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On Convergence Theorems for Strong Fuzzy Variational Henstock Integrals

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Abstract. In this paper, we propose some new concepts for the sequence of fuzzy-number-valued functions by using the notion of inner small variation. Then, we study some the characteristics of the primitives of strong fuzzy variational Henstock integrals defined in m-dimensional Euclidean space using these concepts. Finally, we obtain some convergence theorems for this integrals.

Keywords. inner small variation, high dimensional fuzzy number space, strong fuzzy variational Henstock integral, convergence theorems

1. Introduction

Wu and Gong[1,2] initiated the study of the fuzzy Henstock integral in 2000 to promote the development of fuzzy analysis. In order to study n-dimensional fuzzy numbers, based on the support functions, Zhang et al.[3] proposed the representation theorem of n-dimensional fuzzy numbers two years later, which enable us to further discuss the properties of the primitives of n-dimensional fuzzy integrals.

Since integral theory is one of the important topics as an important part of real analysis. There are lots of study for the integrals defined in m-dimensional Euclidean space. Since m-dimensional Euclidean space does not have linearity, the Vitali's covering theorem could not take effect. To avoid the defect, Lu and Lee[4] propose the notions of the inner cover and the inner variation zero in 1999 to study the properties of the primitive of Henstock integrals defined in m-dimensional Euclidean space. Cabral and Lee[5] further discussed the properties of the primitive of m-dimensional Henstock integrals in 2002. By using same method, Chew[6] discussed the properties of the primitive of McShane integrals in the same year. In 2006, Ye[7] studied the primitive of strong Henstock-Kurzweil integrals.

To study the initial value problem of solutions for the differential equations defined in m-dimensional Euclidean space, we have to put forward the convergence theorems first. In high dimensional real analysis, Paredes, Lee and Chew[8] proved the control convergence theorem of m-dimensional strongly variational integrals with Banach-valued in 2002, while this convergence theorem does not use the concept of inner variation zero. Therefore, it does not distinguish between high-dimensional integrals and one-dimensional integrals. In 2000, Cabral and Lee[9] gave the convergence theorems of real-

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valued strongly variational integrals defined in m-dimensional Euclidean space by using the concept of inner variation zero. For fuzzy analysis, Gong and Shao[10] proved the control convergence theorems by using the concepts of $ACG^*(X)$ and equi-integrability respectively of strong fuzzy Henstock integrals, and established the connection between the two theorems in 2009. Shao, Gong and Li[11] proposed weak fuzzy equi-integrability and further gave the fuzzy Henstock lemma in 2017. They also proposed the control convergence theorem by the fuzzy Henstock lemma of fuzzy Henstock integrals.

The research on fuzzy approximation theory has also achieved good results in recent years. By using smoothing radial basis functions, Gonzalez-Rodelas et al.[12] study the method to approximate the trapezoidal fuzzy numbers in 2022. In the same year, Baez-Sanchez et al.[13] use polygonal fuzzy numbers to approximate fuzzy numbers. Based on the method of power series summability, Baxhaku et al.[14] prove the Korovkin type approximation theorem for positive linear fuzzy sequences in 2022. For the study of fuzzy normed spaces, see[15,16].

The research on the integrals defined in m-dimensional Euclidean space is relatively rare especially for space of n-dimensional fuzzy numbers. In this paper, by using the notion of inner small variation, we define some sequences of n-dimensional fuzzy-valued functions. Then, we describe the primitives characteristics of strong fuzzy variational Henstock integrals defined in m-dimensional Euclidean space using these concepts. Finally, we give some convergence theorems for this integrals by using the concept of inner small variation.

The paper is organized as follows. In Section 2, we provide some basic concepts about n-dimensional fuzzy number space and inner small variation. In Section 3, we define some sequences of fuzzy-valued functions by using the notion of inner small variation and derive some convergence theorems of strong fuzzy variational Henstock integrals. In Section 4, we give some conclusions.

2. Preliminaries

2.1. Fuzzy Number Space

Throughout our paper $\mathbb{R}^n_{\mathscr{F}}$ denotes a set of all fuzzy subsets on $\mathbb{R}^n$. For $0 < \alpha \leq 1$, denote $[\tilde{u}]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$. Suppose $\tilde{u} \in \mathbb{R}^n_{\mathscr{F}}$, satisfies $\tilde{u}$ is normal, convex, upper semi-continuous and $[\tilde{u}]_0 = \overline{\bigcup_{\alpha \in (0,1]} [\tilde{u}]_\alpha}$ is a bounded set, then $\tilde{u}$ is called a fuzzy number. We use $\mathscr{F}(\mathbb{R}^n)$ to denote the fuzzy number space. For addition and scalar multiplication of fuzzy numbers, refer to [17,18].

The distance between two fuzzy numbers is defined $\mathbf{d}_H : \mathscr{F}(\mathbb{R}^n) \times \mathscr{F}(\mathbb{R}^n) \to \mathbb{R}_+ \cup \{0\}$ by

$$\mathbf{d}_H(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} d([\tilde{u}]_\alpha, [\tilde{v}]_\alpha),$$

where

$$\begin{aligned} d([\tilde{u}]_\alpha, [\tilde{v}]_\alpha) &= \inf\{\varepsilon : [\tilde{u}]_\alpha \subset S([\tilde{v}]_\alpha, \varepsilon), [\tilde{v}]_\alpha \subset S([\tilde{u}]_\alpha, \varepsilon)\} \\ &= \max\{\sup_{a \in [\tilde{u}]_\alpha} \inf_{b \in [\tilde{v}]_\alpha} \|a - b\|, \sup_{b \in [\tilde{v}]_\alpha} \inf_{a \in [\tilde{u}]_\alpha} \|a - b\|\}. \end{aligned}$$

By [19], we get that: (1) $(\mathscr{F}(\mathbb{R}^n), \mathbf{d}_H)$ is complete; (2) $\mathbf{d}_H(\tilde{u}+\tilde{w}, \tilde{v}+\tilde{w}) = \mathbf{d}_H(\tilde{u}, \tilde{v})$; (3) $\mathbf{d}_H(h\cdot\tilde{u}, h\cdot\tilde{v}) = |h|\cdot\mathbf{d}_H(\tilde{u}, \tilde{v})$.

Let S^{n-1} be the unit sphere of $\mathbb{R}^n$, i.e. $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, $\langle\cdot,\cdot\rangle$ be the inner product in $\mathbb{R}^n$. Suppose $\tilde{u} \in \mathscr{F}(\mathbb{R}^n), x \in S^{n-1}, r \in [0,1]$, we call $\tilde{u}^*(r,x) = \sup_{a\in[\tilde{u}]_r}\langle a,x\rangle$ the support function of $\tilde{u}$.

2.2. *High Dimensional Fuzzy Integrals*

Let $\mathscr{I}_0 = [\alpha,\beta] \subset \mathbb{R}^m$ with $[\alpha,\beta] = [\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times\cdots\times[\alpha_m,\beta_m]$. Let $I \subset \mathscr{I}_0$ with its Lebesgue measure $\mathfrak{m}(I)$. Given $r > 0$, we denote $\{y \in \mathbb{R}^m : \|y-x\| < r\}$ by $B(x,r)$. We call P the partial partition of $\mathscr{I}_0$ where $P = \{(I_i,x_i); x_i \in \mathscr{I}_0\}$ satisfied that if $i \neq j$ then $\mathfrak{m}(I_i \bigcup I_j) = 0$, and $\bigcup_i I_i \subset \mathscr{I}_0$. Furthermore, if we have $\bigcup_i I_i = \mathscr{I}_0$, we call P a partition of $\mathscr{I}_0$.

Definition 2.1 *Given a gauge $\delta : \mathscr{I}_0 \to (0,+\infty)$. If $x_i \in I_i \subset B(x_i,\delta(x_i))$ for each i, then we say $P = \{(I_i,x_i)\}_{i=1}^m$ is δ-fine.*

Denote by $\Sigma(\mathscr{I}_0)$ the family of compact subintervals of $\mathscr{I}_0$ and let $\tilde{F} : \Sigma(\mathscr{I}_0) \to \mathscr{F}(\mathbb{R}^n)$. If $\tilde{F}(I_1 \bigcup I_2) = \tilde{F}(I_1) + \tilde{F}(I_2)$ when $\mathfrak{m}(I_1 \bigcap I_2) = 0$, then we say $\tilde{F}$ is additive.

Definition 2.2 *Let $\tilde{f} : \mathscr{I}_0 \to \mathscr{F}(\mathbb{R}^n)$ and $\tilde{F} : \Sigma(\mathscr{I}_0) \to \mathscr{F}(\mathbb{R}^n)$. If for $\forall\varepsilon > 0$, $\exists\delta(\xi) > 0$ s.t. for any partition $P = \{(I,\xi)\}$ which is δ-fine*

$$(P)\sum \mathbf{d}_H\big(\tilde{f}(\xi)\cdot\mathfrak{m}(I), \tilde{F}(I)\big) < \varepsilon,$$

then we say $\tilde{f}$ is strongly fuzzy variational Henstock integrable(SFVH integrable for short) on $\mathscr{I}_0$ with the primitive $\tilde{F}$. We write $\tilde{F}(\mathscr{I}_0) = \int_{\mathscr{I}_0}\tilde{f}\mathrm{d}x$ and $\tilde{f} \in SFVH(\mathscr{I}_0)$.

Definition 2.3 *Let $\tilde{f}_n : \mathscr{I}_0 \to \mathscr{F}(\mathbb{R}^n)$ and $\tilde{F}_n : \Sigma(\mathscr{I}_0) \to \mathscr{F}(\mathbb{R}^n)$. If for $\forall\varepsilon > 0$, $\exists\delta(\xi) > 0$ independent of n s.t. for any partition $P = \{(I,\xi)\}$ which is δ-fine, for each n*

$$(P)\sum \mathbf{d}_H\big(\tilde{f}_n(\xi)\cdot\mathfrak{m}(I), \tilde{F}_n(I)\big) < \varepsilon,$$

then we say $\{\tilde{f}_n\}$ is strongly fuzzy variational Henstock equiintegrable on $\mathscr{I}_0$ with the primitive $\{\tilde{F}_n\}$.

2.3. *Inner Small Variation*

By using the idea from [4,5,6,7], we now introduce some notations and concepts as following.

Let $\tilde{f} : \mathscr{I}_0 \to \mathscr{F}(\mathbb{R}^n)$ and $\tilde{F} : \Sigma(\mathscr{I}_0) \to \mathscr{F}(\mathbb{R}^n)$. Given $\eta > 0$, $\delta(x) > 0$, $r \in [0,1]$ and $s \in S^{n-1}$, we define

$$\begin{aligned}\Gamma(\tilde{f},\tilde{F},\delta,\eta,r,s) = \{(I,x); x \in \mathscr{I}_0, \big|\tilde{F}(I)^*(r,s) - \tilde{f}(x)^*(r,s)\cdot\mathfrak{m}(I)\big| \geq \eta\cdot\mathfrak{m}(I) \\ \text{where } (I,x) \text{ is } \delta\text{-fine }\}\end{aligned} \tag{1}$$

For given $r \in [0,1]$ and $s \in S^{n-1}$. When $\tilde{f}$ and $\tilde{F}$ are fixed, we replace $\Gamma(\tilde{f},\tilde{F},\delta,\eta,r,s)$ by $\Gamma(\delta,\eta,r,s)$. Obviously, $\Gamma(\delta,\eta,r,s)$ has the following properties:

1. for given δ, r and s, let $\eta_2 \leq \eta_1$ then we have $\Gamma(\delta, \eta_1, r, s) \subset \Gamma(\delta, \eta_2, r, s)$;
2. for given η, r and s, let $\delta_1(x) \leq \delta_2(x)$, then we have $\Gamma(\delta_1, \eta, r, s) \subset \Gamma(\delta_2, \eta, r, s)$.

Let $E \subset \mathscr{I}_0$, we say $\Gamma(\delta, \eta, r, s)$ is a singular point covering of E if for $\forall x \in E$, $\exists (I, x) \in \Gamma(\delta, \eta, r, s)$. We assume that if η is small enough, there always is $\delta > 0$ s.t. for all $r \in [0,1]$ and $s \in S^{n-1}$, $\Gamma(\delta, \eta, r, s)$ is a singular point covering of $E \subset \mathscr{I}_0$. For simplicity, We replace $\Gamma(\tilde{f}, \tilde{F}, \delta, \eta, r, s)$ by Γ throughout this article.

Definition 2.4 *[7] Let $E \subset \mathscr{I}_0$ and Γ defined above. If for $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.*

$$(P) \sum_{x \in E} \mathfrak{m}(I) < \varepsilon$$

holds for every partial partition $P = \{(I, x)\}$ which is δ-fine satisfy $x \in E$ and $P \subset \Gamma$ of $\mathscr{I}_0$, then we say E is with respect to Γ of inner small variation.

3. Convergence Theorems

Let $\{\tilde{f}_n\}$ and $\{\tilde{F}_n\}$ be given and set

$$\begin{aligned} \Gamma_n = \{(I,x); x \in \mathscr{I}_0, \left|\tilde{F}_n(I)^*(r,s) - \tilde{f}_n(x)^*(r,s) \cdot \mathfrak{m}(I)\right| \geq \eta \cdot \mathfrak{m}(I) \\ \text{where } (I,x) \text{ is } \delta\text{-fine } \} \end{aligned} \tag{2}$$

Definition 3.1 *If for $\forall \varepsilon > 0$, $\exists \delta(\xi) > 0$ independent of n s.t. for any partition $P = \{(I, \xi)\}$ which is δ-fine in Γ_n and $s \in S^{n-1}$*

$$(P) \sum | \tilde{F}_n(I)^*(r,s)) | < \varepsilon \quad \textit{and} \quad (P) \sum | \tilde{f}_n(x)^*(r,s) \cdot \mathfrak{m}(I) | < \varepsilon$$

respect to $r \in [0,1]$ uniformly. Then $\{\tilde{f}_n\}$ is said to satisfied (UI_1) condition.

Theorem 3.1 *$\{\tilde{f}_n\}$ satisfied (UI_1) condition if and only if it is strongly fuzzy variational Henstock equiintegrable on $\mathscr{I}_0$ with the primitive $\{\tilde{F}_n\}$.*

Definition 3.2 *If for $\forall \varepsilon > 0$, $\exists \delta(\xi) > 0$ independent of n s.t. for any partition $P = \{(I, \xi)\}$ which is δ-fine in Γ_n and $s \in S^{n-1}$*

$$(P) \sum | \tilde{F}_n(I)^*(r,s)) | < \varepsilon \quad \textit{and} \quad (P) \sum \mathfrak{m}(I) < \varepsilon$$

respect to $r \in [0,1]$ uniformly. Then $\{\tilde{f}_n\}$ is said to satisfied (UI_2) condition.

Theorem 3.2 *$\{\tilde{f}_n\}$ satisfied (UI_1) condition if it satisfied (UI_2) condition.*

The key points to prove this theorem is to divide $\mathscr{I}_0$ into E_{nk} where

$$E_{nk} = \{x \in \mathscr{I}_0 : \sup_r \sup_s |\tilde{f}_n(x)^*(r,s)| < k, x \notin E_{ni}\}, n, k \in \mathbb{R}^+.$$

Theorem 3.3 *$\{\tilde{f}_n\}$ is strongly fuzzy variational Henstock equiintegrable on $\mathscr{I}_0$ if and only if for $\forall\varepsilon > 0$, $\exists\delta$ s.t. $(P)\sum \mid \tilde{f}_n(x)^*(r,s)\cdot\mathfrak{m}(I) \mid < \varepsilon$ and $(P)\sum \mid \tilde{F}_n(I)^*(r,s)) \mid < \varepsilon$ where $P = \{(I,\xi)\}$ is δ-fine in Γ_n.*

From Theorem 3.2 and Theorem 3.3, we derive the following corollaries.

Corollary 3.1 *$\{\tilde{f}_n\}$ is strongly fuzzy variational Henstock equiintegrable on $\mathscr{I}_0$ if for $\forall\varepsilon > 0$, $\exists\delta$ s.t. $(P)\sum\mathfrak{m}(I) < \varepsilon$ and $(P)\sum \mid \tilde{F}_n(I)^*(r,s)) \mid < \varepsilon$ where $P = \{(I,\xi)\}$ is δ-fine in Γ_n.*

Corollary 3.2 *$\tilde{f}$ is strongly fuzzy variational Henstock integrable on $\mathscr{I}_0$ if and only if for $\forall\varepsilon > 0$, $\exists\delta$ s.t. $(P)\sum \mid \tilde{f}(x)^*(r,s)\cdot\mathfrak{m}(I) \mid < \varepsilon$ and $(P)\sum \mid \tilde{F}(I)^*(r,s)) \mid < \varepsilon$ where $P = \{(I,\xi)\}$ is δ-fine in Γ.*

Definition 3.3 *Let $E \subset \mathscr{I}_0$ is with respect to Γ of inner small variation. If for $\forall\varepsilon > 0$, $\exists\delta$ independent of n on E s.t. for any partial partition $P = \{(I,x)\}$ which is δ-fine satisfy $x \in E$ and $P \subset \Gamma$ of $\mathscr{I}_0$, for $\forall s \in S^{n-1}$*

$$(P)\sum|\tilde{F}_n(I)^*(r,s)| < \varepsilon$$

respect to $r \in [0,1]$ uniformly. Then we say $\{\tilde{F}_n\}$ with respect to Γ satisfies the USL_v *condition on E.*

Note that if we replace "$E \subset \mathscr{I}_0$ is with respect to Γ of inner small variation" by "$E \subset \mathscr{I}_0$ is of measure zero", $\{\tilde{F}_n\}$ is said to satisfy the USL condition.

Definition 3.4 *Let $\tilde{A}_n, \tilde{A} \in \mathscr{F}(\mathbb{R}^n)$. If for $\forall\varepsilon > 0$, $\exists N \in N^+$ s.t. for $\forall s \in S^{n-1}$ and $n > N$*

$$|\tilde{A}_n^*(r,s) - \tilde{A}^*(r,s)| < \varepsilon$$

respect to $r \in [0,1]$ uniformly. Then we say $\tilde{A}_n$ converges to $\tilde{A}$ by support functions.

In this note, we always use the convergence by support functions and write $\tilde{A}_n \to \tilde{A}$ without confusion.

Definition 3.5 *Given $E \in \mathscr{I}_0$ which is with respect to Γ of inner small variation. If $\tilde{f}_n(x) \to \tilde{f}(x)$ where $x \in \mathscr{I}_0/E$, then $\tilde{f}_n \to \tilde{f}$ v.a.e..*

Theorem 3.4 *If $\tilde{f} \in SFVH(\mathscr{I}_0)$ with primitive $\tilde{F}$ and $\tilde{f} = \tilde{g}$ v.a.e., then $\tilde{f} \in SFVH(\mathscr{I}_0)$ with primitive $\tilde{F}$.*

Theorem 3.5 *If fuzzy function sequence $\{\tilde{f}_n\} \subset SFVH(\mathscr{I}_0)$ satisfied the following:*

1. *$\{\tilde{f}_n\}$ satisfy (UI_2) condition and $\tilde{f}_n(x) \to \tilde{f}(x)$ v.a.e. in $\mathscr{I}_0$;*
2. *$\{\tilde{F}_n\}$ with respect to Γ satisfy the* USL_v *condition on E,*

then, we obtain that $\tilde{f} \in SFVH(\mathscr{I}_0)$ and

$$\int_{\mathscr{I}_0}\tilde{f}\mathrm{d}x = \lim_{n\to\infty}\int_{\mathscr{I}_0}\tilde{f}_n\mathrm{d}x.$$

Proof. Since $\tilde{f}_n(x) \to \tilde{f}(x)$ v.a.e. in $\mathscr{I}_0$, there are $E \subset \mathscr{I}_0$ is with respect to Γ of inner small variation satisfied that $E = \{x \in \mathscr{I}_0 : \tilde{f}_n(x) \nrightarrow \tilde{f}(x)$. Define

$$\tilde{f}'_n(x) = \tilde{f}_n(x)\chi_{\mathscr{I}_0/E},$$

then we obtain that $\tilde{f}'_n(x) \to \tilde{f}'(x)$ for all $x \in \mathscr{I}_0$. From Theorem 3.4, we only need to show that $\tilde{f}'(x) \in SFVH(\mathscr{I}_0)$. By Theorem and 3.1 and 3.2, $\{\tilde{f}'\}$ is strongly fuzzy variational Henstock equiintegrable on $\mathscr{I}_0$, that is, for $\forall \varepsilon > 0$, $\exists \delta(\xi) > 0$ independent of n s.t. for any partition $P = \{(I, \xi)\}$ which is δ-fine, for each n

$$(P)\sum \mathbf{d}_H\big(\tilde{f}'_n(\xi)\cdot \mathfrak{m}(I), \tilde{F}'_n(I)\big) < \varepsilon.$$

Therefore, we can find $N > 0$ s.t. for any partition $P' = \{(I, \xi)\}$ which is δ-fine

$$(P')\sum \mathbf{d}_H\big(\tilde{f}'_n(\xi)\cdot \mathfrak{m}(I), \tilde{f}'_m(\xi)\cdot \mathfrak{m}(I)\big) < \varepsilon$$

where $n, m > N$. So, we can obtain that $\{\tilde{F}'_n(I)\}$ is a Cauchy sequence for each $I \in \mathscr{I}_0$. That is $\tilde{F}'_n(I) \to \tilde{F}'(I)$ for each $I \in \mathscr{I}_0$. We compete this proof by above conditions.

Note that if we replace "$\{\tilde{f}_n\}$ satisfied (UI_2) condition" by "$\{\tilde{f}_n\}$ satisfied (UI_1) condition", the above theorem is also holds by Theorem 3.2.

Theorem 3.6 *If fuzzy function sequence $\{\tilde{f}_n\} \subset SFVH(\mathscr{I}_0)$ satisfied the following:*

1. *$\{\tilde{f}_n\}$ satisfy (UI_2) condition and $\tilde{f}_n(x) \to \tilde{f}(x)$ a.e. in $\mathscr{I}_0$;*
2. *$\{\tilde{F}_n\}$ satisfy the* USL *condition.*

Then, we obtain that $\tilde{f} \in SFVH(\mathscr{I}_0)$ and

$$\int_{\mathscr{I}_0} \tilde{f}\mathrm{d}x = \lim_{n\to\infty} \int_{\mathscr{I}_0} \tilde{f}_n \mathrm{d}x.$$

Note that Theorem 3.4 is useful since we can transform "$\tilde{f}_n(x) \to \tilde{f}(x)$ a.e. in $\mathscr{I}_0$" to "$\tilde{f}'_n(x) \to \tilde{f}'(x)$ for all $x \in \mathscr{I}_0$" where $\tilde{f}'_n(x) = \tilde{f}_n(x)\chi_{\mathscr{I}_0/E}$ and $E = \{x \in \mathscr{I}_0 : \tilde{f}_n(x) \nrightarrow \tilde{f}(x)$. Theorem 3.1 allows us to adopt the method based on equiintegrability of sequences for fuzzy-valued functions. In order to obtain this theorem, we demand $\{\tilde{f}_n\}$ corresponding to $\{\tilde{F}_n\}$ since we need for every $s \in S^{n-1}$ $\mid \tilde{f}'_n(x)(r,s)\cdot \mathfrak{m}(I) - \tilde{F}_n(I)(r,s) \mid < \varepsilon$ respect to $r \in [0, 1]$ uniformly.

Corollary 3.3 *If fuzzy function sequence $\{\tilde{f}_n\} \subset SFVH(\mathscr{I}_0)$ satisfied the following:*

1. *$\{\tilde{f}_n\}$ satisfied (UI_1) condition;*
2. *$\tilde{f}_n(x) \to \tilde{f}(x)$ for every $x \in \mathscr{I}_0$.*

Then, we obtain that $\tilde{f} \in SFVH(\mathscr{I}_0)$ and

$$\int_{\mathscr{I}_0} \tilde{f}\mathrm{d}x = \lim_{n\to\infty} \int_{\mathscr{I}_0} \tilde{f}_n \mathrm{d}x.$$

From Theorem 3.2 and Theorem 3.6, we complete this corollary. This theorem corresponding to convergence theorem base on equiintegrability by Theorem 3.1.

4. Conclusion

The purpose of the article is to extend the theory of high-dimensional strong fuzzy variational Henstock integrals, we firstly define some sequences of n-dimensional fuzzy-valued functions using the notion of inner small variation, and discuss some basic properties for this sequences. By using this sequences, we obtain some convergence theorems for this integrals. For future research, we may focus on the applications of this integrals, such as the initial value problem of solutions for fuzzy differential equations for this integrals.

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