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Portfolio Selection Optimization with Hierarchical Fuzzy Conditional Value-at-Risk

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Abstract. Quantitative risk management (QRM,for short) is very important for investors or financial institutions. This paper discusses portfolio selection in fuzzy environments by means of stochastic and fuzzy methods. Two risk measures called hierarchical fuzzy value-at-risk (HFVaR,for short) and hierarchical fuzzy conditional value-at-risk (HFCVaR, for short) are proposed. And then fuzzy portfolio selection models are established based on the risk measure HFCVaR.

Keywords. Portfolio Selection, Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), Hierarchical Fuzzy Value-at-Risk (HFVaR), Hierarchical Fuzzy Conditional Value-at-Risk (HFCVaR)

1. Introduction

The measurement of financial risk has always been the core issue of quantitative risk management (QRM). In 1952, Markowitz [1] Published the pioneering work on quantitative portfolio. Besides variance, there are many other ways to measure the risk of assets. Based on Markowitz's work, researchers have studied and experimented with various risk-measure methods for portfolio selection (For detail, see [2–4]).

Artzner et al. (1999) [5] presented coherent risk measures. Rockafellar and Uryasev (2000) researched a risk measure by means of Conditional Value at Risk (CVaR) or Tail VaR (see [6, 7]). Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are two popular measures of risk in financial engineering. Hans Föllmer et al. (2002) [8] introduced the notion of a convex measure of risk. Hamel et. al (2013) [9] generalized the scalar coherent risk to set-valued case and gave application to the problem of asset pricing.

Due to the complexity of markets and high frequency of transactions, randomness is not sufficient to describe the uncertainties. Therefore, set-valued random variables and fuzzy set-valued random variables are employed to mathematical finance. As a special case, interval random variable is widely used to present the price or return of a risky as-

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set. For example, Zhang and Li (2009) [10] proposed interval-valued mean semi-variance model. Yan et al. (2017) [11] established mean downside semi-variance portfolio selection model. Zhang and Zhang (2022) [12] considered portfolio selection problems based on interval-valued CVaR. In a fuzzy environment, there are also a lot of literatures on portfolio selection problem, see e.g. the book by Fang and Wang (2005) [13]. Enriqueta et al. [14] (2007) discussed fuzzy portfolio optimization under downside risk measures for trapezoidal LR-fuzzy numbers of the same shape, which generalized the mean-absolute semi-deviation by using both interval-valued probabilistic and possibility measure.

In this paper, we extend scalar VaR and CVaR to hierarchical fuzzy VaR (HFVaR,for short) and hierarchical fuzzy CVaR (HFCVaR,for short). The corresponding algorithms to compute HFVaR and HFCVaR are given. Based on HFCVaR, two types of fuzzy portfolio optimization models are established.

The rest of the paper is organized as follows. In Section 2, we introduce the necessary definitions and preliminaries; Section 3 contributes to HFVaR and HFCVaR; Section 4 proposes the fuzzy portfolio selection optimization based on HFCVaR; Section 5 is the conclusion.

2. Preliminaries

In this subsection, we list the knowledge of fuzzy number and interval number needed later. For detail, see ([15, 16]).

Let X be a universe of discourse. A fuzzy set A in X is characterized by a membership function $A(x) : X \to [0,1]$, which assigns to each object $x \in X$ a real number in the interval [0,1], such an $A(x)$ represents the degree of membership of x belonging to A. The set of all fuzzy sets in X is denoted with $\mathcal{F}(X)$.

The α-cuts of a fuzzy sets A is a crisp sets defined as $A_{[\alpha]} = \{x \in X : A(x) \geq \alpha\}$, for $\alpha \in (0,1]$, and $A_{[0]} = cl\{x \in X : A(x) > 0\}$, where cl denotes the closure operator. It is a useful tool for dealing with fuzzy sets or fuzzy numbers.

Definition 2.1. (Trapezoidal fuzzy number) Let $\mathbb{R}$ be the family of all real numbers. A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ is called a Trapezoidal fuzzy number which the membership function $A(x)$ satisfies the following form:

$$A(x) = \begin{cases} 0 & if \quad x \leq a_1, \\ \frac{x-a_1}{a_2-a_1} & if \quad a_1 \leq x \leq a_2, \\ 1 & if \quad a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3} & if \quad a_3 \leq x \leq a_4, \\ 0 & if \quad a_4 \leq x, \end{cases}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1 \leq a_2 \leq a_3 \leq a_4$. We use $A =< a_1, a_2, a_3, a_4 >_T$ to denote a trapezoidal fuzzy number .

Remark 2.1. The family of all fuzzy numbers will be denoted by $\widetilde{\mathbb{R}}$. The α-cuts of fuzzy number A is given by $A_{[\alpha]} = [A_L(\alpha), A_U(\alpha)]$ where, $A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\}$, $A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\}$. The α-cuts of such trapezoidal fuzzy numbers are given by $A_{[\alpha]} = [A_L(\alpha), A_U(\alpha)] = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha]$.

Definition 2.2. (Interval number) Let $\mathbb{R}$ be the set of all real numbers, Let $A = [a^L, a^U] = \{a \in \mathbb{R} : a^L \leq a \leq a^U\}$, A is called interval number.

Let $\mathbb{R}$ be the set of all real numbers, A, B real closed intervals. Let $A = [a^L, a^U] = \{a \in \mathbb{R} : a^L \leq a \leq a^U\}$, $B = [b^L, b^U] = \{b \in \mathbb{R} : b^L \leq b \leq b^U\}$.

$A + B := [a^L + b^L, a^U + b^U]$, $\lambda \cdot A = \lambda[a^L, a^U] := \begin{cases} [\lambda a^L, \lambda a^U], & \text{if} \quad \lambda \geq 0, \\ [\lambda a^U, \lambda a^L], & \text{if} \quad \lambda < 0. \end{cases}$

We only give a brief introduction to interval arithmetic.

Let $m(A) = \frac{1}{2}(a^L + a^U)$, $w(A) = \frac{1}{2}(a^U - a^L)$, which are called the mean and width of the interval number A, respectively. For two interval numbers $A = [a^L, a^U]$ and $B = [b^L, b^U]$, define $A \prec B \Leftrightarrow \begin{cases} m(A) < m(B) \\ m(A) = m(B) \text{ and } a^L < b^L \end{cases}$. $A = B$ if and only if $m(A) = m(B)$, $a^L = b^L$.

Definition 2.3. (c.f. [12]) Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. R is the random interval-valued return of a risky asset. Given confidence level $1 - \beta$ $(0 < \beta < 1)$, the value at risk is defined by

$$IVaR := -\inf\{X \in K_c(\mathbb{R}) : P(R \prec X) = \beta\}, ICVaR := -E(R|R \prec -IVaR),$$

where $K_c(\mathbb{R})$ is the family of all compact and convex subsets of $\mathbb{R}$.

Remark 2.2. $E(A)$ is the expectation for an integrable interval-valued random variable $A = [a^L, a^U]$ defined by Aumann in 1965. That is to say, $E(A) = [E(a^L), E(a^U)]$. Correspondingly, for a given sub-sigma algebra $\mathcal{G} \subset \mathcal{A}$, the conditional expectation $E(A|\mathcal{G}) = [E(a^L|\mathcal{G}), E(a^U|\mathcal{G})]$. For detail, see e.g. [12].

3. Hierarchical fuzzy VaR and hierarchical fuzzy CVaR

Due to the randomness and vagueness of variables, fuzzy set-valued random variable is a good to describe this kind of uncertainty. Similar to interval-valued VaR and CVaR, we can define the VaR and CVaR under fuzzy environment. Here we propose the notions of hierarchical fuzzy VaR and hierarchical fuzzy CVaR by using the α-level set of a fuzzy set.

Now we give the definitions of HFVaR and HFCVaR of fuzzy random variables.

Let $A \in \mathcal{F}(\mathbb{R})$, A is called fuzzy random variable, if $\forall \alpha \in (0, 1]$, $A_{[\alpha]} = [A_L(\alpha), A_U(\alpha)]$ is a fuzzy random interval.

Definition 3.1. (HFVaR) Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. R is the fuzzy random return of a risky asset. Given confidence level $1 - \beta$ $(0 < \beta < 1)$, $\forall \alpha \in (0, 1]$, the α-hierarchical fuzzy value at risk is defined by

$$HFVaR_{[\alpha]} := -\inf\{X_{[\alpha]} \subseteq K_c(\mathbb{R}) : P(R_{[\alpha]} \prec_f X_{[\alpha]}) = 1 - \beta\},$$

where inf is determined by the interval order $\prec_f$.

Definition 3.2. (HFCVaR) Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. R is the fuzzy random return of a risky asset. Given confidence level $1-\beta$ $(0<\beta<1)$, $\forall\alpha \in (0,1]$, $HFVaR_{[\alpha]}$ is the α-hierarchical fuzzy value at risk. The α-hierarchical fuzzy conditional value at risk is defined by $HFCVaR_{[\alpha]} := -E(R_{[\alpha]}|(R_{[\alpha]} \prec_f -HFVaR_{[\alpha]})$

Remark 3.1. $HFCVaR_{[\alpha]}$ satisfies the subadditivity. It is easily obtained by [12]. Calculate HFVaR and HFCVaR according to Algorithm 1.

Algorithm 1 HFVaR and HFCVaR Algorithm

Input: Fuzzy Random Variables R_i, $i = 1,2,\cdots,n$, level of cuts α , confidence level $(1-\beta)$.

Output: $HFVaR_{[\alpha]}$, $HFCVaR_{[\alpha]}$.

1: Given confidence level $(1-\beta)$, level of cuts α , and fuzzy random variables R_i, $i = 1,2,\cdots,n$.
2: $R_{i[\alpha]} := [R^L_{i[\alpha]}, R^U_{i[\alpha]}]$, $i = 1,2,\cdots,n$.
3: Sort $R_{i[\alpha]}$ according to interval order $\prec_f$,$i = 1,2,\cdots,n$.
4: Calculate IVaR,ICVaR of $R_{i[\alpha]}$ according to confidence level $(1-\beta)$
5: $HFVaR_{[\alpha]} := IVaR$, $HFCVaR_{[\alpha]} := ICVaR$.
6: **return** $HFVaR_{[\alpha]}$, $HFCVaR_{[\alpha]}$.

Table 1. Symbol Description

symbol	attribute	description
R	fuzzy random variable (fuzzy number)	The rate of return of security.
R_i	fuzzy random variable (fuzzy number)	The rate of return of the $i-$th security.
R_{ij}	fuzzy random variable (fuzzy number)	The rate of return of the $i-$th security at the j-th period.
x_i	$x_i \in [0,1]$	Proportion invested on security $i, i = 1,2,\cdots,n$.
E		The expectation operator
α	$\alpha \in [0,1]$	The level of cut sets.
$(1-\beta)$	$\beta \in (0,1)$	The confidence level of VaR, IVaR or HFVaR.
$VaR_{(1-\beta)}$		Value-at-Risk under confidence level $(1-\beta)$.
$CVaR_{(1-\beta)}$		conditional value-at-risk under confidence level $(1-\beta)$.
$HFCVaR_{[\alpha]}$		The $\alpha-$hierarchical fuzzy condition value at risk.
$HFVaR_{(1-\beta)[\alpha]}$		The $\alpha-$ HFVaR under confidence level $(1-\beta)$.

4. Portfolio selection optimization based on HFCVaR

In this section, we present two kinds of portfolio selection models based on HFCVaR where the objective is to minimize the downside risk (HFCVaR) constrained by a given expected returns, or the other way around, the objective is to maximize returns constrained by a given expected risks. We use Trapezoidal fuzzy number to represent the returns on the financial securities and suitable definitions of their expected returns.

Given the confidence level $1-\beta$, the level of cuts α, $HFVaR_{ij[\alpha]}$ ($HFCVaR_{ij[\alpha]}$) is the interval value at risk (conditional value at risk, resp.) of the i-th risky asset at the j-th period, $i = 1,\cdots,n$, $j = 1,\cdots,T$. Let x_i $(i = 1,\cdots,n)$ be the proportion of the i-th asset in the portfolio.

Now we build two models of portfolio selection as follows. We consider the optimal portfolio problem under given acceptable maximum of risk level:

- Model 1

$$\begin{array}{ll} Maximize_{\otimes} \; E(R_{P[\alpha]}) = \sum_{i=1}^{n} x_i E(R_{i[\alpha]}), & \\ subjectto & \sum_{i=1}^{n} x_i HFCVaR_{ij[\alpha]} \leq HFCVaR_{0j[\alpha]}, \\ & j = 1, \cdots, T, \\ & \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \; i = 1, \cdots, n \\ & \alpha \in (0, 1]. \end{array} \tag{1}$$

where $E(R_i)$ is the expectation fuzzy returns of the i-th asset, $HFCVaR_{0j[\alpha]} = [HFCVaR^L_{0j[\alpha]}, HFCVaR^U_{0j[\alpha]}]$ is the acceptable maximum risk in the j-period, $j = 1, \cdots, T$, which are subjective and given by the investors.

The key point of solving model (1) is how to defuzzify and transform the fuzzy problem into a precise linear programming.

We transform the model (1) into the following problem (2) according to [12]

$$\begin{array}{ll} Maximize & m(E(R_{P[\alpha]})) = \sum_{i=1}^{n} x_i m(E(R_{i[\alpha]})), \\ s.t. & \sum_{i=1}^{n} x_i HFCVaR^U_{ij[\alpha]} \leq HFCVaR^U_{0j[\alpha]}, \\ & \sum_{i=1}^{n} x_i [m(HFCVaR_{ij[\alpha]}) - \gamma w(HFCVaR_{ij[\alpha]})] \\ & \quad \leq m(HFCVaR_{0j[\alpha]}) + \gamma w(HFCVaR_{0j[\alpha]}), \\ & j = 1, \cdots, T \\ & x_i \geq 0, i = 1, \cdots, n, \alpha \in (0, 1]. \end{array} \tag{2}$$

where $m(A)$ is the midpoint of interval A and $w(A)$ the semi-width of A. $\gamma \in (0, 1)$ is a given index in advance, which can describe the degree of risk appetite of investors. The larger γ, the lower the risk aversion.

- Model 2

$$\begin{array}{ll} Minimize_{\otimes} & \sum_{i=1}^{n} x_i HFCVaR_{i[\alpha]}, \\ subjectto & \sum_{i=1}^{n} x_i E(R_{ij[\alpha]}) \geq R_{0_{j[\alpha]}}, j = 1, \cdots, T \\ & \sum_{i=1}^{n} x_i = 1, \; x_i \geq 0, \; i = 1, \cdots, n, \alpha \in (0, 1]. \end{array} \tag{3}$$

Correspondingly, by [12], the interval-valued linear programming problem (3) is represented as the following real-valued linear programming model (4) below

$$\begin{array}{ll} Minimize & \frac{1}{2}\sum_{i=1}^{n} x_i(HFCVaR_{i[\alpha]}^{L} + HFCVaR_{i[\alpha]}^{U}), \\ s.t. & \sum_{i=1}^{n} x_i E(R_{ij[\alpha]}^{L}) \geq R_{0j[\alpha]}^{L}, \\ & \sum_{i=1}^{n} x_i[m(E(R_{ij[\alpha]})) + \gamma w(E(R_{ij[\alpha]}))] \geq m(R_{0j[\alpha]}) - \gamma w(R_{0j[\alpha]}), \\ & j = 1, \cdots, T, x_i \geq 0, \ for\ i = 1, \cdots, n, \alpha \in (0,1]. \end{array} \tag{4}$$

where m, w and γ have the same meaning as that in (2).

5. Concluding remarks

We treat the returns on assets as a fuzzy number and discuss the optimal portfolio selection under fuzzy environment. The hierarchical fuzzy VaR and hierarchical fuzzy CVaR of fuzzy random variables are defined. Based on HFCVaR, two interval-valued linear programming models are proposed. Aiming at the special fuzzy set of trapezoidal fuzzy numbers, numerical experiments are carried out on the two types of models discussed by using historical data simulation method. Numerical experiments shows that the algorithm is feasible, and the two types of fuzzy portfolio optimization models are effective. But due to the limitation of pages, here we omit this part in present paper.

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