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On Conjunctive and Disjunctive Rational Bivariate Aggregation Functions of Degrees (2,1)

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Abstract. In the last decades, many families of aggregation functions have been presented playing a fundamental role in many research fields such as decision making, fuzzy mathematical morphology, etc. For this reason, it is necessary to study different types of operators to be potentially used in a concrete application as well as the properties they can satisfy. In this paper, conjunctive and disjunctive rational bivariate aggregation functions of degree two in the numerator and degree one in the denominator are studied. In particular, a characterization of conjunctive and disjunctive rational aggregation functions of degrees (2,1) is presented. Moreover, the symmetry property of these operators are investigated.

Keywords. Aggregation function, Rational function, Symmetry

1. Introduction

Aggregation functions have been intensively investigated for about two decades, see [2, 3,4,6], playing a fundamental role in social and scientific sciences. There exist a great quantity of different aggregation functions satisfying different additional properties, each one having its particular importance and interest in different fields of application. In this sense, in this work we focus on two general classes: the class of conjunctive aggregation function (those that take values below the minimum that include well-known t-norms [5] or copulas [7]) and the class of disjunctive aggregation functions (those that take values over the maximum, that include, for instance, t-conorms or co-copulas [5]). All of them are used in meaningful applications in fuzzy logic and approximate reasoning, image processing, probability, statistics and economy [2,3].

In a previous work [1], rational binary aggregation functions, that is, those whose expression is given by the quotient of two bivariate polynomial functions were investigated. In particular, rational binary aggregation functions of degree $(1,1)$ (both bivariate polynomials have degree 1) were characterized. Also, concrete characterizations of those that are symmetric and idempotent were also studied. Following this investigation line,

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in this short paper we will study rational bivariate aggregation functions of degree two in the numerator and degree one in the denominator, with the focus on their characterization. These results are novel with respect to the ones presented in [1].

The paper is structured as follows. After recalling the basic definitions, in Section 3 conjunctive rational binary aggregation functions of degree $(2,1)$ and those that are symmetric will be characterized. Section 4 is devoted to characterize disjunctive rational binary aggregation functions of degree $(2,1)$ and those that are symmetric. The paper ends with some conclusions and future work we want to investigate.

2. Preliminaries

Let us recall some concepts and results that will be used throughout this paper. First, we give the definition of a binary aggregation function.

Definition 1 ([2,3]). *A binary aggregation function $f : [0,1]^2 \to [0,1]$ is a binary mapping that satisfies the following properties:*

i) $f(0,0) = 0$ and $f(1,1) = 1$.
ii) f is increasing in each variable.

A binary aggregation function f is a conjunction when $f(1,0) = f(0,1) = 0$ and a disjunction when $f(1,0) = f(0,1) = 1$. Two additional properties of binary aggregation functions which will be used in this work are the *symmetry*, $f(x,y) = f(y,x)$ for all $x,y \in [0,1]$ and the *idempotency*, $f(x,x) = x$ for all $x \in [0,1]$.

Now, we recall the definition of a rational aggregation function, introduced in [1].

Definition 2 (Definition 3 in [1]). *Consider $n,m \in \mathbb{N}$. A binary operator $R : [0,1]^2 \to [0,1]$ is called a* rational aggregation function of degree (n,m) *if it is an aggregation function and its expression is given by*

$$R(x,y) = \frac{p(x,y)}{q(x,y)} = \frac{\displaystyle\sum_{\substack{0\leq i,j\leq n\\ i+j\leq n}} a_{ij}x^i y^j}{\displaystyle\sum_{\substack{0\leq s,t\leq m\\ s+t\leq m}} b_{st}x^s y^t} \tag{1}$$

for all $x,y \in [0,1]$ where

(i) *$a_{ij} \in \mathbb{R}$ for all $0 \leq i,j \leq n$ and $i+j \leq n$ and there exists at least one a_{ij} with $0 \leq i,j \leq n$ and $i+j = n$ such that $a_{ij} = 1$,*
(ii) *$b_{st} \in \mathbb{R}$ for all $0 \leq s,t \leq m$ and $s+t \leq m$ and there exist some $0 \leq s,t \leq m$ with $s+t = m$ such that $b_{st} \neq 0$,*
(iii) *the polynomials $p(x,y)$ and $q(x,y)$ have no factors in common,*
(iv) *$q(x,y) \neq 0$ for all $x,y \in [0,1]$.*

3. Conjunctive Rational Binary Aggregation Functions of degree (2,1)

In this section we will characterize the conjunctive binary rational aggregation functions of degree $(2,1)$. Thus, the expression of the rational function $R(x,y)$ can be written as

$$R(x,y) = \frac{a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00}}{b_{00} + b_{10}x + b_{01}y}$$

for all $x,y \in [0,1]$ where each $a_{ij}, b_{ij} \in \mathbb{R}$. Based on this expression, next result provides a characterization of conjunctive rational aggregation functions of degree $(2,1)$.

Theorem 1. *A binary operator $R : [0,1]^2 \to [0,1]$ is a conjunctive rational binary aggregation function of degree $(2,1)$ if, and only if, R is given by*

$$R(x,y) = \frac{(b_{00} + b_{10} + b_{01})xy}{b_{00} + b_{10}x + b_{01}y} \tag{2}$$

for all $x,y \in [0,1]$ where the coefficients satisfy $b_{00} > 0$, $b_{00} + b_{01} > 0$, $b_{00} + b_{10} > 0$, $b_{00} + b_{10} + b_{01} > 0$ and each $b_{ij} \in \mathbb{R}$.

The following result characterizes the symmetric conjunctive rational binary aggregation functions of degree $(2,1)$.

Proposition 1. *A binary operator $R : [0,1]^2 \to [0,1]$ is a symmetric conjunctive rational binary aggregation function of degree $(2,1)$ if, and only if, R is given by*

$$R(x,y) = \frac{(2b_{10} + b_{00})xy}{b_{10}(x+y) + b_{00}}$$

for all $x,y \in [0,1]$ where $b_{00} > 0$, $b_{00} + b_{10} > 0$, $b_{00} + 2b_{10} > 0$ and each $b_{ij} \in \mathbb{R}$.

Proposition 2. *There are no idempotent conjunctive rational aggregation functions of degree (2,1).*

4. Disjunctive Rational Binary Aggregation Functions of degree (2,1)

In this section we will characterize disjunctive binary rational aggregation functions of degree $(2,1)$.

Theorem 2. *A binary operator $R : [0,1]^2 \to [0,1]$ is a disjunctive rational binary aggregation function of degree $(2,1)$ if, and only if, R is given by*

$$R(x,y) = \frac{a_{11}xy + a_{10}x + a_{01}y}{(a_{11} + a_{10})x + (a_{11} + a_{01})y - a_{11}} \tag{3}$$

for all $x,y \in [0,1]$ where the coefficients satisfy $a_{11} < 0$, $a_{10} > 0$, $a_{01} > 0$, $a_{10} + a_{01} + a_{11} > 0$, $a_{10} + a_{11} > 0$, $a_{01} + a_{11} > 0$ and each $a_{ij} \in \mathbb{R}$.

Next proposition characterizes the symmetric disjunctive rational binary aggregation functions of degree $(2,1)$.

Proposition 3. *A binary operator* $R : [0,1]^2 \to [0,1]$ *is a symmetric disjunctive rational binary aggregation function of degree* $(2,1)$ *if, and only if, R is given by*

$$R(x,y) = \frac{a_{11}xy + a_{10}(x+y)}{(a_{11}+a_{10})(x+y) - a_{11}}$$

for all $x, y \in [0,1]$ *where the coefficients satisfy* $a_{11} < 0$, $a_{10} > 0$, $2a_{10} + a_{11} > 0$, $a_{10} + a_{11} > 0$ *and each* $a_{ij} \in \mathbb{R}$.

Proposition 4. *There are no idempotent disjunctive rational aggregation functions of degree (2,1).*

The following result relates both families of aggregation functions.

Proposition 5. $R(x,y)$ *is a disjunctive rational aggregation functions of degree (2,1) iff* $1 - R(1-x, 1-y)$ *is a conjunctive rational aggregation functions of degree (2,1).*

5. Conclusions and future work

Following with the investigation on rational aggregation functions started in [1], in this work we have characterized all conjunctive and disjunctive rational binary aggregation functions of degree (2,1) and, in particular, we have also characterized those that are symmetric. As future work, first, we want to further explore different additional properties of these operators such as associativity or the existence of neutral or absorbing elements. Second, we want to analyze their performance in edge detection through fuzzy morphological operators. Finally, rational binary aggregation functions of higher degrees could be worthy to study although the complexity of the results increases drastically.

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