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'Knowing-that' vs. 'Knowing-wh'

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Abstract. Though there is a huge amount of the so-called *epistemic logics* that deal with *propositional attitudes*, i.e., sentences of the form "*a* knows *that P*", their 'wh-cousins' of the form "*a* knows *who* is a *P*", "*a* knows *what* the *P* is", "*a* knows *which Ps* are *Qs*", etc., have been, to the best of my knowledge, almost neglected. A similar disproportion can be observed between the analysis of Yes-No questions, which has been under scrutiny of many erotetic logics, and Wh-questions which have been dealt with just by a few authors. To fill this gap, we have analysed Wh-questions in Transparent Intensional Logic (TIL) and adjusted Gentzen's system of natural deduction to TIL natural language processing; thus, our TIL question-answering system can answer not only Yes-No questions but also derive answers to Wh-questions. In this paper, I am going to apply these results to the analysis of sentences containing a 'knowing-wh' constituent. In addition, I will analyse the relation between '*knowing-that*' and '*knowing-wh'*. For instance, if *a* knows that the Mayor of Ostrava is Mr Macura, can we logically derive that *a* knows *who* is the Mayor of Ostrava? Or, vice versa, if *a* knows *who* is the Mayor of Ostrava and the Mayor of Ostrava is Mr Macura, do these assumptions logically entail that *a* knows that the Mayor of Ostrava is Mr Macura? Though in case of rational *human* agents the answers seem to be a no-doubt YES, perhaps a rather surprising answer is in general negative. We have to specify rules for deriving the relation between knowing-that and knowing-wh, and if a software agent is rational but *resource bounded*, it does not have to have in its ontology the rules necessary to derive the answer. The goal of the paper is the specification of these rules. Hence, when applying these results into the design of a multi-agent system composed of software resource-bounded agents, we have to compute their *inferable knowledge*, which accounts not only for their explicit knowledge but also for their inferential abilities.

Keywords. Knowing-wh; Knowing-that; Transparent Intensional Logic; TIL; Natural Deduction; Inferential rules; Inferential abilities of agents; multi-agent system

1. Introduction

There are many epistemic and doxastic logics which deal with the so-called *propositional attitudes*, i.e., sentences of the form "*a knows that P*" or "*a believes that P*", respectively. These logics are mostly variants of intensional modal logics, whose language contains one or more knowledge operators and whose semantics is given in terms of Kripke models that contain epistemically possible worlds mutually related by accessibility relation. In these logics, '*knowing that*' represented by the operator *K* corresponds to necessity (represented by box operator) and '*believing that*' to possibility (represented by diamond). This modal approach to epistemic logic has been widely applied in philosophy, computer science and artificial intelligence. The sub-category ‘Doxastic Logic’ also includes formal work on belief revision.[1]

These logics provide a handy notation, axioms and rules to deal with knowing/believing that, and a proof system syntactically characterized as a normal modal logic somewhere between K4 and K5 subjective to different opinions about the so-called introspection axioms. Yet, there are many shortcomings as well.

First, there are many different kinds of these logics depending on the chosen axioms and rules. For instance, in the K4 system, positive introspection is presupposed (what is known is known to be known) and in K5 negative introspection as well (it is known what is not known). K5, in particular, is a subject of a lot of criticism. Indeed, how can one know what they don’t know? Hence, to apply some of these logics in a multi-agent system, one must decide which of them would be plausible depending on agents’ abilities, which is often hardly possible, or to design a system in which the agents would switch among particular logics, which is also hardly applicable.

Second, all the intensional modal approaches suffer the problem of logical/mathematical omniscience. It means that once an agent *a* knows that *P* they should also know all the logical consequences of *P* (or, in a restricted Montague-Scott version, all the propositions equivalent to *P*).

Third, intensional, modal epistemic logics do not deal with the limitations of resource-bounded agents accordingly, as they deal with two extreme kinds of knowledge, namely *implicit* and *explicit knowledge. Implicit knowledge* is closed under entailment or under equivalence; it is ascribed to an agent *a* from the outside so that *a* does not have to be *aware* of its knowledge. In particular, *a* does not have to *actively* behave according to it. *Explicit knowledge* is knowledge that *a* is aware of and is able to use.[2]

Logical omniscience is innocuous, as long as we are modelling only implicit knowledge, since the *K*-axiom of rationality simply traces all the logical consequences of a given stock of knowledge that obtain whether the agent is aware of them or not. The axiom does not entail that the agent should know *explicitly* what he implicitly knows. Yet, in order to model knowledge of active, rational but resource-bounded agents, we need the kind of knowledge the agents are aware of and are able to apply in their behaviour. However, it would not be possible for resource-bounded agents to adhere to explicit-knowledge as the principle guiding their policy of drawing inferences because in such a case the agents are actually deprived of any inference abilities. They know only those pieces of knowledge that are explicitly stored in their knowledge base. Nor would it be possible and pragmatically rational for them to (attempt to) adhere to the other extreme of implicit knowledge, for they would have to infer each and every conclusion that would follow from their supply of explicit knowledge. They (we) would be inundated with irrelevant and useless knowledge taxing their (our) resources.

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[1] A first-class summary on epistemic logics can be found in [29]. In recent decades, a great deal of interdisciplinary attention has been paid to epistemic logics by economists and computer scientists who actively develope the field together with logicians and philosophers. The fertile interplay between computer science and epistemic logic has been introduced in [12] and [25].

[2] See Rescher [30], p. 478. Rescher dubs implicit knowledge ‘accessible knowledge’ while his term for explicit knowledge is ‘occurrent knowledge’.

To deal with the omniscience problem, *hyperintensional* epistemic logics are being recently developed.[3] In principle, there are two kinds of such systems. Either hyperintensions are primitive constructs and their behaviour is specified syntactically by axioms and rules, or hyperintensions are designed as structured entities, mostly of procedural character. If the latter, then particular rules controlling the operations with hyperintensions are dictated by their semantics. In our background theory of Transparent Intensional Logic (TIL), we vote for the latter approach. In sentences of the form "*a* knows that *P*" the complement *P* is conceived as the *procedure* producing a PWS-proposition rather than the proposition itself. To model knowledge compatible with resource-bounded agents, we apply the notion of *inferable knowledge*, which is the golden middle way in between the two extremes, namely implicit and explicit knowledge.[4] The basic idea is to calculate the stock of *inferable* knowledge of a given agent, in the following manner. Given an agent *a*, a possible world *w* and an instant of time *t*, the inferable knowledge of *a* at *w, t* functionally depends on *a*'s stock of explicit knowledge in world *w* and time *t* together with a set of inference rules that *a* masters at *w, t*.[5]

While there is a lot of research done in *epistemic logics* into the analysis of *propositional attitudes*, i.e., sentences of the form "*a* knows *that P*", their 'wh-cousins' of the form "*a* knows *who* is a *P*", or "*a* knows *what* the *P* is", "*a* knows *which Ps* are *Qs*", etc., have been, to the best of my knowledge, almost neglected. Says Rendsvig and Saymons [29]:

> While epistemic logicians had traditionally focused on *knowing that,* one finds a range of other uses of knowledge in natural language. As Wang [34] points out, the expressions *knowing how*, *knowing what*, *knowing why* are very common, appearing almost just as frequently (sometimes more frequently) in spoken and written language as *knowing that*. Recently, non-standard epistemic logics of such expressions have been developed, though *knowing who* constructions are present in Hintikka's *Knowledge and Belief* [20].
>
> (ibid., p. 5)

Perhaps the only exception to the current ignorance on *knowing-wh* is the work of Yanjing Wang ([34], [35]) who has developed a formal syntactic theory on this topic. The author does not aim at specifying the meaning of 'knowing-wh'. Rather, he takes *knowing-wh* as a primitive atomic entity and introduces a new modal operator *Kv*. Hence, instead of breaking it down by allowing quantifiers, equalities and other logical constants to occur freely in the language, *Kv* is simply introduced as a new modality. For example, "knowing what a cat is" is rendered by '*Kv cat*' instead of Hintikka's $\exists xK(cat{=}x)$. Wang characterizes this move as being promising, since by restricting the language we may avoid some philosophical issues of first-order modal logic, retain the decidability of the logic, and focus on special logical properties of each particular knowing-wh construct at a high abstraction level.

Yet, much earlier, Hintikka addressed these issues; in the early days of epistemic logic, Hintikka ([20]) elaborated theories of knowing-wh and its relation to questions in terms of the first-order modal logic. For example, "knowing who John is" is formalised as $\exists xK(John{=}x)$, where *K* stands for 'knowing that'. However, partly because Quine's

[3] For an introduction and a brief summary of hyperintensional approaches, see [22].

[4] See [11].

[5] For details, see [11].

objections against philosophical issues in the foundations of first-order modal logic, the development of epistemic logics beyond *knowing that* was strangled.

Perhaps, there are also other reasons for not pursuing the research on *knowing-wh.* First, while there is just one type of the complement of knowing that, possible types of the complements of knowing-wh are much more diverse. Second, it is not clear how to *define* knowing-wh. Sentences with this constituent become meaningless if the verb 'know' is replaced by 'believe', like for instance, "John believes how to play tennis". This fact undermines philosophers' usual conception of knowledge as the 'justified belief'.

Linguistically, this phenomenon occurs frequently when dealing with factive verbs like knowing or regretting. Linguists try to characterize such phenomena from a more general perspective in terms of classifications of verbs, and thus answer the question "which verbs can take an embedded wh-question?" For example, *forget, see, remember* are like *know* in this sense. However, it is a striking cross-linguistic fact that the verb *believe* cannot take any of those embedded questions, in contrast with philosophers' usual conception of knowledge in terms of strengthened justified true belief. Linguists have been trying to give explanations in terms of factivity and other properties of verbs with interesting exceptions. The linguists are also interested in the semantic variability of the same knowing-wh construct in different contexts. Wang ([35]) gives this example. "I know which card is the winning card" can mean I know Ace is the winning card for the game, or I know the card that my opponent holds is the winning card." For philosophers, especially epistemologists, it is crucial to ask whether those knowing-wh statements are also talking about different kinds of knowledge. For example, it has been a frequently debated topic whether knowledge-how can be reduced to knowledge-that. Moreover, knowing why is extremely important for philosophers of science. However, what amounts to know why? Many philosophers think knowing a *scientific explanation* is the key to answering why-questions, and there is a large body of research on it (cf. e.g., (van Fraassen, [13])). Wang [34] characterizes knowing-wh as 'knowing an answer to the corresponding question'. Which brings forward to our attention logics of questions and answers, i.e., erotetic logics. Here a similar disproportion between the attention paid to Yes-No questions and Wh-questions can be observed. While there are many erotetic logics dealing with Yes-No questions,[6] just a few works deal with Wh-questions.[7]

To fill this gap, we have developed a theory of Wh-questions and their analysis in TIL, together with the technique of their answering.[8] Duží & Fait [8] introduce Gentzen's system of natural deduction adjusted for TIL and natural-language processing. The system derives logical consequences of information recorded in the enormous amount of input text data. Thus, the system not only answers the questions by providing explicitly recorded knowledge sought by keywords. It also answers questions in an 'intelligent way' by computing inferable knowledge such that rational human agents would obtain if only it were not beyond their time and space capacities. The analysis of Wh-questions results into λ-terms with a free variable x ranging over entities of type α, which is the type of a possible direct answer. The system provides answers by suitable substitutions of the α-entities extracted from input sentences, the constituents of which match a given

[6] See, for instance [1], [14], [16], [27], [28], [37]. Comprehensive and extensive exposition on current intensional approaches to the semantics of questions can be found in [38]. The first proposal of the analysis of questions and answers in TIL can be found in [31].

[7] See [36], [15] or [23]. A typical representative of such studies is the work of Groenendijk [14].

[8] For details, see [6], [7], [8] and [2].

λ-term. The proposed more detailed classification of Wh-questions restricts the domain of a plausible answer to a subtype of the type α, which makes it easier for the agents to provide a rigorous answer. In addition, it also makes it possible to derive as an answer even more information by applying the semantic rules rooted in the rich semantics of a natural language. In particular, the agents can make use of the relations of requisites and pre-requisites between intensions, or the rules valid for factive verbs like ‘knowing’, ‘regretting’, and so like. Číhalová & Duží [2] apply this theory to the analysis of dynamic activities of agents. Each activity liking ‘going’, ‘building’, etc., is connected with an agent who does the activity, and participants of different kinds. Hence, agents can ask Wh-questions like ‘who’ is doing this or that activity, ‘when’ and ‘where’ does it take place, for ‘whom’ is an agent building something, ‘what’ is being built, etc. The system also accounts for different *tenses* when this or that activity is being done and frequencies of the activity. If an agent obtains a direct answer to a Wh-question, we can conclude that the agent *knows-Wh* the answer. Thus, a natural step in further research is to go on to scrutinize *knowing-wh*, which I am going to do in this paper.

The rest of the paper is organized as follows. Section 2 is a brief summary of quantified epistemic logics. First, I introduce Hintikka’s approach to *knowing-wh* by means of the existentially quantified *knowing-that*. Second, a brief summary of the new approach by Yanjing Wang is presented. Section 3 introduces the fundamentals of Transparent Intenisonal Logic (TIL), which is my background theory. Section 4 presents the main novelty of the paper, i.e., the analysis of ‘knowing-wh’ as relating an agent to the α-value of the asked PWS-intension in a possible world *w* and time *t* of evaluation, or, in the case of mathematics, as relating the agent to the entity produced by the mathematical procedure. I also propose the rules for inferring consequences of such sentences, and the rules relating ‘knowing-wh’ to the corresponding ‘knowing-that’. Concluding remarks can be found in Section 5.

2. Quantified Epistemic Logic and the Logics of Knowing-wh

2.1. Hintikka on Knowing-wh

Jaakko Hintikka can be truthfully characterized as the founder of epistemic logics due to his pioneering work [20]. Hintikka’s notion of knowledge amounts to the elimination of uncertainty. In his 1962 book, he devoted most of the attention to propositional epistemic logic, though his logic is applicable also to doxastic attitudes such as belief. Hintikka’s language is thus that of propositional logic enriched with the operator K_i that makes it possible to create formulas like $K_i\varphi$ meaning “an agent *i* knows that φ”. The language is interpreted on Kripke’s frames with the relation of accessibility $\rightarrow_i \subseteq S\times S$, where S is a non-empty set of possible worlds. The agent *i* knows that φ in a world s if and only if φ is true in all the worlds accessible for *i* from the world s.[9]

Depending on the properties of the accessibility relation, we obtain different systems of modal epistemic logic. Hintikka specifies this relation as being reflexive and transitive. If it is an equivalence relation, we obtain the strongest system *K5*, in which two rather problematic and much discussed axioms 4 and 5 hold. The axiom 4 is $K_i\varphi \supset$

[9] Hintikka himself prefers the term ‘state’ or ‘situation’ to the term ‘possible world’. In the applications of epistemic modal logics to the field of program verification, it is also more natural to talk about possible states in which a running program can occur.

$K_iK_i\varphi$ (positive introspection) and 5 is $\neg K_i\varphi \supset K_i\neg K_i\varphi$ (negative introspection). These systems notoriously suffer the problematic issue of *logical omniscience* (which is due to the rule of necessitation and the axiom of rationality). Despite those problematic issues and a lot of philosophical dispute concerning the axioms 4 and 5, propositional epistemic logics have been successfully applied in many other fields such as artificial intelligence, program verification, distributed and multi-agent systems. In addition, combining Kripke's possible world semantics with other modalities such as temporal ones gave rise to other useful variants of epistemic logics, like *temporal* epistemic logic or *dynamic* epistemic logic, with many applications in theoretical computer science, game theory, or in modelling changes of knowledge and information in agents' knowledge bases.[10]

However, in ordinary vernacular, propositional epistemic '*knowing that*' is actually less frequent than objectual '*knowing which*', '*knowing who*', '*knowing what*', '*knowing how*', etc., for which I use in this paper the term '*knowing-wh*'. And though epistemic propositional logics of 'knowing that' have been flourishing since Hintikka's time, despite the frequency and importance of their objectual 'knowing-wh' cousins, research into this topic has been almost neglected. One of the exceptions is the work of Hintikka who devoted the last chapter of his (1962) book to the analysis of 'knowing who' in terms of *quantified epistemic logic*.[11] Currently, this topic has been opened and studied by Yanjing Wang in his [35], whose results I am going to summarise below.

According to Hintikka (see [18]), a wh-question like "Who is *b*?" amounts to the request for obtaining a piece of information so that the questioner *a* would know who *b* is. This *wh-knowledge* is the *desideratum* of the corresponding *wh-question*. Hence, the study of wh-questions and knowing-wh are closely related to each other, and this view led Hintikka to the opinion that *knowledge acquisition* is even more important than *knowledge justification* which has been the focus of the traditional epistemology. In [20], Hintikka proposed to formalize "*a* knows who *b* is" by the existentially quantified formula $\exists xK_a(b=x)$ and called it *knowledge of objects*. In contrast, the formula $K_a\exists x(b=x)$ is the analysis of *propositional knowledge* (*knowing that*). Thus, the formalism of knowing-wh is based on knowing that, as one and the same operator K_a is used in both cases, which differ only by the scope of the quantifier. Introducing constants and quantifiers into the language calls for a richer structure of possible worlds, according to Hintikka. He conceives possible worlds as not sharing a universal domain of objects, the universe of discourse, for in Hintikka's theory there are non-existent individuals in some possible worlds while existing in others.

Similarly, "I know who murdered *b*" is here formalized as $\exists xK_IM(x,b)$, which is the desideratum of the question "Who murdered *b*?". However, if one knows that *a* murdered *b*, i.e. $K_IM(a,b)$, it is not in general possible to meet the desideratum $\exists xK_IM(x,b)$, because $K_IM(a,b)$ does not have to yield knowing who, as Hintikka argues in [17]; the questioner should also know who *a* actually is, which is called the *conclusiveness condition*. Indeed, as Wang in [35] shows, answering the question "Who gave the first speech?" by "The first speaker" may not be informative at all. From this point of view, the existential generalization rule does not hold: $K_IM(a,b)$ does not entail $\exists xK_IM(x,b)$.

Hintikka also considered more complicated wh-sentences like "I know whom every young mother should trust" (with the intentional meaning "her own mother"). The formalization should then be of the second-order by quantifying over functions; $\exists fK_I\forall x$

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[10] See [12].

[11] In these paragraphs, I draw on a very nice summary of Hintikka's approach by Yanjing Wang in his [35] paper.

$(M(x) \supset T(x,f(x)))$, where M is standing for young mother, f for the function mother-of somebody, T for trusting. But Hintikka wanted to avoid the problem of working in the second-order logic, which resulted into the introduction (together with Gabriel Sandu) of the so-called *Independence Friendly Logic* (see [19]). The authors introduce the independence sign “/” into the language, which is interpreted as letting some quantifiers occurring within the scope of another quantifier be independent of the latter, as if to jump out of the scope. As a result, we obtain a branching structure of quantifies linearly ordered in the formulas.[12] This weird trick, as odd as it seems to be, has one affirmative effect. It makes it possible to stay in between first and second order logic. It can go beyond the first-order epistemic logic, although quantifying is still first order. The above young mother sentence would now be analysed by the formula $K\forall x(\exists y/K)(M(x) \supset T(x,y))$ in which only variables ranging over individuals are quantified. However, the odds of introducing ‘/’ are against its semantics. It cannot be defined within classical model-theoretic semantics; instead, game-theoretic semantics must be applied.

Recent research in quantified epistemic logic is mostly application driven; we can find applications in cryptography with respect to modelling agents’ decoding abilities, in the theory of games, verification of security properties, etc. These epistemic theories are fragments of first-order logic with knowledge and temporal operators. Applications in the theory of multi-agent systems are mostly propositional epistemic logics, where agent names are rigid designators. There must be a finite set of agents who know the identity of each other. In the second-order applications, it is possible to quantify over propositions and formalise sentence like “the agent *a* knows everything *b* knows.”[13]

2.2. Wang’s logics of Knowing-wh

Wang in [35] presents a new proposal of the logic of knowing-wh. He does not share Hintikka’s opinion that knowing-wh should be modelled by existential quantification of the sentences on knowing-that. Instead, Wang introduces a new modal operator Kv for knowing the value of some c. For instance, instead of formalizing “agent a knows what the value of c is” by $\exists x K_a(c = x)$, Wang introduces a simple formula ‘$Kv_a\, c$’, where c belongs to a set C of constant symbols. Thus, Kv is a primitive modality, and the author aims at providing a complete axiomatic system for it. Says Wang: “Following Hintikka, we take a semantics-driven approach for there is usually not enough syntactic intuition on the possible axioms for such knowing-wh constructions. We can discover interesting axioms by axiomatizing the valid formulas w.r.t. semantics.”

The semantics for $Kv_i\, c$ is given by first-order Kripke models with a constant domain. Intuitively, an agent i knows what the value of c is iff c has the same value over all the i-accessible worlds. Having defined the language and semantics, the author aims at finding a complete axiomatization with meaningful axioms and prove some theorems. He deals with three cases of knowing-wh, namely knowing whether, knowing what and knowing how. Similarly, as in other modal logics, several classes of particular axiom schemata are examined depending on the properties of Kripke’s frame accessibility relation.

There is also a list of positive and negative features of such a formal approach. According to Wang, among the advantages there is a balance between expressivity and complexity, as the neat language with the simple modality Kv actually packages (using

[12] For details, see [4].
[13] For details, see [3].

the author's term) a quantifier, knowing that modality and equality together. Such a weaker language is computationally tractable, and some conceptual problems are avoided (or rather hidden). There are also limitations and difficulties of such a formal approach, as the author admits. First, in the language, one cannot formalise complicated sentences in a fully compositional way. Second, though the shared existential form is hidden, it can be sometimes necessary to quantify over higher-order entities like properties or propositions so that different modalities can 'behave' in a completely different way than specified by the allegedly plausible axioms. Finally, as the author himself admits, "in some cases it is highly non-trivial to give a reasonable semantics since we do not understand enough about the meaning of certain knowing-wh yet."

So much for the current formal approach to dealing with the logics of knowing-wh. In my background theory TIL, I will go the other way around. I am going to first define the *meaning* of knowing-wh in full details in the form of a structured meaning procedure encoded by a given sentence, and only then to formalise semantically-driven rules for behaviour of agents in a multi-agent system.

3. Basic Notions of Transparent Intensional Logic

Pavel Tichý, the founder of Transparent Intensional Logic (TIL) was inspired by Frege's semantic triangle.[14] Though Frege did not define the sense of an expression but only characterised it as the 'mode of presentation' of the denoted entity, Tichý defined the sense of an expression, i.e. its *meaning*, as an abstract, algorithmically structured *procedure* that produces the object denoted by the expression, or in rigorously defined cases fails to produce a denotation if there is none.[15] Indeed, in natural language, there are non-denoting terms that have a perfect meaning, like 'the greatest prime number'. Mathematicians had obviously to understand the sense of the term first and only then could they prove that there is no such number. Hence, In TIL, the meaning of an expression is understood as a context invariant *procedure* encoded by a given expression. By context invariant, we mean this. The procedure encoded by an unambiguous expression is one and the same independently of the context in which the expression is used. Yet, if the expression is ambiguous, logic cannot decide its intended meaning. In such a case, we furnish the ambiguous expression with more than one procedure corresponding to its different meanings.

Formally conceived, TIL is a hyperintensional, typed λ-calculus of partial functions. The λ-terms of the TIL language denote *procedures* (which could be approximated by Church's *functions-in-intension*) that produce set-theoretical mappings (functions-in-extension) or lower-order procedures.[16] Tichý coined these meaning procedures *constructions* and I am going to stick to this term. Qua procedural objects, constructions can be *executed* so as to operate on input objects (of a lower-order type) and produce at most one object of the type they are designed to produce, while non-procedural objects

[14] See Tichý's fundamental book on TIL [32].

[15] A similar philosophy of meaning as a 'generalized algorithm' has been proposed by Moschovakis; it can be found in [26]; this conception has been further developed by Loukanova, see [24].

[16] Church ([39], pp. 2-3) discusses two different notions of a function, namely function-in-intension and in extension. Function-in-extension corresponds to the modern notion of function as a set-theoretical mapping, while function-in-intension is given be the meaning of the *rule* of correspondence between arguments and values of the function. Hence, while functions-in-extension are extensionally individuated, two or more functions-in-intension can share the same set-theoretical mapping.

(i.e. non-constructions like individuals, numbers or set-theoretical mappings) cannot be executed.

Tichý defined six kinds of *constructions*, and I will use five of them, as Single Execution is not needed for the purpose of this paper. There are two kinds of *atomic* constructions that present input objects to be operated on by molecular constructions. They are *Trivialisation* and *Variables.*

Variables produce objects dependently on valuations; they are said to *v-construct*. We follow the objective version of Tarski's conception of variables. To each type of the ramified hierarchy of types, countably many variables are assigned. Each type can be organised into countably many valuation arrays. Valuation v then selects a given array and the n^{th}-variable assigned to range over the type produces the n^{th}-entity of this array.

Trivialisation of an object X presents X without the mediation of any other procedures. Using the terminology of programming languages, the Trivialisation of X, denoted by '${}^0X$', is just a *pointer* or *reference* to X. Such a pointer is needed because no non-procedural object, be it an individual or a function-in-extension, can be a constituent of a procedure. The objects on which a construction operates are beyond the construction. Trivialization can present an object of any type, even another construction C. Hence, if C is a construction, ${}^0C$ is said to *present* the construction C to be operated on as a whole within its super-construction. In such a case, C occurs hyperintensionally, i.e. in the *non-executed* mode. Then all the variables occurring in C occur hyperintensionally as well; they are bound by Trivialization and not amenable to logical operations.

The execution of a Trivialisation or a variable never fails to produce an object. However, since TIL is a logic of *partial* functions, the execution of some of the molecular constructions can fail to present an object of the type they are typed to produce. When this happens, we say that a given construction is *v-improper*. This concerns in particular one of the molecular constructions, namely *Composition*, $[X\ X_1 \ldots X_n]$. It is the very *procedure of applying a function f* produced by X (if any) to the tuple argument $\langle a_1, \ldots, a_n \rangle$ (if any) produced by the constructions $X_1, \ldots, X_n$. A Composition is v-improper as soon as f is a *partial function* not defined at its tuple argument, or if one or more of its constituents $X, X_1, \ldots, X_n$ are v-improper.

Another molecular construction is λ-*Closure*, $[\lambda x_1 \ldots x_n\ X]$. It is the very *procedure of producing a function* with the values v-produced by the construction X, by abstracting over the values of the variables $x_1, \ldots, x_n$ to provide functional arguments. No Closure is v-improper for any valuation v, as a Closure always v-constructs a function, which may be, in an extreme case, a degenerate function undefined at all its arguments. For example, if x ranges over real numbers, the Closure $\lambda x\ [{}^0{:}\ x\ {}^00]$ produces such a degenerate function that has no value at any number.

TIL being a *hyperintensional* system, each construction C can occur not only in execution mode designed to produce an object (if any) but also as an object in its own right on which other (higher-order) constructions operate. The Trivialisation of C causes C to occur just presented as an argument, as mentioned above. Yet sometimes, we need to cancel the effect of Trivialisation and trade the mode of C for execution mode. *Double Execution*, ${}^2C$, does just that; it executes C twice over. If C v-constructs a construction D that in turn v-constructs an entity E, then ${}^2C$ v-constructs E. Otherwise, ${}^2C$ is v-improper.

Hence, the following ^{20}-*Elimination* rule is valid for any construction C;

$$^{20}C = C.$$

With constructions of constructions, constructions of functions, functions, and functional values in TIL stratified ontology, we need to keep track of the traffic between multiple logical strata. The *ramified type hierarchy* does just this task. The type of *order 1* includes all objects that are not constructions. Therefore, it includes not only the standard first-order objects of individuals and truth values but also sets, mappings and also functions defined on possible worlds (i.e., the *intensions* germane to possible-world semantics). Definition is inductive, of course. We start with a base, i.e. a collection of non-empty mutually disjoint sets. For the purposes of natural-language analysis, we are usually assuming the following *base* of ground types:[17]

- o: the set of truth-values {**T, F**};
- ι: the set of individuals (the universe of discourse);[18]
- τ: the set of real numbers (doubling as times);
- ω: the set of logically possible worlds (the logical space).

From these basic types, an infinite hierarchy of collections of partial functions is defined by this inductive rule: where $\alpha, \beta_1, \ldots, \beta_n$ are types, then $(\alpha\ \beta_1 \ldots \beta_n)$ is a *functional type*, i.e. the collection of all partial mappings from the Cartesian product $\beta_1 \times \ldots \times \beta_n$ into α.

The type of *order 2* includes the collection of constructions of order 1, i.e. the type $*_1$, which are constructions of first-order objects, and functions that have such constructions in their domain or range. Hence, $*_1$ and $(\alpha\ \beta_1 \ldots \beta_n)$ where some of the types $\alpha, \beta_1, \ldots, \beta_n$ is identical to $*_1$, are types of order 2.

The type of *order 3* includes the collection of constructions of order 2, i.e. the type $*_2$, which are constructions of first- or second-order objects, and functions that have such constructions in their domain or range. Hence, the atomic type $*_2$ and molecular types $(\alpha\ \beta_1 \ldots \beta_n)$ where some of $\alpha, \beta_1, \ldots, \beta_n$ is identical to $*_2$, are types of order 3.

The type of *order n* includes constructions of objects of order m, where $m < n$, and functions that have such constructions in their domain or range; and so on, ad infinitum.

Empirical expressions denote *empirical conditions*, which may or may not be satisfied at the world/time pair selected as points of evaluation. These empirical conditions are modelled as intensions. Intensions are entities of type $(\beta\omega)$: mappings from possible worlds to an arbitrary type β. The type β is frequently the type of the *chronology* of α-objects. These α-chronologies are, in turn, functions mapping time (of type τ) to the type α. Thus α-intensions are frequently functions of type $((\alpha\tau)\omega)$, abbreviated as '$\alpha_{\tau\omega}$'. Where w ranges over ω and t over τ, the following logical form essentially characterises the logical syntax of empirical language: $\lambda w \lambda t\ [\ldots w \ldots . t \ldots]$. Dealing with the two modal parameters, namely possible worlds and times, separately, is connected with many assets, to name at least analysis of physical entities or nomic laws of necessity.

Analytic expressions denote *extensional entities.* They are of a type α where $\alpha \neq (\beta\omega)$ for any type β. In addition, analytic expressions are also those that denote constant α-intensions.

[17] TIL is an open system, and the choice of base depends on the discourse and subject matter under scrutiny. For instance, for the purpose of mathematics, we might vote for another base consisting of $\{o, \eta\}$, where η is the type of natural numbers, as in mathematics possible worlds and times play no role.

[18] We assume that the universe of discourse ι consists of at least two elements, though we leave aside the cardinality of this basic type.

Examples of frequently used α-intensions are: *propositions* of type $o_{\tau\omega}$, *properties of individuals* of type $(o\iota)_{\tau\omega}$, *binary relations-in-intension between individuals* of type $(o\iota\iota)_{\tau\omega}$, *offices* or *roles* of type $\iota_{\tau\omega}$, propositional *attitudes* of type $(o\iota o_{\tau\omega})_{\tau\omega}$ or $(o\iota *_n)_{\tau\omega}$ depending on whether we model implicit or explicit knowledge, respectively.

Logical objects like *truth-functions* and *quantifiers* are extensional: $\wedge$, $\vee$, $\supset$ are of type (ooo), and $\neg$ of type (oo). The *quantifiers* $\forall^{\alpha}$, $\exists^{\alpha}$ are type-theoretically polymorphic total functions of type (o(oα)), for an arbitrary type α, defined as follows. The *universal quantifier* $\forall^{\alpha}$ is a function that associates a class *A* of α-elements with **T** if *A* contains all elements of the type α, otherwise with **F**. The *existential quantifier* $\exists^{\alpha}$ is a function that associates a class *A* of α-elements with **T** if *A* is a non-empty class, otherwise with **F**.

Notational conventions. Below all type indications will be provided outside the formulae in order not to clutter the notation. Moreover, the outermost brackets of Closures will be omitted whenever no confusion can arise. Furthermore, 'X/α' means that an object *X* is (a member) of type α. '$X \to \alpha$' means that *X* is typed to *v*-construct an object of type α. Throughout, it holds that the variables $w \to \omega$ and $t \to \tau$. If $C \to \alpha_{\tau\omega}$ then the frequently used Composition $[[C\, w]\, t]$, which is the *extensionalization* of the α-intension *v*-constructed by *C*, will be encoded as 'C_{wt}'. When no confusion arises, we use the standard infix notation without Trivialisation for the application of logical objects like truth-functions, equalities and quantifiers; thus, instead of '$[^0\forall\lambda x\, B]$', '$[^0\exists\lambda x\, B]$' we often write '$\forall x\, B$', '$\exists x\, B$' for any $B \to o$ to make quantified formulas easier to read. Arithmetic formulas can also be written in the infix way, without Trivializing relations like $\leq$, $=$, $>$, or functions like $+$ or $\times$; for instance, instead of $[^0\!\leq x\ {}^0 50]$ we can write '$[x \leq {}^0 50]$', or instead of $[^0\neg[^0\exists\lambda x\, [^0\!= [^0\!+ x\ {}^0 1]\ x]]]$ we can write '$\neg\exists x\, [[x + {}^0 1] = x]$'.

4. TIL Analysis of Knowing-wh

4.1. Answering Wh-questions

Duží & Fait introduce in [8] the method of deducing answers to Wh-questions by applying an adjusted version of Gentzen's natural deduction system. The adjustments concern in particular the integration of the special rules rooted in the rich semantics of natural language into the process of deriving consequences by means of the standard E/I-rules of natural deduction. These semantic rules are, inter alia, the rules for dealing with property modifiers, factive verbs, presuppositions, etc. Then in [2], Číhalová and Duží applied these results to the analysis of dynamic activities of agents and answering questions on the participants of activities.

Empirical wh-questions denote α-intensions the value of which in a world *w* and time *t* of evaluation the inquirer would like to know. The type α of the value comes in many different variants, and it is determined by the type of a possible direct answer. For instance, a possible direct answer to the question "Who is the mayor of Ostrava?" is Mr. Macura, which is an individual of type ι. Hence, the question denotes an individual office, and the inquirer wants to know who is the holder of the office in a given world and time. The analysis of the question comes down to this construction.

$$\lambda w\lambda t\, [^0\mathrm{I}\, \lambda who\, [who = [^0 Mayor\text{-}of_{wt}\ {}^0 Ostrava]]] \to \iota_{\tau\omega}$$

Types. $who \rightarrow \iota$; $Mayor\text{-}of/(\iota\iota)_{\tau\omega}$; $Ostrava/\iota$; $\mathrm{I}/(\iota(o\iota))$: the singularizer, i.e. the function that assigns to a singleton its only member, otherwise undefined.

For another simple example, consider the question "Which female Czechs are among the first 20 in WTA ranking singles?". Possible answer conveys the set of individuals, currently (written January 24th, 2022) they are Krejčíková, Plíšková and Kvitová. Hence, the question denotes a property of individuals, and the encoded construction is this.

$$\lambda w\lambda t\ [\lambda x\ [[[{}^0Female\ {}^0Czech]_{wt}\ x] \wedge [[{}^0WTA\text{-}ranking\ x] \leq {}^020]]] \rightarrow (o\iota)_{\tau\omega}$$

Types: $Female/((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$: the intersective property modifier that assigns to a property another modified property;[19] $Czech/(o\iota)_{\tau\omega}$; $x \rightarrow \iota$; $WTA\text{-}ranking/(\tau\iota)_{\tau\omega}$: the attribute that associates an individual with a number.

In [8], Duží & Fait introduce a useful logical technique of answering Wh-questions. The answers are obtained by suitable substitutions, i.e. by the method of unification known from the general resolution method. For a simple example, assume that in the agent's knowledge base, there are these formalised sentences.

$\lambda w\lambda t\ [[{}^0WTA\text{-}ranking_{wt}\ {}^0Barty] = {}^01]$
$\lambda w\lambda t\ [[{}^0WTA\text{-}ranking_{wt}\ {}^0Sabalenka] = {}^02]$
$\lambda w\lambda t\ [[{}^0WTA\text{-}ranking_{wt}\ {}^0Krej\check{c}\acute{\imath}kov\acute{a}] = {}^03]$
$\lambda w\lambda t\ [[{}^0WTA\text{-}ranking_{wt}\ {}^0Pl\acute{\imath}\check{s}kov\acute{a}] = {}^04]$
$\lambda w\lambda t\ [[{}^0WTA\text{-}ranking_{wt}\ {}^0Muguruza] = {}^05]$
…
$\lambda w\lambda t\ [[{}^0WTA\text{-}ranking_{wt}\ {}^0Kvitov\acute{a}] = {}^018]$
…
$\lambda w\lambda t\ [{}^0Australian_{wt}\ {}^0Barty]$
$\lambda w\lambda t\ [{}^0Belarusian_{wt}\ {}^0Sabalenka]$
$\lambda w\lambda t\ [{}^0Czech_{wt}\ {}^0Krej\check{c}\acute{\imath}kov\acute{a}]$
$\lambda w\lambda t\ [{}^0Czech_{wt}\ {}^0Pl\acute{\imath}\check{s}kov\acute{a}]$
$\lambda w\lambda t\ [{}^0Spanish_{wt}\ {}^0Muguruza]$
…
$\lambda w\lambda t\ [{}^0Czech_{wt}\ {}^0Kvitov\acute{a}]$
…

In addition, we need the rule that WTA ranking is applicable only to women, hence female players:

(R) $$\forall x\ [\exists n\ [[{}^0WTA\text{-}ranking_{wt}\ x] = n] \supset [{}^0Female_{wt}\ x]]$$

The answer to the above question "Which female Czechs are among the first 20 in WTA ranking singles?" is derived like this. First, we apply the rule of λ-elimination to the facts in our mini-knowledge base, obtaining thus constructions of truth values.[20] Then

[19] For details on property modifiers, see [21] or [5]. The authors introduce the rules of left and right subsectivity; the former is valid for all kinds of modifiers (for instance, a skillful surgeon is skillful, as a surgeon) while the latter is valid for intersective and subsective modifiers (e.g., the skillful surgeon is a surgeon (subsectivity), or a round peg is round and a peg, as round is an intersective modifier).

[20] When applying a proof or inferences in TIL, the first steps eliminate the left-most $\lambda w\lambda t$, which corresponds to two β-conversions. They apply the empirical propositions to the world w and time t of evaluation to obtain a truth-value. Similarly, Wh-question transforms into a construction producing an object of type α. For details, see [8].

we decompose the question into its simple constituents. Finally, the algorithm searches those sentences in the input knowledge base the constituents of which match the constituents of the question. Matching is realised by plausible substitutions, which can be compared to the unification of terms applied in the general resolution method.

Question (raised in a given *w* and *t*):

(Q) $[\lambda x\ [[{}^0Female\ {}^0Czech]_{wt}\ x] \wedge [[{}^0WTA\text{-}ranking_{wt}\ x] \leq {}^020]]]$

Decomposition of the question:

(1) $[[[{}^0Female\ {}^0Czech]_{wt}\ x] \wedge [[{}^0WTA\text{-}ranking_{wt}\ x] \leq {}^020]]$ Q, λ-E
(2) $[[{}^0Female\ {}^0Czech]_{wt}\ x]$!, ∧-E
(3) $[[{}^0WTA\text{-}ranking_{wt}\ x] \leq {}^020]$ 1, ∧-E
(4) $[[{}^0Female_{wt}\ x] \wedge [{}^0Czech_{wt}\ x]]$ 2, left and right subsectivity
(5) $[{}^0Female_{wt}\ x]$ 4, ∧-E
(6) $[{}^0Czech_{wt}\ x]$ 4, ∧-E

Gloss. Since the property modifier *Female* is intersective, Step (4) is applicable.[21]

To answer the question, the algorithm searches a given knowledge base for those sentences the constituents of which match with the derived atomic parts of the question. In addition, basic algebraic operations can be applied. Thus, the first candidate is $[[{}^0WTA\text{-}Ranking_{wt}\ {}^0Barty] = {}^01]$, as $1 \leq 20$. By substituting ${}^0Barty$ for the variable *x* and applying ∀E and ∃E to the rule (R), we have the result $[{}^0Female_{wt}\ {}^0Barty]$ that matches with (5). Another sub-goal to be met is $[{}^0Czech_{wt}\ {}^0Barty]$. Since this goal cannot be met, the algorithm searches for another candidate. The first substitution meeting (3), (5) and (6) is $x = {}^0$*Krejčíková.* Since the question concerns the *set* of individuals, the algorithm searches for another matching sentences, which corresponds to answering the question "Who else"? In the exactly the same way, the answers $x = {}^0$*Plíšková* and $x = {}^0$*Kvitová* are derived. Hence, the exhaustive answer to the question (Q) is the set $\{{}^0$*Krejčíková*, ^{0}*Plíšková*, ^{0}*Kvitová*$\}$.

4.2. Knowing-wh as knowing the answer to a wh-question

I do not analyse knowing-wh and the corresponding wh-question in Hintikka's way within quantified epistemic logic as an existentially quantified formula. Existence of the known object is the consequence, or rather the *presupposition*, of knowing-wh. Moreover, in TIL, once the base of the infinite hierarchy of types has been voted for, it is fixed within the theory. Hence, the universe of discourse is also fixed. We do not have any possibilia or impossibilia as non-existent individuals. Individuals trivially all exist, and *non-trivial existence* is a property of *functions* that a given function has a value at a given argument. For instance, to claim that tangent of π/2 does not exist does not amount to talking about any non-existent number; (which one would it be?). Rather, non-existence is a property of the *function* tangent that it does not have a value at the number π/2. Similarly, the sentence "The King of France does not exist", does not mention any

[21] More on property modifiers, see [5].

non-existent individual that would exist in another possible world. Rather, it conveys information about the office of the French King that currently it goes vacant.

Hence, my analysis of Wh-questions deviates from Hintikka’s one. On the other hand, I agree with Wang [35] that there is a constant domain of individuals in all the possible worlds, and that *knowing-wh* relates an agent to the *value* of an intension or in mathematical cases the value produced by a given construction. Yet, as mentioned above, I am not going to apply formal axiomatic approach without specifying the meaning of ‘knowing-wh’ first.

Assume that the answer to a Wh-question has been derived and the questioner knows the answer. What else can be inferred in such a situation? First, we have to analyse what it amounts for an agent to know the answer. For instance, if John obtains the answer to the question “Who is the Mayor of Ostrava?”, then John knows who the Mayor of Ostrava is. How to analyse this knowledge? There are two possibilities, namely implicit knowledge and explicit knowledge. Let us first deal with the former.

By intensionally knowing-wh, John is related to the Mayor’s office itself. Hence, we have

$$\lambda w\lambda t\ [{}^0\mathit{Know\text{-}wh}_{wt}\ {}^0\mathit{John}\ \lambda w\lambda t\ [{}^0\mathrm{I}\ \lambda \mathit{who}\ [\mathit{who} = [{}^0\mathit{Mayor\text{-}of}_{wt}\ {}^0\mathit{Ostrava}]]]]$$

Types. $\mathit{Know\text{-}wh}/(o\iota\iota_{\tau\omega})_{\tau\omega}$; $\mathit{John}, \mathit{Ostrava}/\iota$; $\mathit{Mayor\text{-}of}/(\iota\iota)_{\tau\omega}$: an empirical attribute.

Yet, we would like to *specify* the relation of *Knowing-wh* in more detail, to refine this concept.

First, there is a presupposition that the Mayor of Ostrava exists. If it were not so, then the answer to the question “Who is the Mayor of Ostrava?” would be ‘nobody’, which actually is the negated presupposition.[22] Second, if John knows who is the Mayer of Ostrava, he must have identified particular individual as the value of the Mayor’s office.

Hence, we can explicate the relation *Knowing-wh* as *knowing the value* of the intension asked for, and define this relation as follows. Let $K^v \to (o\iota\iota_{\tau\omega})_{\tau\omega}$ be the relation of knowing the value of an office $R \to \iota_{\tau\omega}$, $a \to \iota$ an agent who knows the value, $x \to \iota$, and $\mathit{Ident(ified)}/(o\iota\iota\iota_{\tau\omega})_{\tau\omega}$ the relation between the agent, an individual and the office such that a has identified that individual as the value of the office R. Then in any world w and time t of evaluation, the equivalence of the following definition holds:

Def 1; *knowing the value of an office*

$$[K^v_{wt}\ a\ R] =_{\mathrm{df}} \exists x\ [[x = R_{wt}] \wedge [{}^0\mathit{Ident}_{wt}\ a\ x\ R]]$$

Using Def 1, we can specify the first rule for knowing the value.

$$\text{(R1)}\qquad \frac{[K^v_{wt}\ a\ R]}{\exists x\ [x = R_{wt}]}$$

Using the definition of the property *Exist* (of an office of being occupied), i.e., $[{}^0\mathit{Exist}_{wt}\ R] =_{\mathrm{df}} \exists x\ [x = R_{wt}]$, $x \to \iota$, this rule can be reformulated as follows:

[22] Duží and Číhalová in [6] deal with presuppositions of questions. The main idea is this. If the presupposition of a question is not true, then there is no direct answer. Instead, a plausible answer is a complete one, to wit, negated presupposition.

(R2)
$$\frac{[K^v{}_{wt}\ a\ R]}{[{}^0Exist_{wt}\ R]}$$

Obviously, the second rule is this.

(R3)
$$\frac{[K^v{}_{wt}\ a\ R]}{\exists x\ [{}^0Ident_{wt}\ a\ x\ R]}$$

Generalizing a bit to properties of α-entities, let $P \rightarrow (o\alpha)_{\tau\omega}$, $x \rightarrow \iota$, $K^v/(o\iota(o\alpha)_{\tau\omega})_{\tau\omega}$ be the relation of knowing the values of the extensionalised property P in a world w and time t.

Moreover, let *Ident(ified)*$/(o\iota\alpha(o\alpha)_{\tau\omega})_{\tau\omega}$ be the relation between an agent of type ι, an α-entity and a property of α-entities such that the agent identified the α-entity as an element of the population of the property in the world w and time t of evaluation.

There are two possibilities. Either the agent identified only some elements of the population of P in w and t, or all of them (for instance, by obtaining an exhaustive answer to the corresponding wh-question). If the former, we will say that the agent has an incomplete knowledge of (the value of) the property and if the latter, complete knowledge of (the value of) the property.

Def 2; *knowing the value of a property P*

Complete knowledge: $[K^v{}_{wt}\ a\ P]$ iff $[\exists x\ [P_{wt}\ x] \wedge \forall x\ [[P_{wt}\ x] \supset [{}^0Ident_{wt}\ a\ x\ P]]]$

Incomplete knowledge: $[K^v{}_{wt}\ a\ P]$ iff $[\exists x\ [P_{wt}\ x] \wedge \exists x\ [{}^0Ident_{wt}\ a\ x\ P]]$

The first conjunct is the existential presupposition that the population c of the elements of P_{wt} is non-empty; the second conjunct is the condition that the agent a has identified all/some values of this population. For instance, if a obtained only the answer Krejčíková to the above question "Which Czech ladies are among the first 20 in WTA ranking singles?", then the answer is not complete, as a has identified only one element of the set {Krejčíková, Plíšková, Kvitová}. The specification of the rules for this type of knowing the value is obvious.

So far so good. Yet, in which case has the agent identified particular individual as the value of an individual office, a property, or, in general, of an α-intension? Imagine that John would obtain the answer 'the previous boss of the social democrat party' to the question "Who is the Mayor of Ostrava?". Now we can hardly say that John identified the value of the office; all he knows is that the two terms, namely 'the Mayor of Ostrava' and 'the previous boss of the social democrat party' happen to be co-referring in the $\langle w, t\rangle$-pair of evaluation. Yet, John does not know to which individual they co-refer. Or, imagine that the answer conveyed to the question "Which Czech ladies are among the first 20 in WTA ranking singles?" would be 'the Rolland Garros 2021women single winner', 'the player who lost in Wimbledon 2021 final with Ashleigh Barty', 'the Wimbledon 2014 and 2017 women winner'. Again, such an answer is inconclusive. Provided a is not a tennis expert, she does not know the value of the property of being the Czech among the first twenty in WTA ranking single.

Hence, there is a necessary condition for agent's knowing-wh, i.e. knowing the value(s) of the intension asked for, that a has got a *conclusive answer* to the corresponding wh-question in the form of a *definite description* that rigorously refers to the value(s). Most frequently, such a conclusive answer is provided by a *proper* name(s).

Note that the definitions (Def1) and (Def2) comply with this demand, as the *Ident* function relates the agent to the very value of the respective intension. However, if we carelessly specified the second conjunct as $[^0Ident_{wt}\ a\ R_{wt}\ R]$ or $[^0Ident_{wt}\ a\ P_{wt}\ P]$, respectively, the definitions would not be plausible. The first occurrence of the construction of the office R would be in the supposition *de re*, unlike the second one. Hence the principle of the substitution of co-referring terms (or v-congruent constructions at the semantic level) is valid. As a result, we would obtain the conclusion that, for instance, if an agent a obtained an inconclusive answer in the form of another indefinite description, a knows who is R, which is not true, as we just explained above. For example, if John obtained the answer 'the previous boss of the social democrat party', we would obtain the undesirable result that John knows who is the Mayor of Ostrava because he identified 'the previous boss of the social democrat party' as the Mayor of Ostrava.

For completeness, we now specify these definitions and rules for a hyperintensional *Knowing*-wh*. Hyperintensional knowing*-wh relates the agent a to the construction of the intension the value of which a identified. Hence, the type of hyperintensionally knowing the value of the intension asked for is, $K^{*v}/(o\iota *_n)_{\tau\omega}$. As the above definitions and rules relate the agent strictly to the given intension, be it an office R or a property P, the generalization to the hyperintensional case is obvious. We only have to care of the conceptualization of the respective intension whose values are known, as hyperintensional relation is the relation to the one specific construction C of the intension rather than to any other C' equivalent to C, i.e. such that $C = C'$ for any valuation v. Thus, though C and C' produce the same intension, we must keep the perspective of the agent a who sticks to C rather than C'. Here are the rules adjusted to hyperintensional level of knowing-wh.

Additional types. $K^{*v}/(o\iota *_n)_{\tau\omega}$; $Ident^*/(o\iota\iota *_n)_{\tau\omega}$: the relation-in-intension of an agent a who identified an entity x as the value of the office constructed by R.

Def 1*; *knowing* hyperintensionally the value of an office

$$[K^{*v}{}_{wt}\ a\ {}^0R] \text{ iff } \exists x\ [[x = R_{wt}] \wedge [^0Ident^*{}_{wt}\ a\ x\ {}^0R]]$$

(R1*)
$$\frac{[K^{*v}{}_{wt}\ a\ {}^0R]}{\exists x\ [x = R_{wt}]}$$

Obviously, the second rule is this.

(R3*)
$$\frac{[K^{*v}{}_{wt}\ a\ {}^0R]}{\exists x\ [^0Ident^*{}_{wt}\ a\ x\ {}^0R]}$$

The generalised definition of knowing hyperintensionally the value of the property of α-objects is then this.

Def 2*; *knowing hyperintensionally the value of a property*

Complete knowledge: $[K^{*v}{}_{wt}\ a\ {}^0P] \text{ iff } [\exists x\ [P_{wt}\ x] \wedge \forall x\ [[P_{wt}\ x] \supset [^0Ident^*{}_{wt}\ a\ x\ {}^0P]]]$

Incomplete knowledge: $[K^{*v}{}_{wt}\ a\ {}^0P]$ iff $[\exists x\ [P_{wt}\ x] \wedge \exists x\ [{}^0Ident^*{}_{wt}\ a\ x\ {}^0P]]$

Note that in case the population of the property P is infinite, then only incomplete knowledge is possible. This is frequently the case in mathematics.

4.3. Knowing-wh and Knowing-that

4.3.1. Knowing whether

A special case of knowing-wh is *knowing whether*. In [9], §5.1.4, Duží, Jespersen & Materna introduce the analysis of knowing whether in terms of knowing that. Let me briefly recapitulate this analysis. Knowing whether P concerns a proposition or a hyperproposition, i.e., construction of a proposition. The most important difference between *knowing that P* and *knowing whether P* is that the latter is *not factive*: *knowing whether P* is logically compatible with P being false or with P lacking a truth-value. Despite this difference, we can characterise knowing whether by means of knowing that. If one knows whether P it is because any one of the following three options obtains:

- *knowing that P is true*
- *knowing that P is false*
- *knowing that P is undefined (truth-value gap).*

To define knowing whether, we need the definition of the three properties of propositions, namely *True, False and Undef,* all of type $(o o_{\tau\omega})_{\tau\omega}$. They are defined as follows ($P \rightarrow o_{\tau\omega}$):

$$[{}^0True_{wt}\ P]\ v\text{-constructs } \mathbf{T} \text{ if } P_{wt}\ v\text{-constructs } \mathbf{T}\text{, otherwise } \mathbf{F};$$
$$[{}^0False_{wt}\ P]\ v\text{-constructs } \mathbf{T} \text{ if } P_{wt}\ v\text{-constructs } \mathbf{F}\text{, otherwise } \mathbf{F};$$
$$[{}^0Undefined_{wt}\ P] = \neg[{}^0True_{wt}\ P] \wedge \neg[{}^0False_{wt}\ P].$$

Knowing whether requires two definitions, because in the empirical case *knowing* may be either a relation (-in-intension) to a proposition or a relation (-in-intension) to a propositional construction. We use this notation and typing:

$K/(o\iota o_{\tau\omega})_{\tau\omega}$ (‘to know that a proposition is true’)

$K^*/(o\iota {*_n})_{\tau\omega}$ (‘to know that a construction constructs a true proposition’).

Let P, $Q/*_1 \rightarrow o_{\tau\omega}$; $p/*_1 \rightarrow o_{\tau\omega}$; $c/*_2 \rightarrow *_1$; ${}^2c \rightarrow o_{\tau\omega}$; $=_1/(o o_{\tau\omega} o_{\tau\omega})$; $=_2/(o *_1 *_1)$; $\iota/(o_{\tau\omega}(o o_{\tau\omega}))$; $\iota^*/(*_1(o*_1))$. Here ι, ι^* are singularizers, i.e., functions that return the only member of a singleton, otherwise undefined.

Def 3 (*knowing whether P*) Let P be a propositional construction. Then a*n agent a knows whether P iff*

$$\lambda w \lambda t\ [{}^0K_{wt}\ a\ \iota\lambda p\ [p_{wt} \wedge$$
$$[[p =_1 \lambda w \lambda t\ [{}^0True_{wt} P]] \vee [p =_1 \lambda w \lambda t\ [{}^0False_{wt} P]] \vee [p =_1 \lambda w \lambda t\ [{}^0Undef_{wt} P]]]]]$$

Def 3* (*knowing* whether P*) *Let P* be a propositional construction. Then a*n agent a knows* whether P iff*

$$\lambda w \lambda t\, [{}^0K_{wt}\; a\; \iota^*\lambda c\; [[{}^2c]_{wt} \wedge$$
$$[[c =_2 {}^0[\lambda w\lambda t\, [{}^0True_{wt} P]]] \vee [c =_2 {}^0[\lambda w\lambda t\, [{}^0False_{wt} P]]] \vee [c =_2 {}^0[\lambda w\lambda t\, [{}^0Undef_{wt} P]]]]]]$$

Mathematical attitudes invariably demand constructional treatment. Knowing* whether Fermat's Last Theorem is true (i.e., whether the Theorem is a theorem) is to know* which of two constructions constructs **T**. The *analysandum* is the sentence (disregarding tense)

"Fermat knows whether there are positive integers a, b, c, n ($n > 2$) such that $a^n + b^n = c^n$."

Let ν be the type of natural numbers. Let $a, b, c, n, x/*_1 \to \nu$; *Pos(itive integers)*/(oν); 2/ν; *Fermat*/ι; ∀, ∃/(o(oν)); $d/*_2 \to *_1$; ${}^2d \to o$. We write 'x^n' for '$[{}^0Exp\; n\; x]$', *Exp*/(ννν) the power function taking x to its n^{th} power. Since, the value of the Fermat's last Theorem is either true or false, there is no need for the third option (truth-value gap), and the analysis is the Closure

$$\lambda w\lambda t\, [{}^0K^*_{wt}\; {}^0Fermat\; \iota^*\lambda d\, [{}^2d \wedge$$
$$[d =_2 {}^0[\exists abcn\, [[{}^0Pos\; a] \wedge [{}^0Pos\; b] \wedge [{}^0Pos\; c] \wedge [{}^0\!> n\; {}^02] \wedge [{}^0\!= [{}^0\!+ a^n\; b^n]\; c^n]]]$$
$$\vee$$
$$d =_2 {}^0[\forall abcn\, [[{}^0Pos\; a] \wedge [{}^0Pos\; b] \wedge [{}^0Pos\; c] \wedge [{}^0\!> n\; {}^02] \supset \neg[{}^0\!= [{}^0\!+ a^n\; b^n]$$
$$c^n]]]]]].$$

The other cases of *knowing-wh* concern α-intensions where α ≠ o, and their definition in terms of knowing that is not applicable, as we have argued above.

4.3.2. Explication of 'Identifying the value of an α-intension' by means of knowing that

In the previous paragraph 4.2, the function *Ident(ified)*/$(o\iota\iota_{\tau\omega})_{\tau\omega}$ has been introduced as the relation between an agent *a*, an individual and the office such that *a* has identified that individual as the value of the office *R*. The question arises, however, what does it mean 'to *identify* an individual as something'?[23] To answer this question, I will now *explicate Ident by means of knowing-that*.

Assume that John knows who is the Mayor of Ostrava, and the Mayor is Mr Macura. Does this situation entail that John knows that Macura is the Mayor of Ostrava? Though it seems undoubtable, it depends on John's *deduction abilities*. According to **Def 1**, John identified an individual *x* as the value of the Mayor's office in the world *w* and time *t* of evaluation:

$$[K^v_{wt}\; {}^0John\; \lambda w\lambda t\, [{}^0\mathrm{I}\, \lambda who\, [who = [{}^0Mayor\text{-}of_{wt}\; {}^0Ostrava]]]]\ \text{iff}$$
$$\exists x\, [[x = [{}^0Mayor\text{-}of_{wt}\; {}^0Ostrava]] \wedge [{}^0Ident_{wt}\; {}^0John\; x\; \lambda w\lambda t\, [{}^0Mayor\text{-}of_{wt}\; {}^0Ostrava]]]$$

Thus, in any $\langle w,t\rangle$-pair of evaluation, we have:

1) $\exists x\; [[x = [{}^0Mayor\text{-}of_{wt}\; {}^0Ostrava]] \wedge [{}^0Ident_{wt}\; {}^0John\; x\; \lambda w\lambda t\; [{}^0Mayor\text{-}of_{wt}\; {}^0Ostrava]]]$ ∅
2) $[{}^0Mayor\text{-}of_{wt}\; {}^0Ostrava] = {}^0Macura$ ∅

[23] For the sake of simplicity, in this paragraph, I am going to deal with the case of identifying the value of an office. Generalization for properties is obvious.

3) $[{}^0Macura = [{}^0Mayor\text{-}of_{wt}\ {}^0Ostrava]] \wedge [{}^0Ident_{wt}\ {}^0John\ {}^0Macura\ \lambda w\lambda t\ [{}^0Mayor\text{-}of_{wt}\ {}^0Ostrava]]$
$\exists$E, 1, ${}^0Macura/x$
4) $[{}^0Ident_{wt}\ {}^0John\ {}^0Macura\ \lambda w\lambda t\ [{}^0Mayor\text{-}of_{wt}\ {}^0Ostrava]]$
$\wedge$E, 3

To derive that John knows *that* Macura is the Mayor of Ostrava, i.e.

$$\lambda w\lambda t\ [{}^0Know_{wt}\ {}^0John\ \lambda w\lambda t\ [{}^0Macura = \lambda w\lambda t\ [{}^0Mayor\text{-}of_{wt}\ {}^0Ostrava]_{wt}]],$$

we have to *explicate Ident,* i.e., to *postulate* that identifying x as the value of an office amounts to knowing that the value of the office is x. Hence, we specify the ***Meaning Postulates***:

$$[{}^0Ident_{wt}\ a\ x\ R] = [{}^0Know_{wt}\ a\ \lambda w\lambda t\ [x = R_{wt}]]$$

$$[{}^0Ident^*_{wt}\ a\ x\ {}^0R] = [{}^0Know^*_{wt}\ a\ {}^0[\lambda w\lambda t\ [x = R_{wt}]]]$$

Without these postulates, it is not *logically* derivable that knowing the value of an office is equivalent to knowing that this or that individual is the holder of the office. For, assume that John obtained the answer to the wh-question "Who is the Mayor of Ostrava?" formalised in TIL as

$$\lambda w\lambda t\ \lambda x\ [x = [{}^0Mayor\text{-}of_{wt}\ {}^0Ostrava]]$$

that in the world w' and time t' of evaluation $x = {}^0Macura$. Hence, in the $\langle w',t'\rangle$-pair John knows who is the Mayor of Ostrava. To derive that Macura is the Mayor, John has to apply the rule of β-reduction three times

$$[\lambda w\ [\lambda t\ [\lambda x\ [x = [{}^0Mayor\text{-}of_{wt}\ {}^0Ostrava]]\ w']\ t']\ {}^0Macura] =_\beta [{}^0Macura = [{}^0Mayor\text{-}of_{w't'}\ {}^0Ostrava]],$$[24]

and then again β-abstraction, as knowing-that is not the relation to a truth-value but to a proposition; in this case that Macura is the sought Mayor: $\lambda w\lambda t\ [{}^0Macura = [{}^0Mayor\text{-}of_{w't'}\ {}^0Ostrava]]$.

In order to ensure this desirable result even in case the agent does not have the capacity to derive the conclusion, in the next paragraph, we again specify the rules.

4.3.3. Knowing-wh and Knowing-that

Above we have explicated *Ident* by means of *Knowing-that*. Hence, we are now in the position to define the *rules* specifying the mutual relation between *Knowing-that* and *Knowing-wh*.

Let $K^{v}/(o\iota\alpha_{\tau\omega})_{\tau\omega}$ and $K/(o\iota o_{\tau\omega})_{\tau\omega}$ be the relation of knowing-wh and knowing-that, respectively, and let the agent $a \rightarrow \iota$ has *identified* the value of an intension $Int \rightarrow \alpha_{\tau\omega}$. The agent could have identified this value by obtaining the conclusive direct answer in the form of a rigorous definite description to the respective Wh-question, or in any other way.

[24] Though β-reduction is in general not an equivalent conversion in the logic of partial functions such as TIL, these conversions are equivalent. The first two of them are the so-called *restricted* β-reductions which consist just in substituting variables for variables of the same type. The third conversion is equivalent because Trivialization ${}^0Macura$ is never v-improper. For details on β-conversions in TIL, see [10].

Moreover, let this value in the world w and time t of evaluation be $c \rightarrow \alpha$. Then we have this rule.

Knowing-wh ⇒ Knowing-that

$$\frac{[K^{v}{}_{wt}\ a\ Int] \wedge [c = Int_{wt}]}{[K_{wt}\ a\ \lambda w \lambda t\ [c = Int_{wt}]]}$$

Similarly, if the agent a knows that c is the value of *Int* in a $\langle w, t\rangle$-pair of evaluation, then by applying the following rule, a should know-wh the value of *Int* in $\langle w, t\rangle$.

Knowing-that ⇒ Knowing-wh

$$\frac{[K_{wt}\ a\ \lambda w \lambda t\ [c = Int_{wt}]]}{[K^{v}{}_{wt}\ a\ Int] \wedge [c = Int_{wt}]}$$

Generalization for the hyperintensional knowing* is obvious. As these rules capture basic patterns of reasoning with knowing-wh and knowing-that, they might contribute to a smooth communication of agents in a multi-agent system and to avoiding misunderstandings and inconsistencies among the agents.

5. Conclusion

In this paper, I have investigated *knowing-wh* and its relation to *knowing-that.* Unlike formal syntactic theories that deal with knowing-wh as a primitive entity *Kv*, and specify axioms and rules how to deal with it, I fist explicate the meaning of knowing-wh. My starting point is the close relation between knowing-wh and the corresponding wh-question. An agent a knows-wh the value of the intension asked for, iff a has obtained a conclusive answer that rigorously refers to the value, i.e., a has identified it. Then I explicate the meaning of *identifying* the value of an α-intension by means of knowing-that relation. These results vindicate the specification of the sound rules for dealing with such objectual knowledge. The main novelty of the paper are the rules specifying the mutual relation between objectual knowing-wh and propositional knowing-that.

These results, though valuable, are just the first step in the investigation of the objectual knowledge of a value in TIL. Further research will concentrate on the analysis of more complex sentences with the knowing-wh constituent, for instance like those examples adduced by Hintikka, where the quantification over functions and/or propositions is called for. Moreover, we will investigate knowing-how and knowing-why, which would bring our attention also to the 'logic of because'.

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