{\langle (u, e), \mu_R(u, e) \rangle | (u, e) \in U \times E\}$, where $\mu_R: U \times E \to [0,1]$ $\mu_R(u, e) = \mu_{f(e)}(u)$.

Define 2.3[26] Let U be a finite universe of discourse, E be a set of parameters of U, $A \subseteq E$, $F(E)$ be all fuzzy sets on E. When the mapping $\tilde{f}$ satisfies the condition $\tilde{f}: U \to F(E)$, the pair $\varepsilon = (\tilde{f}, E)$ is a pseudo fuzzy soft set over universe U. $\tilde{R}: U \times E \to [0,1]$

is called a pseudo fuzzy soft relation from U to E, which is defined as $\tilde{R} = \{\langle (u,e), \mu_{\tilde{R}}(u,e)\rangle | (u,e) \in U \times E\}$, where $\mu_{\tilde{R}}: U \times E \to [0,1] \mu_{\tilde{R}}(u,e) = \mu_{\tilde{f}(u)}(e)$.

Define 2.4[25] Let(f,E) be a fuzzy soft set on U, the set of all fuzzy soft relations from U to E is denoted as $F_S(U \times E)$. Let$(\tilde{f},E)$ be a fuzzy soft set on U, the set of all pseudo fuzzy soft relations from U to E is denoted as $\tilde{F}_S(U \times E)$. For $R_i \in F_S(U \times E)$, $R_j \in \tilde{F}_S(U \times E)$, if $(u_i,u_j) \in \mathrm{R}$ indicated by $\exists v \in E$ let $(u_i,v) \in R_i$ and $(v,u_j) \in R_j$, which is called that the soft fuzzy relation R over U is synthetic by a fuzzy soft relation from U to E and a pseudo fuzzy soft relation from U to E, is denoted as $\mathrm{R} = R_i \circ R_j$, $\mathrm{R} = \{\langle (u,u), \mu_{\mathrm{R}}(u,u)\rangle | (u,u) \in U \times U\}$, where $\mu_{\mathrm{R}}: U \times U \to [0,1]\ \mu_{\mathrm{R}}(u,u) = \mu_{f(u)}(u)$. If the fuzzy soft relation R over U satisfies reflexivity, symmetry and transitivity, R is called a fuzzy equivalent soft relation on U.

Define 2.5[22] Let U be a finite universe of discourse, $\forall A, B \in F(U)$, a fuzzy set A contained in a fuzzy set B is defined as: $A \subseteq B \Leftrightarrow \forall x \in U, \mu_A(x) \le \mu_B(x)$. The fuzzy inclusion set of A in B is denoted by A^B, the membership function of fuzzy set A^B is defined as: $\mu_{A^B}(x) = \begin{cases} I(\mu_A(x), \mu_B(x)), & \mu_A(x) > 0 \\ 0, & otherwise \end{cases}$. We introduce the measure of α −inclusion error $e_\alpha(A,B)$ of any fuzzy set A in a fuzzy B: $e_\alpha(A,B) = 1 - \frac{power(A \cap A_\alpha^B)}{power(A)}$, among them, $power(A) = \sum_{i=1}^{n} \mu_A(u_i)$ is the cardinality of a fuzzy set A, the α −level cut set of fuzzy set A^B is denoted by A_α^B.

Define 2.6 Let U be a finite universe of discourse, E be a set of parameters of U, (f,E) and$(\tilde{f},E)$ are a fuzzy soft set and a pseudo fuzzy soft set from U to E, R_i and R_j are fuzzy equivalent soft relation and pseudo fuzzy equivalent soft relation from U to E, R is fuzzy equivalent soft relation on U which is synthesized by R_i and R_j, the pair (U,R) is called fuzzy equivalent soft approximation space. $\forall u_i \in U$, the fuzzy equivalent soft partition derived from u_i and R is represented as a fuzzy set, which membership function is defined as $U_i(u) = \mathrm{R}(u_i,u)\ i = 1,2,\ldots,n$. The boundary s,t satisfies the conditions: $0 \le s < t \le 1$, for any $A \in F(U)$, the s-lower approximation set of A and the t-upper approximation set of A in the fuzzy equivalent soft approximation space are defined as follows:

$$\underline{\mathrm{R}}_I{}^s(A)(u_i) = \begin{cases} \inf\limits_{u \in S_{is}} I(U_i(u), A(u)) & \exists \alpha_s = sup\{\alpha \in (0,1] | e_\alpha(U_i,A) \le 1-s\} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$S_{is} = \mathrm{supp}(U_i \cap (U_i{}^A)_{\alpha_s}) \tag{2}$$

$$e_\alpha(U_i,A) = 1 - \frac{power(U_i \cap (U_i{}^A)_\alpha)}{power(U_i)} \quad i = 1,2,\ldots,n. \tag{3}$$

$$\overline{\mathrm{R}}^T{}_t(A)(u_i) = \begin{cases} \sup\limits_{u \in S_{it}} T(U_i(u), A(u)) & \exists \alpha_t = sup\{\alpha \in (0,1] | e_\alpha{}'(U_i,A) < 1-t\} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$S_{it} = \text{supp}(U_i \cap (U_i \cap A)_{\alpha_t}) \tag{5}$$

$$e_\alpha{}'(U_i, A) = 1 - \frac{power(U_i \cap (U_i \cap A)_\alpha)}{power(U_i)} \quad i = 1,2,\ldots,n. \tag{6}$$

Among them, T is the T-module operator and I is the implication operator, the pair $(\underline{\mathrm{R}}_I{}^S(A), \overline{\mathrm{R}}^T{}_t(A))$ is called variable precision fuzzy soft rough set of A.

3. The uncertainty measurements of variable precision fuzzy soft rough set

3.1. Uncertainty measure for fuzzy equivalent soft approximation space

Define 3.1.1 In the fuzzy equivalent soft approximation space, there exists an exact subset $U_s = \{u_1, u_2, \ldots, u_m \; m \leq n\}$ in the universe $U = \{u_1, u_2, u_3, \ldots, u_n\}$, A subset U_s conditions to be fulfilled: $U_s = \text{supp}(U_i \cap (U_i{}^A)_{\alpha_s})$, where $F(U_s)$ denoted by all fuzzy sets on U_s. For any $A \in F(U_s)$, $\forall u_i \in U_s$, the fuzzy equivalent soft partitions derived from u_i and R are represented as fuzzy sets, their membership function are defined as $U_i(u) = \mathrm{R}(u_i, u) \; i = 1,2,\ldots,m$. In the following definitions, the symbol $U_i(u)$ is defined in the same way.

Define 3.1.2 In the fuzzy equivalent soft approximation space (U_s, R), the information granularity of variable precision fuzzy soft rough set under fuzzy soft equivalent relation R is defined as: $GK(\mathrm{R}) = \frac{1}{|U_s|^2} \sum |U_i(u)|$.

Theorem 3.1.1 Let $U_s = \{u_1, u_2, u_3, \ldots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations. $\forall u_i \in U_s$, the fuzzy equivalent soft partition derived from u_i and P is represented as a fuzzy set, its membership function is defined as $\mathrm{P}_i(u) = \mathrm{P}(u_i, u) \; i = 1,2,\ldots,m$. In the following definitions, the symbol $\mathrm{P}_i(u)$ is defined in the same way. When $\mathrm{R} \subseteq \mathrm{P}$, $GK(\mathrm{R}) \leq GK(\mathrm{P})$.

Proof: Because $\mathrm{R} \subseteq \mathrm{P}$, it can be known $\forall u_i \in U_s\big(U_i(u) \subseteq \mathrm{P}_i(u)\big)$, $\forall u_i \in U_s(|U_i(u)| \leq |\mathrm{P}_i(u)|)$, then $GK(\mathrm{R}) = \frac{1}{|U_s|^2} \sum |U_i(u)| \leq \frac{1}{|U_s|^2} \sum |\mathrm{P}_i(u)| \leq GK(\mathrm{P})$.

Define 3.1.3 In the fuzzy equivalent soft approximation space (U_s, R), the rough entropy of variable precision fuzzy soft rough set under fuzzy soft equivalent relation R is defined as: $E_r(\mathrm{R}) = -\sum \frac{1}{|U_s|} log_2 \frac{1}{|U_i(u)|}$.

Theorem 3.1.2 Let $U_s = \{u_1, u_2, u_3, \ldots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations. When $\mathrm{R} \subseteq \mathrm{P}$, $E_r(\mathrm{R}) \leq E_r(\mathrm{P})$.

Proof: Because $\mathrm{R} \subseteq \mathrm{P}$, it can be known $\forall u_i \in U_s(log_2|U_i(u)| \leq log_2|\mathrm{P}_i(u)|)$, $E_r(\mathrm{R}) = -\sum \frac{1}{|U_s|} log_2 \frac{1}{|U_i(u)|} \leq -\sum \frac{1}{|U_s|} log_2 \frac{1}{|\mathrm{P}_i(u)|} \leq E_r(\mathrm{P})$.

Define 3.1.4 In the fuzzy equivalent soft approximation space (U_s, R), the information entropy based on Shannon entropy of variable precision fuzzy soft rough set under fuzzy soft equivalent relation R is defined as: $H(\mathrm{R}) = -\frac{1}{|U_s|} \sum log_2 \frac{|U_i(u)|}{|U_s|}$.

Theorem 3.1.3 Let $U_s = \{u_1, u_2, u_3, \ldots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations. When $\mathrm{R} \subseteq \mathrm{P}$, $H(\mathrm{R}) \geq H(\mathrm{P})$.

Proof: Because $\mathrm{R} \subseteq \mathrm{P}$, it can be known, $\forall u_i \in U_s \; (-log_2|U_i(u)| \geq -log_2|\mathrm{P}_i(u)|)$, then $H(\mathrm{R}) = -\frac{1}{|U_s|} \sum log_2 \frac{|U_i(u)|}{|U_s|} \geq -\frac{1}{|U_s|} \sum log_2 \frac{|\mathrm{P}_i(u)|}{|U_s|} = H(\mathrm{P})$.

Define 3.1.5 In the fuzzy equivalent soft approximation space (U_s, R), the information entropy of variable precision fuzzy soft rough set under fuzzy soft equivalent relation R is defined as: $E(\mathrm{R}) = \frac{1}{|U_s|}\sum\left(1 - \frac{|U_i(u)|}{|U_s|}\right)$.

Theorem 3.1.4 Let $U_s = \{u_1, u_2, u_3, \dots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations. When $\mathrm{R} \subseteq \mathrm{P}$, $E(\mathrm{R}) \geq E(\mathrm{P})$.

Proof: Because $\mathrm{R} \subseteq \mathrm{P}$, it can be known $\forall u_i \in U_s(|U_s| - |U_i(u)| \geq |U_s| - |\mathrm{P}_i(u)|)$, then $E(\mathrm{R}) = \frac{1}{|U_s|}\sum\left(1 - \frac{|U_i(u)|}{|U_s|}\right) \geq \frac{1}{|U_s|}\sum\left(1 - \frac{|\mathrm{P}_i(u)|}{|U_s|}\right) = E(\mathrm{P})$.

Define 3.1.6 The combined entropy and conditional entropy of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R and P are defined as:$E(\mathrm{P};\mathrm{R}) = \frac{1}{|U_s|}\sum\left(1 - \frac{|U_i(u)\cap \mathrm{P}_i(u)|}{|U_s|}\right)$, $E(\mathrm{P}|\mathrm{R}) = \frac{1}{|U_s|}\sum\left(\frac{|U_i(u)|}{|U_s|} - \frac{|U_i(u)\cap \mathrm{P}_i(u)|}{|U_s|}\right)$.

Theorem 3.1.5 Let $E(\mathrm{R})$ is the information entropy of variable precision fuzzy soft rough set under fuzzy soft equivalent relation R in finite universe $U_s = \{u_1, u_2, u_3, \dots, u_m\}$, then $E(\mathrm{P}|\mathrm{R}) = E(\mathrm{P};\mathrm{R}) - E(\mathrm{R})$.

Proof: $E(\mathrm{P}|\mathrm{R}) - E(\mathrm{P};\mathrm{R}) = \frac{1}{|U_s|}\sum\left(\frac{|U_i(u)|}{|U_s|} - 1\right) = -E(\mathrm{R})$.

3.2. Uncertainty measure for approximated set

Define 3.2.1 In the fuzzy equivalent soft approximation space (U_s, R), $\forall u_i \in U_s$, the fuzzy equivalent soft partitions derived from u_i and R are represented as fuzzy sets, their membership function are defined as $U_i(u) = \mathrm{R}(u_i, u)\ i = 1,2,\dots,m$. For $\forall A \in F(U_s)$, variable precision fuzzy soft rough set of A is denoted by $(\underline{\mathrm{R}}_I{}^s(A), \overline{\mathrm{R}}^T{}_t(A))$, where $\underline{\mathrm{R}}_I{}^s(A)$ represents the s-lower approximate set of A, $\overline{\mathrm{R}}^T{}_t(A)$ represents the t- upper approximate set of A. The roughness of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R is defined as: $\rho_\mathrm{R} = 1 - \frac{|\underline{\mathrm{R}}_I{}^s(A)|}{\left|\overline{\mathrm{R}}^T{}_t(A)\right|}$.

Theorem 3.2.1 Let $U_s = \{u_1, u_2, u_3, \dots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations. When $\mathrm{R} \subseteq \mathrm{P}$, $\rho_\mathrm{R} \leq \rho_\mathrm{P}$.

Proof: Because $\mathrm{R} \subseteq \mathrm{P}$, it can be known $\forall u_i \in U_s(U_i(u) \subseteq \mathrm{P}_i(u))$. Assuming that the set S_{is} and the set S_{it} are the same under the fuzzy equivalent soft relation R and P, then according to the definition discussion:

Let $I(x,y) = \max(1-x, y)$, $T(x,y) = \min(x,y)$. Because $\forall u_i \in U_s\ \ (U_i(u) \subseteq \mathrm{P}_i(u))$, it can be known $\forall u_i \in U_s\ (1 - U_i(u) \supseteq 1 - \mathrm{P}_i(u))\ i = 1,2,\dots,m$.

(1)When $\max(1 - U_i(u), A(u)) = A(u)$, $A(u) \geq 1 - U_i(u) \geq 1 - \mathrm{P}_i(u)$, then $\max(1 - \mathrm{P}_i(u), A(u)) = A(u)$.

(2)When $\max(1 - U_i(u), A(u)) = 1 - U_i(u)$, $1 - U_i(u) \geq A(u)$, $1 - U_i(u) \geq 1 - \mathrm{P}_i(u)$, at this time, no matter what value $\max(1 - \mathrm{P}_i(u), A(u))$ takes, it is less than or equal to $1 - U_i(u)$. In summary, $\underline{\mathrm{R}}_I{}^s(A) \supseteq \underline{\mathrm{P}}_I{}^s(A)$.

(3)When $\min(U_i(u), A(u)) = A(u)$, $A(u) \leq U_i(u) \leq \mathrm{P}_i(u)$, $\min(\mathrm{P}_i(u), A(u)) = A(u)$.

(4)When $\min(U_i(u), A(u)) = U_i(u)$, $U_i(u) \leq A(u)$, $U_i(u) \leq \mathrm{P}_i(u)$, at this time, no matter what value $\min(\mathrm{P}_i(u), A(u))$ takes, it is more than or equal to $U_i(u)$. In summary, $\overline{\mathrm{R}}^T{}_t(A) \subseteq \overline{\mathrm{P}}^T{}_t(A)$.

Accordingly, $\rho_\mathrm{R} = 1 - \frac{|\underline{\mathrm{R}}_I{}^s(A)|}{\left|\overline{\mathrm{R}}^T{}_t(A)\right|} \leq 1 - \frac{|\underline{\mathrm{P}}_I{}^s(A)|}{\left|\overline{\mathrm{P}}^T{}_t(A)\right|} = \rho_\mathrm{P}$.

Define 3.2.2 In the fuzzy equivalent soft approximation space(U_s, R), $\forall u_i \in U_s$, the roughness based rough entropy of the variable precision fuzzy soft rough set under the fuzzy equivalent soft relation R is defined as $C(\mathrm{R}) = \rho_{\mathrm{R}} \cdot GK(\mathrm{R})$, where ρ_{R} is the roughness of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R, $GK(\mathrm{R})$ is the information granularity of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R.

Theorem 3.2.2 Let $U_s = \{u_1, u_2, u_3, \dots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations. When $\mathrm{R} \subseteq \mathrm{P}$, $C(\mathrm{R}) \leq C(\mathrm{P})$.

Proof: From theorem 3.1.1 and theorem 3.2.1 we know that when $\mathrm{R} \subseteq \mathrm{P}$, $\rho_{\mathrm{R}} \leq \rho_{\mathrm{P}}$, $GK(\mathrm{R}) \leq GK(\mathrm{P})$, then $C(\mathrm{R}) = \rho_{\mathrm{R}} \cdot GK(\mathrm{R}) \leq \rho_{\mathrm{P}} \cdot GK(\mathrm{P})$.

Define 3.2.3 In the fuzzy equivalent soft approximation space (U_s, R), $\forall u_i \in U_s$, for $\forall A \in F(U_s)$, the rough membership degree of u_i with respect to A is defined as: $\mathrm{R}(A)(u_i) = \frac{\sum_{u \in U_s} \min\{\mathrm{R}(u_i,u), A(u)\}}{\sum_{u \in U_s} \mathrm{R}(u_i,u)}$, where $A(u)$ represents the membership degree of element u in set A, then the fuzzy entropy of the variable precision fuzzy soft rough set under fuzzy soft equivalent relation R is defined as: $F\mathrm{R}(A) = -\frac{2}{|U_s|} \sum_{u_i \in U_s} \mathrm{R}(A)(u_i) log_2 \mathrm{R}(A)(u_i)$.

Theorem 3.2.2 In the fuzzy equivalent soft approximation space(U_s, R), the fuzzy entropy of a definable exact set is 0.

Proof: There exists a definable exact set X in the fuzzy equivalent soft approximation space (U_s, R). Since the set X is an exact set, $X(\mathrm{u})=1$ when $\forall u \in X$, $X(u) = 0$ when $\forall u \notin X$. For $\forall u_i \in U_s$, when $u \in X$, $\mathrm{R}(X)(u_i) = \frac{\sum_{u \in U_s} \min\{\mathrm{R}(u_i,u), X(u)\}}{\sum_{u \in U_s} \mathrm{R}(u_i,u)} = 1$. For $\forall u_i \in U_s$, when $u \notin X$, $\mathrm{R}(X)(u_i) = \frac{\sum_{u \in U_s} \min\{\mathrm{R}(u_i,u), X(u)\}}{\sum_{u \in U_s} \mathrm{R}(u_i,u)} = 0$ .Accordingly, $F\mathrm{R}(X) = -\frac{2}{|U_s|} \sum_{u_i \in U_s} \mathrm{R}(X)(u_i) log_2 \mathrm{R}(X)(u_i) = 0$.

Theorem 3.2.3 In the fuzzy equivalent soft approximation space(U_s, R), for any $A, B \in F(U_s)$, if $A \subseteq B$, $F\mathrm{R}(A) \geq F\mathrm{R}(B)$.

Proof: Because $A \subseteq B$, $\forall u_i \in U_s$, it can be known $\mathrm{R}(A)(u_i) = \frac{\sum_{u \in U_s} \min\{\mathrm{R}(u_i,u), A(u)\}}{\sum_{u \in U_s} \mathrm{R}(u_i,u)} \leq \frac{\sum_{u \in U_s} \min\{\mathrm{R}(u_i,u), B(u)\}}{\sum_{u \in U_s} \mathrm{R}(u_i,u)} = \mathrm{R}(B)(u_i)$, $-\mathrm{R}(A)(u_i) log_2 \mathrm{R}(A)(u_i) \geq -\mathrm{R}(B)(u_i) log_2 \mathrm{R}(B)(u_i)$, also $F\mathrm{R}(A) \geq F\mathrm{R}(B)$.

3.3. The relationship between uncertainty measures of variable precision fuzzy soft rough set

Theorem 3.3.1 In the fuzzy equivalent soft approximation space(U_s, R), the information entropy based on Shannon entropy $H(\mathrm{R})$ of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R and rough entropy $E_r(\mathrm{R})$ of variable precision fuzzy soft rough set under fuzzy equivalent soft relation R have the following relationship: $H(\mathrm{R}) + E_r(\mathrm{R}) = log_2|U_s|$.

Proof: $H(\mathrm{R}) = -\frac{1}{|U_s|} \sum log_2 \frac{|U_i(u)|}{|U_s|} = -\frac{1}{|U_s|} \sum (log_2|U_i(u)| - log_2|U_s|) - \left(-\sum \frac{1}{|U_s|} log_2 \frac{1}{|U_i(u)|}\right) + \sum \frac{1}{|U_s|} log_2|U_s| = -E_r(\mathrm{R}) + log_2|U_s|$.

Theorem 3.3.2 In the fuzzy equivalent soft approximation space (U_s, R), the information entropy $E(\mathrm{R})$ of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R and the information granularity $GK(\mathrm{R})$ of the variable precision fuzzy soft rough set under fuzzy equivalent soft relation R have the following relationship: $E(\mathrm{R}) + GK(\mathrm{R}) = 1$.

Proof: $E(\mathrm{R}) + GK(\mathrm{R}) = \frac{1}{|U_s|}\sum(1 - \frac{|U_i(u)|}{|U_s|}) + \frac{1}{|U_s|}\sum\frac{|U_i(u)|}{|U_s|} = \frac{1}{|U_s|}|U_s| = 1.$

4. Example

Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ represents the set of six different houses, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ represents the set of attributes possessed by each house, which respectively represent the price, location, basic decoration, lighting, surrounding environment and basic construction. In order to assess the six houses comprehensively, the fuzzy soft set(f, E) is used to represent the evaluation result of one of the experts as shown in Table 2, and the pseudo fuzzy soft set $(\tilde{f}, E)$ is used to represent the evaluation result of the other expert as shown in Table 3.

Table 2. fuzzy soft set

U/E	e_1	e_2	e_3	e_4	e_5	e_6
u_1	1	0.85	0.50	0.69	0.69	0.84
u_2	0.85	1	0.50	0.65	0.65	0.84
u_3	0.50	0.50	1	0.50	0.50	0.50
u_4	0.69	0.50	0.50	1	0.61	0.69
u_5	0.69	0.69	0.50	0.69	1	0.69
u_6	0.84	0.84	0.50	0.69	0.69	1

Table 3. pseudo fuzzy soft set

U/E	e_1	e_2	e_3	e_4	e_5	e_6
u1	1	0.10	0.50	0.69	0.69	0.81
u2	0.35	1	0.50	0.40	0.40	0.69
u3	0.20	0.50	1	0.25	0.39	0.40
u4	0.62	0.69	0.34	1	0.50	0.69
u5	0.41	0.69	0.50	0.41	1	0.69
u6	0.30	0.50	0.50	0.30	0.60	1

Synthesizing fuzzy equivalent soft relation on U,

$$\mathrm{R} = \begin{pmatrix} 1 & 0.85 & 0.50 & 0.69 & 0.69 & 0.84 \\ 0.85 & 1 & 0.50 & 0.69 & 0.69 & 0.84 \\ 0.50 & 0.50 & 1 & 0.50 & 0.50 & 0.50 \\ 0.69 & 0.69 & 0.50 & 1 & 0.69 & 0.69 \\ 0.69 & 0.69 & 0.50 & 0.69 & 1 & 0.69 \\ 0.84 & 0.84 & 0.50 & 0.69 & 0.69 & 1 \end{pmatrix}$$

Let set $A \in F(U)$, $A = {}^{0.60}/_{u_1} + {}^{0.65}/_{u_2} + {}^{0.65}/_{u_3} + {}^{0.70}/_{u_4} + {}^{0.80}/_{u_5} + {}^{0.85}/_{u_6}$ represents a collection of house that is rated as good, let $I(x, y) = \max(1 - x, y)$, $T(x, y) = \min(x, y)$, $s = 0.75$, $t = 0.55$, for example by u_3, the s-lower approximate set of A and the t- upper approximate set of A are:

$\underline{\mathrm{R}}_I{}^{0.75}(A) = \{{}^{0.60}/_{u_1},\ {}^{0.65}/_{u_2},\ {}^{0.65}/_{u_3},\ {}^{0.65}/_{u_4},\ {}^{0.65}/_{u_5},\ {}^{0.65}/_{u_6}\}$

$\overline{\mathrm{R}}^T{}_{0.55}(A) = \{{}^{0.84}/_{u_1},\ {}^{0.84}/_{u_2},\ {}^{0.65}/_{u_3},\ {}^{0.70}/_{u_4},\ {}^{0.80}/_{u_5},\ {}^{0.85}/_{u_6}\}$

when $\alpha_s = 0.65$, $U_s = \{u_2, u_3, u_4, u_5, u_6\}$, in the fuzzy equivalent soft approximation space(U_s, R), the fuzzy equivalent soft relation R over U_s is

$$\mathrm{R}=\begin{pmatrix} 1 & 0.50 & 0.69 & 0.69 & 0.84 \\ 0.50 & 1 & 0.50 & 0.50 & 0.50 \\ 0.69 & 0.50 & 1 & 0.69 & 0.69 \\ 0.69 & 0.50 & 0.69 & 1 & 0.69 \\ 0.84 & 0.50 & 0.69 & 0.69 & 1 \end{pmatrix}$$

Let set $A \in F(U)$, $A = {}^{0.65}/_{u_2} + {}^{0.65}/_{u_3} + {}^{0.70}/_{u_4} + {}^{0.80}/_{u_5} + {}^{0.85}/_{u_6}$ represents a collection of house that is rated as good, under this condition, the s-lower approximate set of A and the t- upper approximate set of A are:

$\underline{\mathrm{R}}_I{}^{0.75}(A) = \{{}^{0.65}/_{u_2},\ {}^{0.65}/_{u_3},\ {}^{0.65}/_{u_4},\ {}^{0.65}/_{u_5},\ {}^{0.65}/_{u_6}\}$

$\overline{\mathrm{R}}^{T}{}_{0.55}(A) = \{{}^{0.84}/_{u_2},\ {}^{0.65}/_{u_3},\ {}^{0.70}/_{u_4},\ {}^{0.80}/_{u_5},\ {}^{0.85}/_{u_6}\}$

In the fuzzy equivalent soft approximation space(U_s, R), the information granularity of variable precision fuzzy soft rough set $GK(\mathrm{R}) = 0.70$, rough entropy $E_r(\mathrm{R}) = 1.81$, information entropy based on Shannon entropy $H(\mathrm{R}) = 0.51$, information entropy $E(\mathrm{R}) = 0.30$, roughness $\rho_{\mathrm{R}} = 0.15$, roughness based rough entropy $C(\mathrm{R}) = 0.11$, fuzzy entropy $F\mathrm{R}(A) = 0.26$. If any $B \in F(U_s)$, $A \subseteq B$, $B = {}^{0.70}/_{u_2} + {}^{0.70}/_{u_3} + {}^{0.70}/_{u_4} + {}^{0.80}/_{u_5} + {}^{0.85}/_{u_6}$, fuzzy entropy $F\mathrm{R}(B) = 0.24$.

Let $U_s = \{u_1, u_2, u_3, \dots, u_m\}$ is a finite universe, R and P are fuzzy soft equivalent relations, and $\mathrm{R} \subseteq \mathrm{P}$, in the fuzzy equivalent soft approximation space(U_s, P), the fuzzy equivalent soft relation P over U_s is

$$\mathrm{P}=\begin{pmatrix} 1 & 0.50 & 0.69 & 0.69 & 0.84 \\ 0.50 & 1 & 0.50 & 0.50 & 0.50 \\ 0.69 & 0.50 & 1 & 1 & 0.69 \\ 0.69 & 0.50 & 1 & 1 & 0.69 \\ 0.84 & 0.50 & 0.69 & 0.69 & 1 \end{pmatrix}$$

Let set $A \in F(U)$, $A = {}^{0.65}/_{u_2} + {}^{0.65}/_{u_3} + {}^{0.70}/_{u_4} + {}^{0.80}/_{u_5} + {}^{0.85}/_{u_6}$ represents a collection of house that is rated as good, under this condition, the s-lower approximate set of A and the t- upper approximate set of A are:

$\underline{\mathrm{P}}_I{}^{0.75}(A) = \{{}^{0.65}/_{u_2},\ {}^{0.65}/_{u_3},\ {}^{0.65}/_{u_4},\ {}^{0.65}/_{u_5},\ {}^{0.65}/_{u_6}\}$

$\overline{\mathrm{P}}^{T}{}_{0.55}(A) = \{{}^{0.84}/_{u_2},\ {}^{0.65}/_{u_3},\ {}^{0.80}/_{u_4},\ {}^{0.80}/_{u_5},\ {}^{0.85}/_{u_6}\}$

In the fuzzy equivalent soft approximation space(U_s, P), the information granularity of variable precision fuzzy soft rough set $GK(\mathrm{P}) = 0.72$, rough entropy $E_r(\mathrm{P}) = 1.86$, information entropy based on Shannon entropy $H(\mathrm{P}) = 0.46$ information entropy $E(\mathrm{P}) = 0.29$, roughness $\rho_{\mathrm{P}} = 0.17$, roughness based rough entropy $C(\mathrm{P}) = 0.12$, the combined entropy $E(\mathrm{P}; \mathrm{R}) = 0.29$, conditional entropy $E(\mathrm{P}|\mathrm{R}) = 0$.

To sum up, the example results satisfy all the above theorems, which verifies the feasibility and effectiveness of the measurement methods.

5. Conclusion

With the in-depth study of different data models, the uncertainty of how to recognize and express knowledge has become the key to the study of mathematical models. The variable precision fuzzy soft rough set model is proposed. It combines the knowledge expression characteristics of the soft set theory with the robustness of the variable precision fuzzy rough set model, which is helpful to analyze the mathematical model built on the premise of fuzzy soft relations, solving the problems of constructing the model based on actual fuzzy data. Based on the variable precision fuzzy soft rough set model, this paper proposes uncertainty measurement methods based on the idea of entropy, and deeply studies the mathematical properties of the variable precision fuzzy soft rough set model. The next step of this paper will be based on the variable precision intuitionistic fuzzy soft rough set model, carry out fuzzy extension research, so as to solve the problem of analyzing and processing fuzzy data under complex knowledge expression in practical application fields.

Acknowledgement

This research was supported by the Qinghai Key Laboratory of the Internet of Things (2020-ZJ-Y16) and the National Key Research and Development Project (2020YFC1523305).

References

[1] Shannon CE. The Mathematical Theory of Communication. The Bell System Technical Journal. 1948 Jul;27(3):379-423.
[2] Shi HT, Li YX, Ding MS, Li XB, Li ZX, Zhang ZF. Wind Power Prediction Based on Rough Set Conditional Entropy Weighted Markov Chain. Proceedings of the 2021 IEEE Sustainable Power and Energy Conference; 2021 Dec 23-25; New York, USA: IEEE Press; c2022. p. 169-72.
[3] Sun L, Wang LY, Ding WP, Qian YH, Xu JC. Feature Selection Using Fuzzy Neighborhood Entropy-Based Uncertainty Measures for Fuzzy Neighborhood Multigranulation Rough Sets. IEEE Transactions on Fuzzy Systems. 2021 Jan;29(1):19-33.
[4] Pal JK, Ray SS, Cho SB, Pal SK. Fuzzy-Rough Entropy Measure and Histogram Based Patient Selection for miRNA Ranking in Cancer. IEEE/ACM Transactions on Computational Biology and Bioinformatics. 2018 Mar;15(2):659-72.
[5] Wang Z, Zheng X, Pan HG, Li DY. Information Entropy Multi-Decision Attribute Reduction Fuzzy Rough Set for Dust Particulate Imagery Characteristic Extraction. IEEE Access. 2020 Apr;8(1):77865-74.
[6] Zhang QH, Xiao Y. New Attribute Reduction on Information Entropy. Journal of Frontiers of Computer Science and Technology. 2013 Apr;7(04):359-67.
[7] Xiong JX, Wu JZ, Wang QH. Decision Cost Attribute Reduction of Hybrid Data Based on Neighborhood Mutual Information Entropy. Journal of Chinese Computer Systems. 2021 Aug;42(8):1584-90.
[8] Liang JY, Chin KS, Dang CY, Yam CMR. A New Method for Measuring Uncertainty and Fuzziness in Rough Set Theory. International Journal of General Systems. 2002 Apr;31(4):331-42.
[9] Liang JY, Shi ZZ. The Information Entropy, Rough Entropy and Knowledge Granulation in Rough Set Theory. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 2004 Jan;12(1):37-46.
[10] Qian YH, Liang JY. Combination Entropy and Combination Granulation in Incomplete Information System. Lecture Notes in Computer Science. 2006 Jul;4062(1):184-90.
[11] Yang WP, Lin ML. Information entropy in intuitionistic fuzzy information systems. Computer Engineering and Applications. 2014 Nov;50(11):130-34.
[12] Mi JS, Leung Y, Wu WZ. An Uncertainty Measure in Partition-Based Fuzzy Rough Sets. International Journal of General Systems. 2007 Jan;34(1):77-90.
[13] Chen BB, Fan JL, Lei B, Gao MF. SLIC superpixel granulation-based rough entropy image segmentation algorithm. Transducer and Microsystem Technologies. 2022 Feb;41(2):105-7.
[14] Molodtsov D. Soft set theory—First results. Computers & Mathematics with Applications. 1999 Feb;37(4-5):19-31.
[15] Maji P, Bis R, Roy A R. Fuzzy soft sets. Fuzzy Math. 2001 Jan;9(3):589-602.
[16] Khan A, Zhu YG. A Novel Approach to Parameter Reduction of Fuzzy Soft Set. IEEE Access. 2019 Sep;7:128956-128967.

[17] Feng F, Li CX, Dav BJ, Ali IM. Soft sets combined with fuzzy sets and rough sets: A tentative approach. Soft Computing. 2010 Jul;14(9):899-911.

[18] Ziarko W. Variable Precision Rough Sets Model. Journal of Computer & Systems Sciences. 1993 Feb;46(1):39-59.

[19] Katzberg JD, Ziarko W. Variable Precision Rough Sets with Asymmetric Bounds. Rough Sets, Fuzzy Sets and Knowledge Discovery. 1994 Oct;167-77.

[20] Xing RF, Liu W, Fan YY. Research On Diagnosis of Tumor Based on Variable Precision Rough Set Theory. Science & Technology Vision. 2019 Jan;2019(03):43+7.

[21] Wu SZ, Duan C, She ZY. Image Multi-Threshold Segmentation Algorithm Using Variable Precision Rough Set and Particle Swarm Optimization. Journal of Northwest Normal University (Natural Science). 2019 Apr;55(02):65-72.

[22] Alicja MR, Leszek R. Variable Precision Fuzzy Rough Sets. Lecture Notes in Computer Science. 2004 Dec;3100(1):144-60.

[23] Zhang AQ, Ha MH, Fan YJ. Variable Precision Fuzzy Rough Set Model Based on Fuzzy Covering. Proceedings of the International Conference on Innovative Computing Information & Control; 2008 Jun 18-20; New York, USA: IEEE Press; c2008. p. 399-399.

[24] Gong ZT, Zhang XX. Variable Precision Intuitionistic Fuzzy Rough Sets Model and Its Application. International Journal of Machine Learning & Cybernetics. 2014 Apr;5(2):263-80.

[25] Li LH, Li S, Li Y, Yang YF. Fuzzy sets and rough sets. Beijing: Tsinghua University Press; 2015. 196 p.

[26] Sun BZ, Ma WM. Soft Fuzzy Rough Sets and Its Application in Decision Making. Artificial Intelligence Review: An International Science and Engineering Journal. 2014 Jan;41(1):67-80.