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Robust Estimation Method Using Successive Approximation Algorithm to Correct Errors

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Abstract. In measurement practice, the residuals in least squares adjustment usually show various abnormal discrete distributions, including outliers, which is not conducive to the optimization of final measured values. In this paper, according to the physical mechanism of deviation, dispersion and outlier of repeated observations, it can be seen that abnormal distribution and outlier are normal measurement phenomena, and weakening the influence of outlier is an incorrect research direction. Then, by revealing the advantages of functional model processing, this paper puts forward the error correction idea of using the approximate function model to approach the actual function model step by step, and forms a new theoretical method to optimize the final measured values, which greatly improves the quality of measured values. This is a new measurement theory idea that is completely different from mainstream robust estimation research.

Keywords. Robust estimation, Least square method, Gross error, Outlier, Function model.

1. Introduction

In measurement practice, the residuals obtained by the least squares adjustment usually show various abnormal discrete distributions, including outliers, so that the adjustment result is not an optimal result. In order to solve these problems, academia has carried out a lot of research and formed many robust estimation methods, and there are tens of thousands of relevant documents [1-4]. However, based on the error classification theory, these mainstream studies believe that outlier comes from wrong measurements and should weaken its influence, and no one cares about the real physical mechanism behind the phenomena of deviation, dispersion and outlier. Moreover, only considering precision but ignoring trueness, mainstream research inevitably sacrifices trueness, so as to obtain a false high accuracy and an unnecessary cumbersome mathematical process.

Different from the mainstream research aimed at weakening the influence of outliers, and starting from the error classless philosophy[5-13], this paper will analyze the actual physical mechanism that errors cause the phenomenon of deviation, dispersion and outlier, prove that outlier (abnormal distribution) is a normal measurement phenomenon and its influence should not be weakened, give a successive approximation algorithm using error's function model to realize error correction, and realize the

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optimization of final measured values. Moreover, its mathematical process is simple and logical.

That is, the mainstream research of robust estimation is to weaken the influence of outliers, but this paper is to effectively use abnormal distribution and outliers. They are two completely opposite research directions.

Concerning the research method, this paper will use true values plus errors to simulate repeated observations, and verify the effectiveness of the algorithm by the ability that the final measured values respond to the true values. We will see that weakening the influence of outliers or eliminating outliers usually sacrifices trueness. Besides, we will also see the theoretical rigor and practical effectiveness of the successive approximation algorithm.

Next, through the regularity and randomness of errors, the author reveals the physical mechanism of deviation, dispersion and outlier of observation errors, and then derives the robust estimation algorithm that uses the function model to approximate successively.

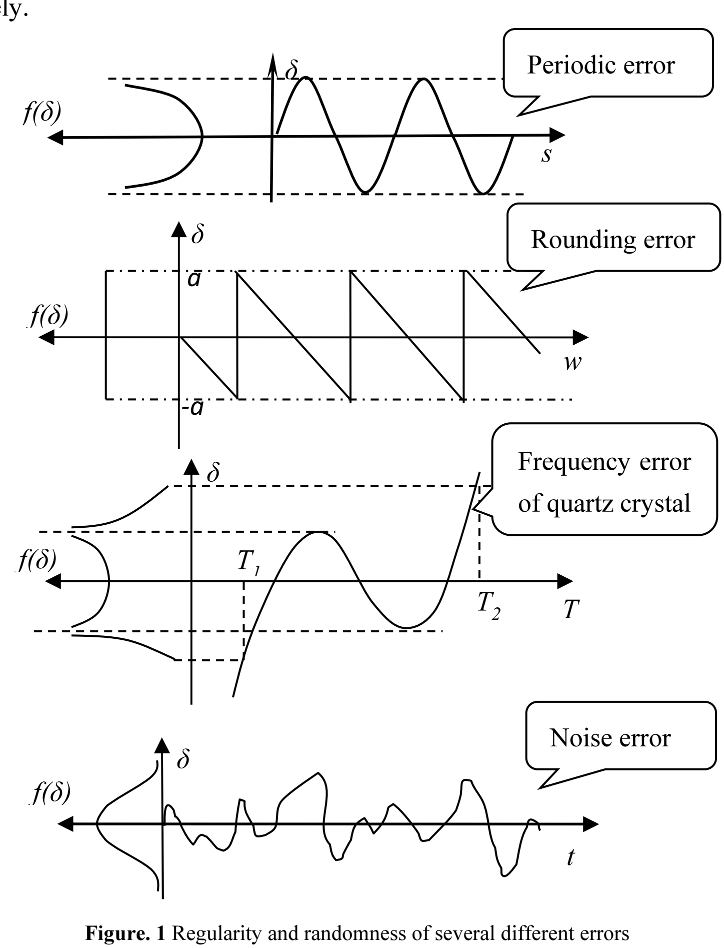

Figure. 1 Regularity and randomness of several different errors

2. Regularity and randomness of error

Let's review a new error epistemology from the regularity and randomness of errors.

The regularity of error means that there is a functional relationship between error and some measurement conditions. For example, the periodic error of geodimeter is a sinusoidal function of distance condition, the AC interference error in voltage measurement is a sinusoidal function of time condition, the frequency error of quartz crystal is a function of temperature condition, the rounding error is a sawtooth law function of true value, the electronic noise error is a random function of time condition, and so on.

The randomness of error means that all possible values of error form a random distribution, or the error exists in a limited probability interval. For example, periodic error and AC interference error follow U-shaped distribution, rounding error follows rectangular distribution, quartz crystal frequency error follows M-shaped distribution, electronic noise error follows normal distribution, etc., as shown in Figure 1.

That is, the regularity and randomness of errors are the results of observing errors from different perspectives, and errors are the unity of regularity and randomness. In other words, having both regularity and randomness, the error cannot be classified by regularity and randomness, the traditional classical error classification theory is a philosophical mistake, and we need to study the error processing method with an error classless epistemology [5-13].

3. Deviation, dispersion and outlier of observation error in repeated measurements

Now let's see the physical mechanism behind the phenomena of deviation, dispersion and outlier.

In measurement practice, repeated measurement conditions are in a changing state. For example, in leveling network survey, the instrument erection conditions (leveling, height, direction, temperature, etc.) of each route are different from each other; in traverse network survey, the instrument erection conditions and distance conditions observed by each traverse are also different from each other; in GNSS network survey, the positions of satellites in each observation period are different from each other, and the signal propagation conditions of each survey station are different from each other.

When the measurement conditions associated with the regular error change, it will inevitably drive the error to change, which is the physical mechanism of the dispersion caused by the regular error. For example, when the distance condition changes in repeated measurement, the periodic error will lead to the dispersion of observation error sequence; When the time condition changes in repeated measurement, the AC interference error will lead to the dispersion of observation error sequence; When the temperature condition changes in repeated measurement, the frequency error of quartz crystal will lead to the dispersion of observation error sequence; When the range conditions change in repeated measurement, the rounding error will lead to the dispersion of observation error sequence; When the time condition changes in repeated measurement, the noise error will lead to the dispersion of observation error sequence; and so on.

On the contrary, when the measurement conditions associated with the regular error remain unchanged in the repeated measurement, the error will remain constant in the repeated measurement, resulting in the overall deviation of the observation error sequence.

However, the actual measurement conditions are usually neither even change nor absolutely unchanged, but an uneven change, which will inevitably drive the corresponding regular errors to produce uneven changes. This is the physical mechanism of abnormal distribution or even outlier of errors in measurement practice. For example, the serious imbalance of phase condition change of periodic error and AC interference error will cause them to form outlier distribution, the serious imbalance of discarding four and leaving five will cause outlier distribution of rounding error, the serious imbalance of temperature condition change will cause outlier distribution of quartz crystal frequency error, and so on.

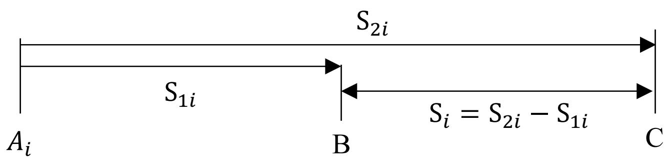

Figure .2 Measuring distance by difference method

For example, using difference method to measure distance, as shown in Figure.2, table 1 simulates a repeated differential observation data with a true value of $S_{A_iC} - S_{A_iB} = 8m$ by using the periodic error $\delta_i = 5\sin(\frac{d_i}{20} \times 2\pi + \frac{\pi}{4})\ (mm)$ of a geodimeter. It can be seen that each observation value Si is different from each other, but the mean value is 8.0014m, indicating that both dispersion and deviation coexist, and its distribution is also uneven.

Table 1 Simulation observation values of periodic error of geodimeter

i	S_{A_iB}	S_{A_iC}	$S_{2i} = S_{A_iB} + \delta_i$	$S_{1i} = S_{A_iC} + \delta_i$	$S_i = S_{1i} - S_{2i}$
1	10	18	9.9965	18.0008	8.0043
2	12	20	11.9951	20.0035	8.0084
3	33	41	32.9951	41.0045	8.0094
4	27	35	27.0008	34.9965	7.9957
5	22	30	22.0049	29.9965	7.9916
6	28	36	27.9992	35.9977	7.9985
7	30	38	29.9965	38.0008	8.0043
8	36	44	35.9977	44.0045	8.0068
9	38	46	38.0008	46.0023	8.0015
10	26	34	26.0023	33.9955	7.9932
11	34	42	33.9955	42.0049	8.0094
12	16	24	15.9977	24.0045	8.0068
13	18	26	18.0008	26.0023	8.0015
14	19	27	19.0023	27.0008	7.9985
15	42	50	42.0049	49.9965	7.9916
$S_{A_iC} - S_{A_iB} = 8m$					

Moreover, when the number of samples is small, the random superposition of several errors can also appear outlier phenomenon.

4. Error processing with function model and random model

The dispersion and outliers of repeated observation errors come from regular errors, or even the superposition of a variety of different regular errors. Therefore, the observation error sequence can be regarded as both regular errors and randomly distributed errors. Naturally, both using the error function model to correct the error and incorporating the error into the random model to realize the self-compensation of the error are effective schemes to realize the adjustment.

Example 1: use the observation data in Table 1 to find the best measured value with random model and functional model respectively.

Assuming that the unknown true value is Y, the error equation treated according to the random model is:

$$V_i = S_{1i} - S_{2i} - Y \tag{4-1}$$

According to the least square method, the best measured value is:

$$y = \frac{\sum_{i=1}^{n}(S_{1i} - S_{2i})}{n} = 8.0014(m) \tag{4-2}$$

It can be seen that the final error is 1.4mm, which is much smaller than the 5mm amplitude of periodic error.

The function model of periodic error is:

$$\delta_i = A\sin(\frac{d_i}{20}\times 2\pi + \phi) = P\sin\frac{d_i}{20}\times 2\pi + Q\cos\frac{d_i}{20}\times 2\pi \tag{4-3}$$

The error equation treated according to the functional model is:

$$\begin{aligned} V_i &= S_{1i} - \delta_{1i} - (S_{2i} - \delta_{2i}) - Y \\ &= S_{1i} - S_{2i} - P(\sin\varphi_{1i} - \sin\varphi_{2i}) - Q(\cos\varphi_{1i} - \cos\varphi_{2i}) - Y \\ &= S_{1i} - S_{2i} - P(\sin\frac{S_{1i}}{20}\times 2\pi - \sin\frac{S_{2i}}{20}\times 2\pi) \\ &\quad - Q(\cos\frac{S_{1i}}{20}\times 2\pi - \cos\frac{S_{2i}}{20}\times 2\pi) - Y \end{aligned} \tag{4-4}$$

Making $S_i = S_{1i} - S_{2i}$, $A_i = \sin\frac{S_{1i}}{20}\times 2\pi - \sin\frac{S_{2i}}{20}\times 2\pi$, and $B_i = \cos\frac{S_{1i}}{20}\times 2\pi - \cos\frac{S_{2i}}{20}\times 2\pi$, the error equation (4-4) becomes:

$$V_i = S_i - A_iP - B_iQ - Y \tag{4-5}$$

According to the least square method, the normal equations are:

$$\begin{pmatrix} n & \sum A_i & \sum B_i \\ \sum A_i & \sum A_i^2 & \sum A_iB_i \\ \sum B_i & \sum A_iB_i & \sum B_i^2 \end{pmatrix}\begin{pmatrix} y \\ p \\ q \end{pmatrix} = \begin{pmatrix} \sum S_i \\ \sum A_iS_i \\ \sum B_iS_i \end{pmatrix} \tag{4-6}$$

Substituting the data, the results are:

$$\begin{pmatrix} y \\ p \\ q \end{pmatrix} = \begin{pmatrix} 8.00000 \\ 0.00353 \\ 0.00353 \end{pmatrix} \tag{4-7}$$

In this way, the measured value returns to the true value, and the amplitude is $A = \sqrt{p^2 + q^2} = 0.00499\,(m)$ and phase is $\phi = \arctan\frac{p}{q} = \frac{\pi}{4}$, which also return to the true values completely.

In short, the error can not only be corrected by its function model, but also be incorporated into the random model to realize its self-compensation.

5. The harm of eliminating outliers

However, in actual measurement, there are many and miscellaneous sources of errors, and it is impossible to fully understand the functional law of each error, so most of the functional model processing like example 1 is unrealistic. Moreover, some errors cannot be treated with a strict function model as example 1 to realize error correction. For example, the rounding error in Figure 1 is the sawtooth law of the true value, but the true value is precisely unknown. This is quite different from the periodic error in example 1, the random model seems to be the only way out, but the random model processing has to face the dilemma of unbalanced error distribution. The following is a simulation case to illustrate.

Example 2: The true mass values of three objects A, B and C are 5.1g, 4.2g and 7.2g respectively. Now, the readings of the precision balance are rounded to the gram bit for the combined measurement of the three objects. The original mass observation values must be as shown in Table 2. Now, we calculate the best measured value of each object mass according to the least square method to observe the response of the measured value to the true value.

Table 2. Combined observations simulated by the rounding error of the balance

Measuring method	A	B	C	A+B	A+C	B+C	A+B+C
Observed values (g)	5	4	7	9	12	11	17

Assuming that the masses of the three objects are X_1, X_2 and X_3 respectively, the error equations are:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 7 \\ 9 \\ 12 \\ 11 \\ 17 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{5-1}$$

According to the least square method, the solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5.125 \\ 4.125 \\ 7.125 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} -0.125 \\ -0.125 \\ -0.125 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.625 \end{pmatrix}. \tag{5-2}$$

According to the cognition of mainstream research, v_7 in the equations (5-2) must be judged as a gross error, which is caused by wrong measurement and should be eliminated. After the elimination, the least square method is used again to obtain:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5-3}$$

It can be seen that this is actually counterproductive. The residual looks very comfortable and can give a very high precision evaluation, but the error of the measured value is actually greater than that without elimination! That is, v_7 is actually a normal measurement error, the root cause of outliers is the imbalance of measurement data collection, and the high precision obtained by eliminating outliers is at the expense of trueness.

Example 3: Table 3 shows a set of simulated observation data of three-stage baseline measurement by geodimeter, assuming that points A, B, C and D are on the same straight line, the true values of three distances AB, BC and CD are all 15.0000m, and assuming that the geodimeter has only periodic error $5\sin(\frac{d_i}{20} \times 2\pi + \frac{\pi}{3})\ (mm)$. Let's observe the harm of eliminating outlier by solving the best measured values of line segments AB, BC and CD.

Table 3. Observation values simulated by geodimeter's periodic error

i	Line segments	True values (m)	Error values (mm)	Simulated observations (m)
1	AB	15	-2.5	14.99750
2	BC	15	-2.5	14.99750
3	CD	15	-2.5	14.99750
4	AC	30	-4.3	29.99567
5	BD	30	-4.3	29.99567
6	AD	45	2.5	45.00250

1) Deal with periodic error with random model

Assuming that the true values of the three distances are Y_1, Y_2 and Y_3 respectively, the error equations are:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{pmatrix} = \begin{pmatrix} 14.99750 \\ 14.99750 \\ 14.99750 \\ 29.99567 \\ 29.99567 \\ 45.00250 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \tag{5-4}$$

According to the least square principle, the final measured values are:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 15.00000 \\ 14.99783 \\ 15.00000 \end{pmatrix} \tag{5-5}$$

Therefore

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 14.99750 \\ 14.99750 \\ 14.99750 \\ 29.99567 \\ 29.99567 \\ 45.00250 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 15.00000 \\ 14.99783 \\ 15.00000 \end{pmatrix} = \begin{pmatrix} -2.5 \\ -0.33 \\ -2.5 \\ -2.17 \\ -2.17 \\ 4.67 \end{pmatrix} \tag{5-6}$$

According to the cognition of mainstream research, v_6 in the equation (5-6) is considered as a gross error, which is caused by wrong measurement operation and needs to be eliminated. In this way, the observation error equations become:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix} = \begin{pmatrix} 14.99750 \\ 14.99750 \\ 14.99750 \\ 29.99567 \\ 29.99567 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \tag{5-7}$$

According to the least square method, the final measured values are:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14.99767 \\ 14.99783 \\ 14.99767 \end{pmatrix} \tag{5-8}$$

Substituting the measured values into the observation error equation, the residuals are:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} -0.17 \\ -0.33 \\ -0.17 \\ 0.17 \\ 0.17 \end{pmatrix} \tag{5-9}$$

It can also be seen that after the "gross error" is eliminated, the residuals are indeed much denser, but the measured values y_1 and y_3 deviate more from the true value! This also proves that the so-called elimination of gross errors actually sacrifices trueness.

2) Deal with periodic error with functional model

The functional model of periodic error is:

$$\delta_i = M\sin(\frac{d_i}{20}\times 2\pi+\phi) = P\sin\frac{d_i}{20}\times 2\pi + Q\cos\frac{d_i}{20}\times 2\pi \tag{5-10}$$

Among equation (5-10), $P = M\cos\phi$, $Q = M\sin\phi$.

Making $a_i = \sin\frac{d_i}{20}\times 2\pi$, $b_i = \cos\frac{d_i}{20}\times 2\pi$, the observation error equations are:

$$\begin{pmatrix} V_1\\ V_2\\ V_3\\ V_4\\ V_5\\ V_6 \end{pmatrix} = \begin{pmatrix} d_1\\ d_2\\ d_3\\ d_4\\ d_5\\ d_6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & a_1 & b_1\\ 0 & 1 & 0 & a_2 & b_2\\ 0 & 0 & 1 & a_3 & b_3\\ 1 & 1 & 0 & a_4 & b_4\\ 0 & 1 & 1 & a_5 & b_5\\ 1 & 1 & 1 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} Y_1\\ Y_2\\ Y_3\\ P\\ Q \end{pmatrix} \tag{5-11}$$

Substituting the observed values, there are:

$$\begin{pmatrix} V_1\\ V_2\\ V_3\\ V_4\\ V_5\\ V_6 \end{pmatrix} = \begin{pmatrix} 14.9975\\ 14.9975\\ 14.9975\\ 29.99567\\ 29.99567\\ 45.0025 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & -1.00000 & -0.00079\\ 0 & 1 & 0 & -1.00000 & -0.00079\\ 0 & 0 & 1 & -1.00000 & -0.00079\\ 1 & 1 & 0 & 0.00136 & -1.00000\\ 0 & 1 & 1 & 0.00136 & -1.00000\\ 1 & 1 & 1 & 1.00000 & -0.00079 \end{pmatrix} \begin{pmatrix} Y_1\\ Y_2\\ Y_3\\ P\\ Q \end{pmatrix} \tag{5-12}$$

According to the least square principle, the final measured values are:

$$\begin{pmatrix} y_1\\ y_2\\ y_3\\ p\\ q \end{pmatrix} = \begin{pmatrix} 15.00000\\ 15.00000\\ 15.00000\\ 0.002498\\ 0.004337 \end{pmatrix} \tag{5-13}$$

In this way, the amplitude of the periodic error is $m = \sqrt{p^2+q^2} = 5(mm)$, and the phase is $\phi = \arctan\frac{q}{p} = \frac{\pi}{3}$.

It can be seen that the periodic error is completely corrected, the measured values restore the true values, and the outlier here is also a normal error without any trouble.

6. Successive approximation algorithm

It has been confirmed that outliers should not be eliminated. Then, in order to benefit the final measured value, can we use the approximate function model of error? Of course, the answer is yes.

The periodic error in examples 1 and 3 can be treated by functional model because it is the function of phase, and the phase is the function of measurement serial number i, so its essence is that the error is the function of i. However, the error in example 2 is also a function of i, and the error in any measurement is a function of i, but the problem is only that their functional relationship cannot be expressed as an accurate mathematical model as examples 1 and 3. Although they cannot be expressed as an accurate mathematical model, it also should be effective to use approximate mathematical model.

According to this idea, let's see the distribution of V_i in example 2. After the adjustment, it is found that v_7 is 0.625, which is an outlier. The reason for the outlier is that the rounding error (rectangular distribution) with sawtooth regularity is unevenly sampled. Therefore, once there are two distinct residual groups, we can reasonably believe that there is a regular error component in the residuals, and its contribution to the two groups is the same, but the sign is opposite. Then, we can use the functional model of this error component to improve the observation error equations, so as to further approximate the two groups of residuals, whose principle is similar to Fourier series approximation. In this way, the observation error equations of example 2 are improved to:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 7 \\ 9 \\ 12 \\ 11 \\ 17 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ C_1 \end{pmatrix} \tag{6-1}$$

According to the least square method, there are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ c_1 \end{pmatrix} = \begin{pmatrix} 5.204545 \\ 4.204545 \\ 7.204545 \\ 0.318182 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} 0.11364 \\ 0.11364 \\ 0.11364 \\ -0.09091 \\ -0.09091 \\ -0.09091 \\ 0.06818 \end{pmatrix}. \tag{6-2}$$

It can be seen that the residual is obviously approached, its distribution is improved, and the outlier phenomenon disappears. However, it is obvious that there are still two groups of positive and negative residuals. We can continue this idea and make the approximation algorithm again, so the error equations become:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 7 \\ 9 \\ 12 \\ 11 \\ 17 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ C_1 \\ C_2 \end{pmatrix} \tag{6-3}$$

According to the least square method, the solutions are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5.2 \\ 4.2 \\ 7.2 \\ 0.3 \\ 0.1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6-4}$$

Similarly, this method is also used for example 3. After going through two rounds approximation algorithm, there are:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 14.99963 \\ 14.99963 \\ 14.99963 \\ 0.00287 \\ 0.00073 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6-5}$$

The authenticity of the measured values is obviously better than that treated according to the random model.

Now, this method is also applied to example 1. After four rounds approximation algorithm, the following results are obtained:

$$\begin{pmatrix} y \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 8.000032 \\ 0.005290 \\ 0.002386 \\ 0.001136 \\ 0.000326 \end{pmatrix} \tag{6-6}$$

Compared with the measured value of 8.0014 processed by the random model in the original example, the error of the measured value of 8.000032 is reduced by 45 times. The function model parameter value $c_1 \sim c_4$ shows obvious convergence, which also shows the effectiveness of the method.

In the three cases, the observed values are simulated by the true values plus errors, and the effectiveness of the algorithm is tested by the ability of the final measured value to respond to the true value. In terms of effect, the final measured values of the three cases have been greatly improved compared with the pure random model. Now, let's see the application effect of a practical case.

Example 4: Table 4 is a set of observation data that using six segment baselines method calibrate additive and multiplicative constant error of a geodimeter, and the successive approximation algorithm is used to solve the best estimation of additive and multiplicative constant error.

Table 4. Observation data for calibrating the additive and multiplicative constant error of geodimeter

Instrument model: DTM-310 Serial number: 010086
Test location: Wuhan baseline field for length calibration

i	Baseline values D_i (m)	Observed values D_i' (m)	Error values y_i(mm)
1	264.23900	264.24038	-1.38
2	192.22441	192.22620	-1.79
3	312.32239	312.32532	-2.93
4	408.41467	408.41696	-2.29
5	432.43573	432.43840	-2.67
6	960.97569	960.97867	-2.98
7	456.46341	456.46539	-1.98
8	120.09798	120.09947	-1.49
9	216.19026	216.19136	-1.10
10	240.21132	240.21255	-1.23
11	480.50624	480.50906	-2.82
12	768.75128	768.75395	-2.67
13	576.56139	576.56419	-2.80
14	120.11334	120.11362	-0.28
15	360.40826	360.41033	-2.07
16	648.65330	648.65567	-2.37
17	24.02106	24.02290	-1.84
18	264.31598	264.31837	-2.39
19	552.56102	552.56371	-2.69
20	240.29492	240.29698	-2.06
21	528.53996	528.54266	-2.70

Assuming that the additive and multiplicative constant errors of the geodimeter are K and R respectively, the error equation is:

$$V_i = y_i - K - D_i R \tag{6-7}$$

According to the least square method, the measured values of the two errors are:

$$\begin{pmatrix} k \\ r \end{pmatrix} = \begin{pmatrix} -1.28mm \\ -2.16 \times 10^{-6} \end{pmatrix} \tag{6-8}$$

The standard deviations are:

$$\begin{pmatrix} m_0 \\ m_{\Delta k} \\ m_{\Delta r} \end{pmatrix} = \begin{pmatrix} \pm 0.52mm \\ \pm 0.23mm \\ \pm 5.1 \times 10^{-7} \end{pmatrix} \tag{6-9}$$

Now, according to the approximation algorithm, after four rounds of approximation algorithm, we get:

$$\begin{pmatrix} k \\ r \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1.083mm \\ -2.589 \times 10^{-6} \\ 0.479mm \\ 0.271mm \\ 0.143mm \\ 0.094mm \end{pmatrix} \tag{6-10}$$

The standard deviations are:

$$\begin{pmatrix} m_0 \\ m_{\Delta k} \\ m_{\Delta r} \\ m_{\Delta c_1} \\ m_{\Delta c_2} \\ m_{\Delta c_3} \\ m_{\Delta c_4} \end{pmatrix} = \begin{pmatrix} \pm 0.073mm \\ \pm 0.033mm \\ \pm 7.3 \times 10^{-8} \\ \pm 0.017mm \\ \pm 0.017mm \\ \pm 0.017mm \\ \pm 0.017mm \end{pmatrix} \tag{6-11}$$

It can be seen that the quality of the results is greatly improved because the approximation algorithm corrects the regular errors such as the residual periodic error of the geodimeter, which greatly weakens their impact on the final measured values.

In short, the core of understanding this principle is to abandon the traditional thinking that the residual must be white noise, and recognize that the residual itself has regularity[7]. The theoretical basis of this method is that various regular errors are the root causes of dispersion and outlier, the residuals not only follow random distribution (including abnormal distribution) but also have regularity, and the error can be treated not only according to random model, but also according to function model. Its principle is to use the approximate function model of error to gradually approximate the actual function model of error to realize error correction. Its essence is to fit the error according to the function model $V(i) = C_1 f_1(i) + C_2 f_2(i) + \cdots$, so it can effectively overcome the imbalance of sampling. Of course, the premise of the effectiveness of this method is that the overall characteristics of the error have been sampled and cannot be seriously missing (eliminated).

It should be pointed out that according to this algorithm, when the redundant observations are sufficient, the precision can be much higher than that of random model processing, because when there are enough redundant observations, the law details of the residuals can be displayed and the model fitting can be accurate; when there are few repeated observations, the regularity of the residuals cannot be fully displayed, the approximation times are limited, and the fitting effect will naturally be limited.

7. About the real gross error

Now, another problem is: how to distinguish the real gross errors caused by wrong measurement? The answer is, to take the index of maximum permissible error (MPE) of the measuring instrument (sensor) as the judgment basis. When the standard deviation given by the least square method is consistent with this index, of course, it can be judged that all observed values are normal; on the contrary, due to wrong operation (including instrument failure), the error can reach thousands of times of the nominal tolerance index of the instrument, which is very easy to find in the adjustment and does not need too complex mathematical principles at all.

For example, there was a fault phenomenon of "ten meters" error in the early phase geodimeter, and its error value was an integral multiple of the precision ruler length (the precision ruler length of the early rangefinder was mostly 10 meters, which rarely appeared after improving the instrument design). The physical principle of its formation is that many electric rulers of different lengths are used in the phase geodimeter for measurement, when the measurement error of the long ruler is greater than 1 / 2 of the length of the fine ruler, there will be errors in the connection between the measured values of the fine measurement and the rough measurement. However, this error can reach thousands of times of the nominal precision of the instrument, and can even appear in several observation equations of the traverse network many times. When the standard deviation given by the least square method reaches thousands of times of the nominal limit difference of the instrument, I believe anyone can know that there is a problem with the data. Therefore, this gross error in a traverse network can be easily identified without complex mathematical methods. The processing method is usually to send the instrument for repair, inspection and re measurement. I believe no one dares to use some mathematical method to save this kind of measurement data containing a large number of gross errors.

That is, there is no necessary relationship between outliers and gross errors, and outliers are not necessarily gross errors.

8. Conclusion

Through the physical mechanism of deviation, dispersion and outlier of repeated observations, this paper expounds that the discrete error sample sequence follows both random distribution and regularity, and can realize adjustment not only by self-compensation, but also by function model correction. Therefore, a function model correction algorithm using the approximate function model of error to realize successive approximation is derived, which greatly improves the quality of measured values, and can effectively overcome the influence of outlier error. Effectively using outliers and clearly opposing weakening their influence, this new theoretical idea challenges the mainstream robust estimation research, and completely changes the research direction of measurement theory.

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