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Solving Interval Investment Problem in Vague Environment Using Dynamic Programming Approach

Hamiden Abd El- Wahed KHALIFA[a, b], W. A. AFIFI[c] and Pavan KUMAR[d,1]
[a] *Department of Operations Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt*
[b] *Department of Mathematics, College of Science and Arts, Al- Badaya, Qassim University, Saudi Arabia*
[c] *Mathematics and Statistics Department, College of Science, Taibah University, Yanbu, Saudi Arabia*
[d] *Division of Mathematics, School of Advanced Science and Languages, VIT Bhopal University, Sehore-466114, India*

Abstract. In financial planning problems, the determination of the best investment is one of the interesting optimization models. In the proposed work, an investment problem (IP) is introduced in vague environment. The vagueness in return parameter is characterized by normalized heptagonal fuzzy number (HFN). One of the suitable interval approximations, namely, an inexact rough interval of a normalized HFN is utilized. Afterward, the inexact rough interval investment problem is considered. A dynamic programming (DP) approach is developed, which is applied for optimizing the fuzzy investment problem. The ideology of ''rough interval number'' is suggested in the mathematical modeling framework of the proposed problem to show the rough data as an inexact rough interval of piecewise quadratic fuzzy numbers. Afterward, the DP approach is applied to solve and compute a rough interval solution. Finally, a numerical example is yielded for the utility of the approach to apply on real-world problem for the decision-maker. The obtained results consist of the total optimal return with inexact rough intervals on a $ 10 million investments is as follows: $ [[1.69, 2.08]: [1.75, 1.91]] millions.

Keywords. Dynamic programming; Inexact interval; Investment; Normalized heptagonal fuzzy numbers; Optimization; Uncertainty

1. Introduction

The optimization of investment problems (IPs) has been widely applied in practice such as project management. Normally, some problems concerned with a decision-maker (DM) and planners are as follows:

- Whether the project would be finished before a given deadline? and

[1] Corresponding author, Pavan KUMAR, VIT Bhopal University, Sehore-466114, India; Email address: pavankmaths@gmail.com

- How we should invest capital?

provided that qualities of the project may not be under the normal level. Since the cost is one of the most important factors the DM is concerned, the IP appears to hold the balance in real project management. An IP under uncertainty has been widely studied in the literature [1, 2]. Several of the central issues that face sovereign wealth funds were studied [3], and it has proved that the fixed costs are modeled proportionally to capital stock [4]. In group decision-making, the process of three-way decisions was presented with an interval-valued fuzzy decisions by theoretic rough sets [5].

In mathematical optimization, linear programming (LP) has a vital role. An LP approach was highlighted as an application to approximate the DP model [6]. Thereafter, a DP approach was suggested for the optimization of workforce planning decisions in the industry [7]. A dynamic programming (DP) model for scheduling with cancellations was introduced with an application to chemotherapy appointment booking [8]. In the last few decades, several researchers studied the heterogeneity of investment strategies, and consequently return, across different types of institutional investors. There has been comparatively less empirical analysis of agency problems at severing funds largely due to the non-availability of data [9]. A DP approach was proposed to determine the optimum train speed profiles under the restrictions of speed and passage points [10-11]. The DP was adopted to the evaluation when all investments in the set have multiple possible values. In addition, the rate of return changes with the change in the amount invested [12-13]. Using the lexicographic order, the neutrosophic complex programming was studied and obtained the optimal solution [14-15]. In literature, two types of fixed costs were studied: The first assumes a lump-sum cost that has to be paid to set up a project and the second assumes fixed costs per unit time that are independent of the level of investment, and are incurred at each point in time for non-zero investment [16].

In literature, the research article [17] evaluated the amount of investment in a decision support model. They adopted the uncertainties using the intervals and probabilities. The work in [18] introduced the enhancement of capacitated transportation model under fuzzy sense. The work in [19] studied the IP model with chaos return. In addition, the work in [20] investigated a solution method to optimize the fuzzy portfolio selection model. The heptagonal fuzzy numbers (HFNs) were studied by [21-22] to solve the critical path problem as well as the vendor selection problem. In 2020, The work referenced in [23] presented an overview of the interval and fuzzy portfolio selection problems. They formulated the portfolio selection problem as a bi-criteria optimization model. The work in [24] proposed the inexact rough interval fuzzy LP approach with an application to agricultural irrigation systems. Recently, several methods have been developed using intervals as well as fuzzy set theory. For instance, the work in [25] elaborated the investment opportunities using the interval-valued fuzzy approach. In their work, they attempted to decrease the estimation error due to any uncertainty. The research in [26] presented the HFNs using value and ambiguity index.

In this paper, the main objective is to study an inexact rough interval investment problem. As far as the contribution of this paper concerned, a DP approach is adopted to evaluate the developed model. The process of optimization is illustrated by an example.

The remainder portion of this article is organized as follows: In Section 2, some preliminary concepts are elaborated. In Section 3, the model statement is introduced. In Section 4, a numerical example is given. The results and discussion are presented in Section 5. In the end, Section 6 concludes the work.

2. Preliminaries

Definition 1. [26] A fuzzy set $\widetilde{P}$ defined on the set of reals $\mathbb{R}$ is said to be a fuzzy number when its membership function: $\mu_{\widetilde{P}}(\alpha): \mathbb{R} \to [0,1]$, have the following properties:

1) $\mu_{\widetilde{P}}(\alpha)$ is an upper semi-continuous function;
2) $\mu_{\widetilde{P}}(\delta\alpha + (1-\delta)\beta) \geq \min\{\mu_{\widetilde{P}}(\alpha), \mu_{\widetilde{P}}(\beta)\}\ \forall\ \alpha, \beta \in \mathbb{R}$ and $0 \leq \delta \leq 1$;
3) $\widetilde{P}$ is normal, i.e., $\exists\ \alpha_0 \in \mathbb{R}$ for which $\mu_{\widetilde{P}}(\alpha_0) = 1$;
4) $\mathrm{Supp}\left(\widetilde{P}\right) = \{\alpha \in \mathbb{R}: \mu_{\widetilde{P}}(\alpha) > 0\}$ is referred as the support of $\widetilde{P}$. In addition, the closure, designated by $\mathrm{cl}(\mathrm{Supp}(\widetilde{P}))$, is compact set.

Definition 2. [26] A fuzzy number $\tilde{A}_H(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ is a heptagonal fuzzy number (HFN), for $p_1, p_2, p_3, p_4, p_5, p_6, p_7 \in \mathbb{R}$, provided that membership function is given by:

$$\mu_{\tilde{A}_H}(x) = \begin{cases} \frac{1}{3}\left(\frac{y-p_1}{p_2-p_1}\right) & \text{for } p_1 \leq y \leq p_2, \\ \frac{1}{3}+\frac{1}{3}\left(\frac{y-p_2}{p_3-p_2}\right) & \text{for } p_2 \leq y \leq p_3, \\ \frac{2}{3}+\frac{1}{3}\left(\frac{y-p_3}{p_4-p_3}\right) & \text{for } p_3 \leq y \leq p_4, \\ 1-\frac{1}{3}\left(\frac{y-p_4}{p_5-p_4}\right) & \text{for } p_4 \leq y \leq p_5, \\ \frac{2}{3}-\frac{1}{3}\left(\frac{y-p_5}{p_6-p_5}\right) & \text{for } p_5 \leq y \leq p_6, \\ \frac{1}{3}\left(\frac{y-p_6}{p_7-p_6}\right) & \text{for } p_6 \leq y \leq p_7, \\ 0, & \text{for } y < p_1 \text{ and } y > p_7. \end{cases}$$

Definition 3. [26] Let $\widetilde{A}_H = (p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ and $\widetilde{B}_H = (q_1, q_2, q_3, q_4, q_5, q_6, q_7)$. Then, the arithmetic operators are presented as follows:

Addition: $\widetilde{A}_H \oplus \widetilde{B}_H = (p_1, p_2, p_3, p_4, p_5, p_6, p_7) \oplus (q_1, q_2, q_3, q_4, q_5, q_6, q_7)$
$= (p_1 + q_1, p_2 + q_2, p_3 + q_3, p_4 + q_4, p_5 + q_5, p_6 + q_6, p_7 + q_7)$,

Subtraction: $\widetilde{A}_H \ominus \widetilde{B}_H = (p_1, p_2, p_3, p_4, p_5, p_6, p_7) \ominus (q_1, q_2, q_3, q_4, q_5, q_6, q_7)$
$= (p_1 - q_7, p_2 - q_6, p_3 - q_5, p_4 - q_4, p_5 - q_3, p_6 - q_2, p_7 - q_1)$,

Scalar multiplication: $k\widetilde{A}_H = \begin{cases} k(p_1, p_2, p_3, p_4, p_5, p_6, p_7), k \geq 0, \\ k(p_7, p_6, p_5, p_4, p_3, p_2, p_1), k < 0. \end{cases}$

Definition 4. [26] A rough interval approximation, represented by x^R, for the HFN $\widetilde{A}_H = (p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ is referred by an interval including the prescribed value of lower as well as upper bounds, provided the distribution details of x are given:

$$A^R = \left[A_\alpha^{(UAI)} : A_\alpha^{(LAI)}\right], \tag{1}$$

In this definition, $A_\alpha^{(LAI)} = \inf\left\{x \in \mathbb{R}: \mu_{\tilde{A}} \geq \frac{1}{3}\right\}$, and $A_\alpha^{(UAI)} = \sup\left\{x \in \mathbb{R}: \mu_{\tilde{A}} \geq \frac{1}{3}\right\}$, are the respective upper as well as lower approximation intervals of A^R. For $\tilde{A}_H(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$, the rough interval of $\tilde{A}_H$ is equal to $A^R = \left[[p_2, p_6]: [p_3, p_5]\right]$.

Definition 5. [24] Given $A^R = \left[A_\alpha^{(UAI)}: A_\alpha^{(LAI)}\right]$ and $B^R = \left[B_\alpha^{(UAI)}: B_\alpha^{(LAI)}\right]$ are two rough intervals, provided that $A^R > 0$ and $B^R > 0$. Then, the algebraic operators $\{+, -, \times, \div\}$ are presented by:

$$A^R \oplus B^R = \left[\left[A_\alpha^{(UAI)} + B_\alpha^{(UAI)}\right]: \left[A_\alpha^{(LAI)} + B_\alpha^{(LAI)}\right]\right] \tag{2}$$

$$A^R \ominus B^R = \left[\left[A_\alpha^{(UAI)} - B_\alpha^{(UAI)}\right]: \left[A_\alpha^{(LAI)} - B_\alpha^{(LAI)}\right]\right] \tag{3}$$

$$A^R \otimes B^R = \left[\left[A_\alpha^{(UAI)} \times B_\alpha^{(UAI)}\right]: \left[A_\alpha^{(LAI)} \times B_\alpha^{(LAI)}\right]\right] \tag{4}$$

$$A^R \oslash B^R = \left[\left[A_\alpha^{(UAI)} \,/\, B_\alpha^{(UAI)}\right]: \left[A_\alpha^{(LAI)} \,/\, B_\alpha^{(LAI)}\right]\right] \tag{5}$$

Also, $A_\alpha^{(UAI)} = \left[A_\alpha^{-(UAI)}, A_\alpha^{+(UAI)}\right], A_\alpha^{(LAI)} = \left[A_\alpha^{-(LAI)}, A_\alpha^{+(LAI)}\right]$,
$B_\alpha^{(UAI)} = \left[B_\alpha^{-(UAI)}, B_\alpha^{+(UAI)}\right]$, and $B_\alpha^{(LAI)} = \left[B_\alpha^{-(LAI)}, B_\alpha^{+(LAI)}\right]$,
provided that $A_\alpha^{-(UAI)}, A_\alpha^{+(UAI)}, A_\alpha^{-(LAI)}, A_\alpha^{+(LAI)}, B_\alpha^{-(UAI)}, B_\alpha^{+(UAI)}, B_\alpha^{-(LAI)}, B_\alpha^{+(LAI)}$ are the crisp values, which denote the respective lower as well as upper bounds for $A_\alpha^{(UAI)}, A_\alpha^{(LAI)}, B_\alpha^{(UAI)}$ and $B_\alpha^{(LAI)}$. Then, we have

$$A^R \oplus B^R = \left[\left[A_\alpha^{-(UAI)} + B_\alpha^{-(UAI)}, A_\alpha^{+(UAI)} + B_\alpha^{+(UAI)}\right]: \left[A_\alpha^{-(LAI)} + B_\alpha^{-(LAI)}, A_\alpha^{+(LAI)} + B_\alpha^{+(LAI)}\right]\right] \tag{2'}$$

$$A^R \ominus B^R = \left[\left[A_\alpha^{-(UAI)} - B_\alpha^{+(UAI)}, A_\alpha^{+(UAI)} - B_\alpha^{-(UAI)}\right]: \left[A_\alpha^{-(LAI)} - B_\alpha^{+(LAI)}, A_\alpha^{+(LAI)} - B_\alpha^{-(LAI)}\right]\right] \tag{3'}$$

$$A^R \otimes B^R = \left[\left[A_\alpha^{-(UAI)} \times B_\alpha^{-(UAI)}, A_\alpha^{+(UAI)} \times B_\alpha^{+(UAI)}\right]: \left[A_\alpha^{-(LAI)} \times B_\alpha^{-(LAI)}, A_\alpha^{+(LAI)} \times B_\alpha^{+(LAI)}\right]\right] \tag{4'}$$

$$A^R \oslash B^R = \left[\left[A_\alpha^{-(UAI)}/B_\alpha^{+(UAI)}, A_\alpha^{+(UAI)}/B_\alpha^{-(UAI)}\right]: \left[A_\alpha^{-(LAI)}/B_\alpha^{+(LAI)}, A_\alpha^{+(LAI)}/B_\alpha^{-(LAI)}\right]\right] \tag{5'}$$

Definition 6. [24] Let $A^R = \left[A_\alpha^{(UAI)}: A_\alpha^{(LAI)}\right]$ and $B^R = \left[B_\alpha^{(UAI)}: B_\alpha^{(LAI)}\right]$ be the order relations are as follows

(a) $A^R \leq B^R \Leftrightarrow A_\alpha^{+(UAI)} \leq B_\alpha^{+(UAI)}$ and $A_\alpha^{-(UAI)} \leq B_\alpha^{-(UAI)}$ (6)

(b) $A^R < B^R \Leftrightarrow A^R \leq B^R$ and $A^R \neq B^R$ (7)

3. Problem formulation

3.1. Justification for taking HFNs

The use of HFN in mathematical modelling is comparatively more complicated than the other fuzzy numbers like compared the Triangular or Trapezoidal Fuzzy numbers. However, HFN provides an extra possibility to denote the imperfect knowledge that leads to model some of the real-life models in a more adequate way. HFN gives the flexibility to the DM to make the decision using two different heights of HFN. In addition, the HFN denotes the information in a detailed way, and also the vagueness might be handled in more realistically.

3.2. Model development

Consider that investor has at his disposal \$ N millions to invest in L possible production programs I, II, …, L. The expectation of profit for some time period p is unknown. Nevertheless, they are to be estimated and provided as an inexact rough interval number. The main objective is to allocate the investment in the available L assets in seeking a way to get the maximum of the total expectation of return, for a fixed level of risk. Naturally, the investor cannot exceed his / her available wealth \$ N million. We now define the following notations:
$f_1(x)$: Profit function for investing in I,
$f_2(x)$: Profit function for investing in II
⋮ ⋮
$f_n(x)$: Profit function for investing in L,
$F_{1,2}(l)$: Optimal profit, when amount l is invested simultaneously in I as well as II,
$F_{1,2,3}(l)$: Optimal profit, when amount l is invested simultaneously in I, II, and III,
⋮
$F_{1,2,3,\ldots n}(l)$: Optimal profit, where l is invested in I, II, III and L together.
Moreover, we recall the following symbols:

$$x \wedge y = \min(x, y)\,;\, x \vee y = \max(x, y)\,;\text{ and } x + y = \text{sum}(x, y)\,.$$

In the case of using DP approach, at least one criterion must be implemented to yield the best possible outcome. The initial criterion used here for comparison of two HFNs following the corresponding crisp value. In other words, we can write

$$\frac{p_1 + p_2 + p_3 + 2p_4 + p_5 + p_6 + p_7}{8} > \frac{q_1 + q_2 + q_3 + 2q_4 + q_5 + q_6 + q_7}{8}$$
$$\Rightarrow \quad \tilde{A}_H > \tilde{B}_H\,, \tag{8}$$

where $\hat{A}_H = \frac{p_1 + p_2 + p_3 + 2p_4 + p_5 + p_6 + p_7}{8}$ and $\hat{B}_H = \frac{q_1 + q_2 + q_3 + 2q_4 + q_5 + q_6 + q_7}{8}$.
The second criterion based on maximal level of presumption is implemented when required as follows:

$$p_4 > q_4 \longrightarrow \tilde{A}_H > \tilde{B}_H \tag{9}$$

The third criterion, the divergence criterion is implemented when required as follows:

$$(\text{Total divergence } \tilde{A}_H) > (\text{ Total divergence } \tilde{B}_H) \Rightarrow \tilde{A}_H > \tilde{B}_H. \tag{10}$$

The total divergence is compensative evaluation of risk. In this case, the expected loss is equivalent to the expected pay off.

4. Numerical example

Consider an investor has at his disposal \$ 10 million to invest purpose in the production programs designated by I, II, III and IV. For the three years, the mean profits are not known. However, it is observed that they are estimated and given in exact rough intervals as shown in Table 2. First of all, we compute $F_{1,2}(l = 2)$ as follows:

(a) $f_1(0) \oplus f_1(2) = 0 \oplus \left[[0.20, 0.26]: [0.21, 0.25]\right] = [[0.20, 0.26] : [0.21, 0.25]]$ (11)

(b) $f_1(1) \oplus f_2(1) = \left[[0.25, 0.30]: [0.26, 0.29]\right] \oplus \left[[0.20, 0.26]: [0.21, 0.25]\right]$
$= [[0.45, 0.56] : [0.47, 0.54]]$ (12)

(c) $f_1(2) \oplus f_2(0) = [[0.25, 0.30] : [0.26, 0.29]] \oplus 0 = [[0.25, 0.30] : [0.26, 0.29]]$ (13)

Comparing the intervals in (a), (b), and (c), we obtain that the optimal value of profit or the best policy is determined using the investment amount of \$ 1 million I, and \$1 million in II with the total profit (or optimum policy) being $\$[[0.45, 0.56] : [0.47, 0.54]]$ millions. We evaluate the optimal profits in investments in I and II for different values of l as:

$$F_{1,2}(l) = \max_{x+y=l} \left(f_1(x) \oplus f_2(y)\right) \quad (14)$$

We compute $F_{1,2,3}(l)$, the optimal return on investments in I, II, and III for different numerical data for l by the expression:

$$F_{1,2,3}(l) = \max_{x+y=l} \left(f_{1,2}(x) \oplus f_3(y)\right) \quad (15)$$

The results of these computations are given in Table 5. Now, let us compute $F_{1,2,3}(l)$, the optimal return on the investments in I, II, III and IV for different values of l by:

$$F_{1,2,3,4}(l) = \max_{x+y=l} \left(f_{1,2,3}(x) \oplus f_4(y)\right) \quad (16)$$

The computational results are given in the following Tables 1-5.

Table 1. Normalized HFN return on an investment for a period of three years

Inves-tment	Profit investment in I	Profit investment in II	Profit investment in III	Profit investment in IV
0	0	0	0	0
1	(0.22, 0.25, 0.26, 0.28, 0.29, 0.30, 0.42)	(0.15, 0.20, 0.21, 0.25, 0.25, 0.26, 0.28)	(0.10, 0.12, 0.125, 0.13, 0.14, 0.16, 0.17)	(0.16 , 0.19, 0.195, 0.20, 0.22, 0.24, 0.25)
2	(0.38, 0.40, 0.425, 0.41, 0.45, 0.48, 0.52)	(0.31, 0.33, 0.34, 0.35, 0.40, 0.43, 0.45)	(0.20, 0.21, 0.22, 0.23, 0.25, 0. 26, 0.28)	(0.33, 0.35, 0.36, 0.37, 0.39, 0.42, 0.48)
3	(0.55, 0.58, 0.59, 0.60, 0. 65, 0.71, 0.73)	(0.45, 0.48, 0.50, 0.55, 0.56, 0. 60)	(0.40, 0.43, 0.44, 0.45, 0.47, 0.52, 0.55)	(0.30, 0.35, 0.36, 0.37, 0.46, 0.48, 0.50)
4	(0.65, 0.70, 0.725, 0.71, 0.80, 0.85, 0.90)	(0.45, 0.50, 0.52, 0.55, 0.60, 0.67, 0.70)	(0.40, 0.45, 0.46, 0.47, 0.50, 0.51, 0.52)	(0.38, 0.40, 0.42, 0.44, 0.50, 0.52, 0.53)
5	(0.75, 0. 81, 0. 83, 0.84, 0.85, 1.01, 1.03)	(0.58, 0.60, 0.61, 0.65, 0.75, 0. 76, 0.80)	(0.50, 0.53, 0.54, 0.55, 0.65, 0.66, 0.70)	(0.50, 0.51, 0.52, 0.525, 0.53, 0.58, 0.60)
6	(0.90, 0.95, 0.97, 0.98, 1.05, 1.11, 1.15)	(0.65, 0.70, 0.72, 0.73, 0.85, 0.90, 0.95)	(0.65, 0.70, 0.71, 0.72, 0.73, 0. 74, 0.77)	(0.50, 0.55, 0.56, 0.565, 0.57, 0.58, 0.60)
7	(0.90, 0.95, 1.06, 1.07, 1.11, 1.16, 1.2)	(0.80, 0.83, 0.84, 0.85, 0.87, 0.90, .95)	(0.73, 0.76, 0. 77, 0.79, 0.81, 0.83, 0.85)	(0.50, 0.56, 0.57, 0.575, 0.58, 0.59, 0.60)
8	(1.00, 1.10, 1.22, 1.25, 1.27 1.30, 1.35),	(0.80, 0.85, 0.86, 0.87, 0.89, 0.90, 1.0)	(0.80, 0.89, 0.92, 0.93, 0.94, 0.95, 0.98)	(0.53, 0.85, 0. 59, 0.595, 0.60, 0.61, 0.63)

9	(1.11, 1.124, 1.30, 1.33, 1.35, 1.42, 1.45)	(0.85, 0.88, 0.89, 0.90, 0.91, 0.93, 0.94)	(0.90, 0.95, 0.96, 0.98, 1.00, 1.02, 1.05)	(0.55, 0.58, 0.59, 0. 595, 0.60, 0.61, 0.63)
10	(1.15, 1.35, 1.39, 1.40, 1.47, 1.50, 1.55)	(0.85, 0.90, 0.91, 0.92, 0.93, 0.94, 1.00)	(0.95, 0.98, 1.00, 1.03, 1.05, 1.08, 2.00)	(0.55, 0.59, 0.60, 0.61, 0.63, 0.64, 0.65)

Table 2. An inexact rough interval returns on an investment for a period of three years

Invest ment	Profit investment in I	Profit investment in II	Profit investment in III	Profit investment in IV
0	0	0	0	0
1	[[0.25, 0.30] : [0.26, 0.29]]	[[0.20, 0.26] : [0.21, 0.25]]	[[0.12, 0.16] : [0.13, 0.14]]	[[0.19, 0.24] : [0.20, 0.22]]
2	[[0.40, 0.48] : [0.41, 0.45]]	[[0.33, 0.43] : [0.35, 0.40]]	[[0.21, 0.26] : [0.22, 0.25]]	[[0.35, 0.42] : [0.36, 0.39]]
3	[[0.58, 0.71] : [0.59, 0.65]]	[[0.48, 0.60] : [0.50, 0.56]]	[[0.43, 0.52] : [0.45, 0.47]]	[[0.35, 0.48] : [0.36, 0.46]]
4	[[0.70, 0.85] : [0.71, 0.80]]	[[0.50, 0.67] : [0.55, 0.60]]	[[0.45, 0.51] : [0.46, 0.50]]	[[0.40, 0.52] : [0.42, 0.50]]
5	[[0.81, 1.01] : [0.83, 0.85]]	[[0.60, 0.76] : [0.96, 0.75]]	[[0.53, 0.66] : [0.54, 0.65]]	[[0.51, 0.58] : [0.52, 0.53]]
6	[[0.95, 1.11] : [0.97, 1.05]]	[[0.70, 0.90] : [0.72, 0.85]]	[[0.70, 0.74] : [0.71, 0.73]]	[[0.55, 0.58] : [0.56, 0.57]]
7	[[0.95, 1.16] : [1.06, 1.11]]	[[0.83, 0.90] : [0.84, 0.87]]	[[0.76, 0.83] : [0.77, 0.81]]	[[0.56, 0.59] : [0.57, 0.58]]
8	[[1.10, 1.30] : [1.22, 1.27]]	[[0.85, 0.90] : [0.86, 0.89]]	[[0.89, 0.95] : [0.92, 0.94]]	[[0.58, 0.61] : [0.59, 0.60]]
9	[[1.24, 1.42] : [1.30, 1.35]]	[[0.88, 0.93] : [0.89, 0.91]]	[[0.95, 1.02] : [0.96, 1.00]]	[[0.58, 0.61] : [0.59, 0.60]]
10	[[1.35, 1.50] : [1.39, 1.47]]	[[0.90, 0.94] : [0.91, 0.93]]	[[0.98, 1.08] : [1.00, 1.05]]	[[0.59, 0.64] : [0.60, 0.63]]

Table 3. Optimal policy using an inexact rough interval with investments in I & II

l	$f_1(x)$	$f_2(x)$	$F_{1,2}(l)$	Best policy for I including II
0	0	0	0	(0, 0)
1	[[0.25, 0.30] : [0.26, 0.29]]	[[0.20, 0.26] : [0.21, 0.25]]	[[0.25, 0.30] : [0.26, 0.29]]	(1, 0)
2	[[0.40, 0.48] : [0.41, 0.45]]	[[0.33, 0.43] : [0.35, 0.40]]	[[0.45, 0.56] : [0.47, 0.54]]	(1, 1)
3	[[0.58, 0.71] : [0.59, 0.65]]	[[0.48, 0.60] : [0.50, 0.56]]	[[0.60, 0.74] : [0.62, 0.70]]	(2, 1)
4	[[0.70, 0.85] : [0.71, 0.80]]	[[0.50, 0.67] : [0.55, 0.60]]	[[0.78, 0.97] : [0.80, 0.90]]	(3, 1)
5	[[0.81, 1.01] : [0.83, 0.85]]	[[0.60, 0.76] : [0.69, 0.75]]	[[0.91, 1.10] : [0.94, 1.05]]	(3, 2)
6	[[0.95, 1.11] : [0.97, 1.05]]	[[0.70, 0.90] : [0.72, 0.85]]	[[1.06, 1.31] : [1.09, 1.21]]	(3, 3)
7	[[0.95, 1.16] : [1.06, 1.11]]	[[0.83, 0.90] : [0.84, 0.87]]	[[1.18, 1.45] : [1.21, 1.36]]	(4, 3)
8	[[1.10, 1.30] : [1.22, 1.27]]	[[0.85, 0.90] : [0.86, 0.89]]	[[1.29, 1.61] : [1.33, 1.41]]	(5, 3)
9	[[1.24, 1.42] : [1.30, 1.35]]	[[0.88, 0.93] : [0.89, 0.91]]	[[1.43, 1.71] : [1.47, 1.61]]	(6, 3)
10	[[1.35, 1.50] : [1.37, 1.47]]	[[0.90, 0.94] : [0.91, 0.93]]	[[1.45, 1.78] : [1.47, 1.65]]	(6, 4)

Table 4. Optimal policy by an inexact rough interval with investments in I, II and III

l	$F_{1,2}(l)$	$f_3(x)$	$F_{1,2,3}(l)$	Best policy for I & II
0	0	0	0	(0, 0, 0)
1	[[0.25, 0.30] : [0.26, 0.29]]	[[0.12, 0.16] : [0.13, 0.14]]	[[0.25, 0.30] : [0.26, 0.29]]	(1, 0, 0)
2	[[0.45, 0.56] : [0.47, 0.54]]	[[0.21, 0.26] : [0.22, 0.25]]	[[0.45, 0.56] : [0.47, 0.54]]	(1, 1, 0)
3	[[0.60, 0.74] : [0.62, 0.70]]	[[0.43, 0.52] : [0.45, 0.47]]	[[0.60, 0.74] : [0.62, 0.70]]	(2, 1, 0)
4	[[0.78, 0.97] : [0.80, 0.90]]	[[0.45, 0.51] : [0.46, 0.50]]	[[0.78, 0.97] : [0.80, 0.90]]	(3, 1, 0)
5	[[0.91, 1.10] : [0.94, 1.05]]	[[0.53, 0.66] : [0.54, 0.65]]	[[0.91, 1.10] : [0.94, 1.05]]	(3, 2, 0)
6	[[1.06, 1.31] : [1.09, 1.21]]	[[0.70, 0.74] : [0.71, 0.73]]	[[1.03, 1.26] : [1.07, 1.17]]	(2, 1, 3)
7	[[1.18, 1.45] : [1.21, 1.36]]	[[0.76, 0.83] : [0.77, 0.81]]	[[1.21, 1.49] : [1.25, 1.37]]	(3, 1, 3)
8	[[1.29, 1.61] : [1.33, 1.41]]	[[0.89, 0.95] : [0.90, 0.94]]	[[1.34, 1.66] : [1.39, 1.52]]	(3, 2, 3)
9	[[1.43, 1.71] : [1.47, 1.61]]	[[0.95, 1.02] : [0.96, 1.00]]	[[1.49, 1.83] : [1.54, 1.68]]	(3, 3, 3)
10	[[1.45, 1.78] : [1.47, 1.65]]	[[0.98, 1.08] : [1.00, 1.05]]	[[1.61, 1.97] : [1.66, 1.83]]	(4, 3, 3)

Table 5. Optimal policy using an inexact rough interval with investments in I, II and III

l	$F_{1,2,3}(l)$	$f_4(x)$	$F_{1,2,3,4}(l)$	Best policy for I including II
0	0	0	0	(0, 0, 0, 0)
1	[[0.25, 0.30] : [0.26, 0.29]]	[[0.19, 0.24] : [0.20, 0.22]]	[[0.25, 0.30] : [0.26, 0.29]]	(1, 0, 0, 0)
2	[[0.45, 0.56] : [0.47, 0.54]]	[[0.35, 0.42] : [0.36, 0.39]]	[[0.45, 0.56] : [0.47, 0.54]]	(1, 1, 0, 0)
3	[[0.60, 0.74] : [0.62, 0.70]]	[[0.35, 0.48] : [0.36, 0.46]]	[[0.64, 0.80] : [0.67, 0.76]]	(1, 1, 0, 1)
4	[[0.78, 0.97] : [0.80, 0.90]]	[[0.40, 0.52] : [0.42, 0.50]]	[[0.79, 0.98] : [0.82, 0.92]]	(2, 1, 0, 1)
5	[[0.91, 1.10] : [0.94, 1.05]]	[[0.51, 0.54] : [0.52, 0.53]]	[[0.97, 1.21] : [1.00, 1.12]]	(3, 1, 0, 1)
6	[[1.03, 1.26] : [1.07, 1.17]]	[[0.55, 0.58] : [0.56, 0.57]]	[[1.13, 1.39] : [1.16, 1.29]]	(3, 1, 0, 2)
7	[[1.21, 1.49] : [1.25, 1.37]]	[[0.56, 0.59] : [0.57, 0.58]]	[[1.26, 1.56] : [1.30, 1.44]]	(3, 2, 0, 2)
8	[[1.34, 1.66] : [1.39, 1.52]]	[[0.58, 0.61] : [0.59, 0.60]]	[1.40, 1.73] : [1.45, 1.59]]	(3, 1, 3, 1)
9	[[1.49, 1.83] : [1.54, 1.68]]	[[0.58, 0.61] : [0.59, 0.60]]	[[1.56, 1.91] : [1.61, 1.76]]	(3, 1, 3, 2)
10	[[1.61, 1.97] : [1.66, 1.83]]	[[0.59, 0.64] : [0.60, 0.63]]	[[1.69, 2.08] : [1.75, 1.91]]	(3, 2, 3, 2)

5. Results and discussion

In this Section, we present the results and discussion. The optimal investment amount of $ 10 million based on the comparative study presented in Table 4. It is observed that the amount $ 3 million in I including an inexact rough interval best possible return can be expressed as follows:

$ [[0.85, 0.71]: [0.59, 0.65]] millions,

$ 2 million in II with an inexact rough interval optimal value of return

$ [[0.33, 0.34]: [0.35, 0.40]] millions,

$ 3 million in III with an inexact rough interval optimal value of return

$ [[0.43, 0.52]: [0.45, 0.47]] millions,

$ 2 million in IV with an inexact rough interval optimal value of return

$ [[0.35, 0.42]: [0.36, 0.39],

Thus, the total optimal return with inexact rough intervals on an amount of $ 10 million investments is as follows: $ [[1.69, 2.08]: [1.75, 1.91]] millions.

6. Concluding remarks

In the present study, an IP with inexact rough intervals has been introduced. A DP approach has been applied to obtain an inexact rough interval optimal return. In existing approaches, the DM faces a problem including ambiguity in the data of the problem, where the proposed approach resolves this issue by handling the data with roughness. These are the main advantages of the suggested approach. The entire process of optimization has illustrated by a numerical example. The researchers well applied the DP in the investment managements and found the good result to the optimal return to the investor when they invest the money in institution or small business. At last, we say that the approach is good and give the idea to investor to invest the money to the small business. The use of R statistics for solving the investment problem in vague environment is a good choice and it is a very solid tool. Also, the use of DP is very successful. There are many future research directions. The suggested process has can be extended to other types of investment problems by introducing time, discounting, special constraints, etc. Some directions of further research include stochastic parameters, intuitionistic fuzzy sets, fuzzy random variable, etc.

References

[1] A. B. Abe, and J. C. Eberly, A unified model of investment under uncertainty. American Economic Review, 84, 1369-1384 (1994).

[2] E. E. Ammar, and H. A. Khalifa, Characterization of optimal solutions of uncertainty investment problem. Applied Mathematics and Computation, 160, 111-124 (2005).

[3] S. Bernstein, J. Lerner, and A. Schoar, The investment strategies of sovereign wealth funds. J. of Economics Perspective, 27, 219-239 (2013).

[4] R. W. Cooper, and J. Haltiwanger, On the nature of capital adjustment costs. Review of Economic Studies, 73, 611-634 (2006).

[5] Y. Dajun, L. Decui, and P. Hu, Three-way decisions with interval-valued intuitionistic fuzzy decision-theoretic rough sets in group decision-making. Symmetry, 10, 1-23 (2018).

[6] D. P. Farias, and B. V. Roy, The linear programming approach to approximate dynamic programming. Operations Research, 51, 850-865 (2004).

[7] A. A. F. Fini, A. Akbarnezhad, T. H. Rashidi, and W. S. T. Travis, Dynamic programming approach toward optimization of workforce planning decisions. J. of Construction Engineering and Management, 144(2), 04017113-14 (2018).

[8] Y. Gocgun, Dynamic scheduling with cancellations: an application to chemotherapy appointment booking. Int. J. of Optimization and Control: Theories & Applications, 8(2), 161-169 (2018).
[9] P. A. Gompers, and M. Andrew, Institutional invests and equity prices. Quarterly Journal of Economics, 116(1), 229-259 (2001).
[10] J. T. Haahr, D. Pisinger, and M. Sabbaghian, A dynamic programming approach for optimizing train speed profiles with speed restrictions and passage points. Transportation Research Part B: Methodological, 99, 167-182 (2017).
[11] M. A. Hussah, On solving fuzzy investment problem using dynamic programming. Int. J. of Computer Applications, 181(22), 14-20 (2018).
[12] K. Jabbarova, and N. Hasanova, An application of the VIKOR method to decision making in investment problem under Z-valued information. In book: 13th Int. Conf. on Theory and Application of Fuzzy Systems and Soft Computing- ICAFS-2018.
[13] C. Kahraman, D. Ruan, and E. C. Bozdag, Optimization of multilevel investments using dynamic programming based on fuzzy cash flows. Fuzzy Optimization and Decision Making, 2, 101-122 (2003).
[14] A. Kaufmann, and M. M. Gupta, Fuzzy Mathematical Models in Engineering and Management Science, Elsevier Science Publishing Company Inc., 1988, New York.
[15] H. A. Khalifa, P. Kumar, F. Smarandache, On optimizing neutrosophic complex programming using lexicographic order. Neutrosophic Sets and Systems, 32, 330-343 (2020).
[16] J. Lee, K. and Shin, The role of a variable input in the relationship between investment and uncertainty. American Economic Review, 90, 667-680 (2000).
[17] I. Aynur, and A. I. Jabbarova, Solution for the investment decision making problem through interval probabilities. Procedia Computer Science, 102, 465-468 (2016).
[18] H. A. Khalifa, G. A. Majed, and P. Kumar, Enhancement of capacitated transportation problem in fuzzy environment. Advances in Fuzzy Systems, Volume 2020; Article ID 8893976 (2020).
[19] H. A. Khalifa, A study on investment problem in chaos environment. J. of Applied Research on Industrial Engineering, 6(3), 177-183 (2020).
[20] H. A. Khalifa, G. A. Majed, and P. Kumar, A new approach for the optimization of portfolio selection problem in fuzzy environment. Advances in Mathematics: Scientific Journal, 9(9), 7171–7190 (2020).
[21] H. A. Khalifa, G. A. Majed, and P. Kumar, On determining the critical path of activity network with normalized heptagonal fuzzy data. Wireless Communications and Mobile Computing, Volume 2021, Article ID 6699403, 14 pages (2021).
[22] H. A. Khalifa, P. Kumar, and A. H. Bayoumi, An inexact rough interval of normalized heptagonal fuzzy numbers for solving vendor selection problem. Applied Mathematics & Information Sciences, 15(3), 317-324 (2021).
[23] Krzysztof K, Ludmila D, Pavel S. A simple view on the interval and fuzzy portfolio selection problems. Entropy, 22(932) (2020).
[24] H. Lu, G. Huang, and L. He, An inexact rough interval fuzzy linear programming method for generating conjunctive water allocation strategies to agricultural irrigation systems. Applied Mathematical Modeling, 35, 4330-4340 (2011).
[25] M. L. Naiyer, C. Ercan, and Y. Muhammed, Evaluation of investment opportunities with interval-valued fuzzy topsis method. Applied Mathematics and Nonlinear Sciences, 5(1), 461-474 (2020).
[26] K. Rathi, S. Balamohan, Representation and ranking of fuzzy numbers with heptagonal membership function using value and ambiguity index. Applied Mathematical Sciences, 8(87), 4309-4321 (2014).