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Ordinal Conditional Functions for Abstract Argumentation

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Abstract. We interpret and formalise *ordinal conditional functions (OCFs)* in abstract argumentation frameworks based on ideas and concepts defined for *conditional logics*. There, these functions are used to rank interpretations, and we adapt them to rank extensions instead. Using conflict-freeness and admissibility as two essential principles to define the semantics of OCFs, we obtain a framework that allows to rank sets of arguments wrt. their plausibility. We analyse the properties of this framework in-depth, and in doing so we establish a formal bridge between the approaches of abstract argumentation and conditional logics.

Keywords. Abstract Argumentation, Ranking Functions

1. Introduction

Abstract argumentation frameworks (AF) [1] have gathered research interest as a model for rational decision-making in presence of conflicting information. Using AFs, *arguments* and *attacks* can be represented as nodes and edges, respectively, of a directed graph. In order to reason over AFs *extension semantics* were defined, which are functions such that a set of arguments can be considered jointly acceptable. Recently Skiba et al. [2] generalised this reasoning process to rank sets of arguments based on their plausibility. Another used reasoning formalism is *conditional logic*, which studies conditionals like "if A then B" written as $(B \mid A)$. So given the information that A is true it is more "believable" that B is true, than B being not true. In order to define a value of believability, *ordinal conditional functions* (OCF) (also known as *ranking functions*) were defined [3]. These functions can be used to rank possible worlds according to their plausibility. One example of an OCF is the *System Z* ranking function [4], which exhibits particularly good reasoning properties.

In recent years, the relationship between argumentation and conditional logic was investigated in [5,6,7,8] and by Weydert [9,10]. While abstract argumentation usually only provides a criterion to determine whether a set of arguments is jointly accepted or not, OCFs on the other hand can rank possible worlds according to their plausibility. In this paper, we want to use these ideas from conditional logic to reason in abstract argumentation. Our goal is to rank sets of arguments according to their plausibility, i. e., we can state not only whether a set of arguments is accepted or not, but also state that a set is more plausible than another one. In particular, we can rank sets of arguments, which are not jointly acceptable w.r.t. extension semantics, for example, we can say that out of two conflicting sets one of them is more plausible. One potential application of such a ranking is decision-making in presence of constraints, where a solution (represented as a set of arguments) satisfies constraints that cannot be satisfied by a set of arguments under extension semantics. One possible way to still make a decision would be to select the most plausible sets of arguments, which are satisfying the constraints.

To achieve such a ranking of sets of arguments, we will use the guiding principles of admissible reasoning for abstract argumentation frameworks namely *conflict-freeness* and *admissibility* to develop ordinal conditional functions for abstract argumentation. In order to connect abstract argumentation and conditional logics we interpret the set of attacks as a set conditionals. Since there can be a number of functions satisfying the defined principles, we develop a model-based reasoning technique inspired by System Z. This System Z ranking function allows us to model plausibility values for each set of arguments I being *in* while a different set O is *out*. These plausibility values can be used to rank sets of arguments, and therefore continue recent work about extension-ranking semantics started in [2].

This paper is structured as follows. We recall all necessary preliminaries about abstract argumentation and conditional logics in Section 2. Section 3 introduces OCFs for abstract argumentation. In Section 4, we look at OCFs based on System Z. A extension-ranking semantics is introduced in Section 5 as well as an in-depth investigation of the properties of that semantics is presented. We conclude this paper in Section 6 with a discussion about related work.

2. Preliminaries

In this section, we recall all necessary definitions of abstract argumentation and conditional logics.

2.1. Abstract Argumentation

Argumentation frameworks [1] are a formalism that allows the representation of conflicts between pieces of information using arguments and attacks between arguments.

Definition 1. An *abstract argumentation framework* (AF) is a directed graph $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ where A is a finite set of *arguments* and R is an *attack relation* $\mathsf{R} \subseteq \mathsf{A} \times \mathsf{A}$.

An argument a is said to *attack* an argument b if $(a,b) \in \mathsf{R}$. We say that an argument a is *defended by a set* $S \subseteq \mathsf{A}$ if every argument $b \in \mathsf{A}$ that attacks a is attacked by some $c \in S$. For $a \in \mathsf{A}$ define $a^- = \{b \mid (b,a) \in \mathsf{R}\}$ and $a^+ = \{b \mid (a,b) \in \mathsf{R}\}$, so the set of attackers of a and the set of arguments attacked by a. For a set of arguments $S \subseteq \mathsf{A}$ we extend these definitions to S^+ and S^- via $S^+ = \bigcup_{a \in S} a^+$ and $S^- = \bigcup_{a \in S} a^-$, respectively. For two graphs $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ and $\mathsf{AF}' = (\mathsf{A}', \mathsf{R}')$, we define $\mathsf{AF} \cup \mathsf{AF}' = (\mathsf{A} \cup \mathsf{A}', \mathsf{R} \cup \mathsf{R}')$.

To reason with abstract argumentation frameworks a number of different semantical notions have been developed, like the *extension-based* or the *labelling-based* approaches, for an overview see [11]. Both these approaches are handling sets of arguments, which can be considered jointly acceptable. The extension-based semantics are relying on two basic concepts: *conflict-freeness* and *defence*.

Definition 2. Given $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$, a set $E \subseteq \mathsf{A}$ is:

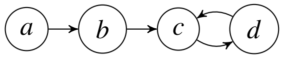

Figure 1. Abstract argumentation framework AF from Example 1.

- *conflict-free* iff $\forall a,b \in E$, $(a,b) \notin \mathsf{R}$;
- *admissible* iff it is conflict-free, and it defends its elements.

We use cf(AF) and ad(AF) for denoting the sets of conflict-free and admissible sets of an argumentation framework AF, respectively. The semantics proposed by Dung [1] are then defined as follows.

Definition 3. Given $\mathsf{AF} = (\mathsf{A},\mathsf{R})$, an admissible set $E \subseteq \mathsf{A}$ is a *complete* extension (co) iff it contains every argument that it defends; a *preferred* extension (pr) iff it is a $\subseteq$-maximal complete extension; the unique *grounded* extension (gr) iff it is the $\subseteq$-minimal complete extension; a *stable* extension (st) iff $E^+ = \mathsf{A} \setminus E$.

The sets of extensions of an argumentation framework AF, for these four semantics, are denoted (respectively) co(AF), pr(AF), gr(AF) and st(AF). Based on these semantics, we can define the status of any (set of) argument(s), namely *skeptically accepted* (belonging to each σ-extension), *credulously accepted* (belonging to some σ-extension) and *rejected* (belonging to no σ-extension). Given an argumentation framework AF and an extension semantics σ, we use (respectively) $\mathrm{sk}_\sigma(\mathsf{AF})$, $\mathrm{cr}_\sigma(\mathsf{AF})$ and $\mathrm{rej}_\sigma(\mathsf{AF})$ to denote these sets of arguments.

Example 1. Consider the argumentation framework $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ depicted as a directed graph in Figure 1, with the nodes corresponding to the arguments $\mathsf{A} = \{a,b,c,d\}$, and the edges corresponding to the attacks $\mathsf{R} = \{(a,b),(b,c),(c,d),(d,c)\}$. The sets $\{a\}$, $\{a,c\}$ and $\{a,d\}$ are complete extensions of AF, while only $\{a,c\}$ and $\{a,d\}$ are stable.

For more details about these semantics (and other ones defined in the literature), we refer the interested reader to [1,11].

2.2. Conditional Logics

In order to define the usual *propositional language* $\mathscr{L}(At)$ over At we use a set of atoms At and connectives $\wedge$ (and), $\vee$ (or) and $\neg$ (negation). The function $\omega : At \to \{T,F\}$ defines an *interpretation* (or *possible world*) ω for $\mathscr{L}(At)$. $\Omega(At)$ denotes the set of all interpretations. An interpretation ω *satisfies* an atom $a \in At$ ($\omega \models a$), iff $\omega(a) = T$. As a *conditional* we consider structures like $(\psi \mid \phi)$, which can be read as "if ϕ then (usually) ψ". Informally speaking, an interpretation ω *verifies* a conditional $(\psi \mid \phi)$ iff it satisfies both antecedent (ϕ) and conclusion (ψ) ($(\psi \mid \phi)(\omega) = 1$); it *falsifies* iff it satisfies the antecedent but not the conclusion ($(\psi \mid \phi)(\omega) = 0$); otherwise the conditional is *not applicable*. A conditional is satisfied by ω if it does not falsify it.

We use *ordinal conditional functions (OCFs)* (also called *ranking functions*) [3], $\kappa : \Omega(At) \to \mathbb{N} \cup \{\infty\}$ to denote a plausibility degree of interpretations and define $\kappa(\phi) := min\{\kappa(\omega) \mid \omega \models \phi\}$. An OCF κ *satisfies* a set Δ of conditionals, if for each $(\psi \mid \phi) \in \Delta$, $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$, i. e., verifying a conditional is more plausible than falsifying it. Since the set of all satisfying OCFs may be difficult to handle, one usually relies on model-based inference for reasoning. In this paper, we will focus on the System Z ranking function [4] as an example for model-based inference.

Definition 4. A conditional $(\psi \mid \phi)$ is *tolerated* by a finite set of conditionals Δ if there is a possible world ω, which verifies $(\psi \mid \phi)$ and does not falsify any conditional $(\psi' \mid \phi') \in \Delta$. The *Z-Partitioning* $(\Delta_0,...,\Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$
- $\Delta_1,...,\Delta_n$ is the Z-Partitioning of $\Delta \setminus \Delta_0$

For $\delta \in \Delta : Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $(\delta_0,...,\delta_n)$ is the Z-Partitioning of Δ. We define a ranking function κ^Z_Δ via $\kappa^Z_\Delta(\omega) = max\{Z_\Delta(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $max\, \emptyset = -1$.

Example 2 ([4])**.** Consider the following set of conditionals Δ about the flying ability of penguins.

δ_1: "birds fly" $(f \mid b)$ δ_2: "penguins are birds" $(b \mid p)$
δ_3: "penguins do not fly" $(\neg f \mid p)$

The Z-Partitioning of Δ is $\Delta_0 = \{\delta_1\}$ and $\Delta_1 = \{\delta_2, \delta_3\}$, because Δ_0 can be tolerated by all conditionals, while δ_2 and δ_3 cannot be tolerated by Δ. We can calculate the plausibility value of interpretations ω. For example, a flying penguin ($\omega(p) = \omega(b) = \omega(f) = T$) receives a value of $\kappa^Z_\Delta(\omega) = 1$.

3. Ordinal Conditional Functions for Abstract Argumentation

In this section we define OCFs for abstract argumentation. We define a function κ with two parameters (I and O) to calculate a numerical plausibility value. These parameter are sets of arguments where the first set I is considered *in*, and the second set is considered *out*. So $\kappa(I,O) = 0$ means that the set I being *in* and the set O being *out* is not surprising. Note that our OCF need two parameters instead of only one, like in conditional logics, since abstract argumentation is missing the notion of negation.

Definition 5. Let $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ be an AF. A *OCF* for AF is a function $\kappa : 2^{\mathsf{A}} \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$.

For sets $I, O \subseteq \mathsf{A}$ we abbreviate

$$\kappa(I,O) = min\{\kappa(S) | I \subseteq S, S \cap O = \emptyset\}$$

$$\kappa(I,O) = \infty \text{ if } I \cap O \neq \emptyset$$

Example 3. Consider $\mathsf{AF}_2 = (\{a,b\},\{(a,b)\})$. One exemplary OCF $\kappa^C(I,O)$ returns the number of conflicts in I, i. e., for $\{a,b\}$ $\kappa^C(\{a,b\},\emptyset) = 1$. For any other set $S \subset \{a,b\}$ like $S = \{a\}$ we have $\kappa^C(S,\emptyset) = \kappa^C(S,\{b\}) = 0$ and $\kappa^C(S,S) = \infty$.

The following definitions are inspired by OCFs in conditional logic. However, while conditional logic semantics follow a single principle regarding conditional acceptance ("a conditional is accepted if its verification is more plausible than its violation"), for admissible reasoning in abstract argumentation we have two guiding principles:

- An argument should not be accepted if one of its attackers is accepted.

i	$\kappa^{-1}(i)$
3	$(\{a,b\},\emptyset)$
2	$(\emptyset,\{a\})$, $(\emptyset,\{b\})$, $(\emptyset,\{a,b\})$
1	$(\emptyset,\emptyset)$, $(\{b\},\{a\})$, $(\{b\},\emptyset)$
0	$(\{a\},\emptyset)$, $(\{a\},\{b\})$

Table 1. Example OCF for Example 4. Note κ is only partially defined.

- An argument should be accepted if all its attackers are not accepted.

The first principle is also called *conflict-freeness*, i. e., a set does not contain two arguments, which share an attack. So conflicting sets are less plausible than conflict-free sets. The second principle is *admissibility*, so a set, which defends itself from all possible attackers, is at least as plausible as set not defending itself. Implementing these two principles for OCFs gives us:

Definition 6. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an AF and $a, b \in \mathsf{A}$.

- An OCF κ *accepts* an attack (a,b) with $a \neq b$ if $\kappa(\{a\},\{b\}) < \kappa(\{a,b\},\emptyset)$.
- An OCF κ *possibly reinstates* an argument $a \in \mathsf{A}$ if $\kappa(S \cup \{a\}, a^-) \leq \kappa(S, \{a\} \cup a^-)$ for all $S \subseteq \mathsf{A}$ with $S \cap (a^- \cup a^+) = \emptyset$.

Intuitively, for an OCF to accept an attack (a,b) means that it is more plausible that argument a is *in* and b is *out*, than both a and b being *in* at the same time. For an OCF to possibly reinstate an argument a means that if all attackers of a are *out*, then it is at least as plausible that a is *in* than *out*.

Next we want to denote when an AF is satisfied by an OCF, i.e. when we can define an OCF satisfying all principles defined above for an AF.

Definition 7. An OCF κ *satisfies* an argumentation framework $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ if it accepts all attacks in R and possibly reinstates all arguments in A.

Example 4. Consider $\mathsf{AF}_2 = (\{a,b\}, \{(a,b)\})$. So the following statements have to hold:

1. $\kappa(\{a\},\{b\}) < \kappa(\{a,b\},\emptyset)$
2. $\kappa(\{a\},\emptyset) \leq \kappa(\emptyset,\{a\})$
3. $\kappa(\{b\},\{a\}) \leq \kappa(\emptyset,\{a,b\})$

Table 1 depicts an OCF that satisfies AF_2.

Note that if an AF contains a self-attacking argument a, then there is no OCF that satisfies it. Because to accept attack (a,a) it has to hold that $\kappa(\{a\},\{a\}) < \kappa(\{a\},\emptyset)$, which is impossible, since $\kappa(\{a\},\{a\}) = \infty$.

4. The System Z Ranking Function for Abstract Argumentation

In this section, we want to define an OCF inspired by System Z. The basic idea of System Z is that a conditional $(B \mid A)$ is *tolerated* by a set of conditionals, if it is confirmed by a world ω and no other conditional is refuted. In our setting of abstract argumentation we interpret an attack from argument a to argument b as the conditional relationship "if a is acceptable then b should not be acceptable". So the whole attack relation can be interpreted as a set of conditionals. To tolerate an attack, we have to find a set of arguments, which verifies an attack while not violating any other attack. In addition, we use a similar idea to the admissible semantics from Dung. Recall, a set is admissible iff all arguments contained in the set are defended by the set. We add another condition for a set S to tolerate an attack, namely *attack admissibility*, which states that if all attackers of an argument are not in S, then this argument should be included in S.

We begin with defining, when an attack is satisfied by a set S.

Definition 8. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework.

- A set $S \subseteq \mathsf{A}$ *verifies* an attack (a,b) iff $a \in S$ and $b \notin S$.
- A set $S \subseteq \mathsf{A}$ *violates* an attack (a,b) iff $a \in S$ and $b \in S$.
- A set $S \subseteq \mathsf{A}$ *satisfies* an attack (a,b) iff it does not violate it.

Intuitively speaking, a set satisfies an attack if this set does not contain any two conflicting arguments. So for an AF $\mathsf{AF}_3 = (\{a,b,c\},\{(a,b),(b,c)\})$, we can observe, that the set $S_1 = \{a\}$ verifies the attack (a,b) and does not violate the attack (b,c), while the set $S_2 = \{a,b\}$ verifies the attack (b,c), however S_2 violates attack (a,b).

To satisfy attack admissibility of an argument, we know that, if all the attackers of the argument are *out*, then the argument itself should be *in*.

Definition 9. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework.

- A set $S \subseteq \mathsf{A}$ *verifies attack admissibility* of $a \in \mathsf{A}$ iff $a \in S$ and $b \notin S$ for all $b \in a^-$.
- A set $S \subseteq \mathsf{A}$ *violates attack admissibility* of $a \in \mathsf{A}$ iff $a \notin S$ and $b \notin S$ for all $b \in a^-$.
- A set $S \subseteq \mathsf{A}$ *satisfies attack admissibility* of $a \in \mathsf{A}$ iff it does not violate it.

We recall $\mathsf{AF}_3 = (\{a,b,c\},\{(a,b),(b,c)\})$, we see that the set $S_3 = \{a,c\}$ verifies attack admissibility of argument c, because the only attacker of c, b is not part of S_3 and one of b's attackers is contained in S_3.

Now we combine both these definitions and define when an attack can be tolerated.

Definition 10. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework. A set $P \subseteq \mathsf{R}$ *tolerates* an attack (a,b) iff there is a set $S \subseteq \mathsf{A}$ that

1. verifies (a,b),
2. satisfies each attack in P, and
3. satisfies attack admissibility of each $c \in \mathsf{A}$

So in order to tolerate an attack, we need to find a set of arguments S s.t. S is conflict-free and every argument contained in S has to be defended. Recall $\mathsf{AF}_3 = (\{a,b,c\},\{(a,b),(b,c)\})$, then the attack (b,c) is not tolerated by $\{(a,b),(b,c)\}$. Because, for (b,c) to be verified for any set S, it has to hold that $b \in S$. Then to not violate (a,b) a is not allowed to be contained in S. However, then we have the problem that S does not contain any attackers of a, meaning that attack admissibility of a is violated.

With these definitions, we can define the OCF κ^Z for an AF AF.

Definition 11. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework. Then the Z-attack-Partitioning $(\mathsf{R}_0, \ldots, \mathsf{R}_n)$ with $\mathsf{R}_0 \cup \ldots \cup \mathsf{R}_n \subseteq \mathsf{R}$ is defined as

i	$\kappa^{-1}(i)$
2	$(\{b,c\},X),(\{a,b,c\},X),(\{b,c,d\},X),(\{a,b,c,d\},X)$
1	$(\{a,b\},X),(\{c,d\},X),(\{a,b,d\},X),(\{a,c,d\},X)$
0	$(\emptyset,X),(\{a\},X),(\{b\},X),(\{c\},X),(\{d\},X),(\{a,c\},X),(\{b,d\},X),(\{a,d\},X)$

Table 2. The OCF κ^Z, where for every pair (I,X) $X \subseteq \mathsf{A}$ is any set s.t. $I \cap X = \emptyset$.

- $\mathsf{R}_0 = \{r \in \mathsf{R} \mid \mathsf{R} \text{ tolerates } r\}$
- $(\mathsf{R}_1,\dots,\mathsf{R}_n)$ is the Z-attack-Partitioning of $\mathsf{R} \setminus \mathsf{R}_0$

For $r \in \mathsf{R}$ define $Z_{\mathsf{R}}(r) = i$ if $r \in \mathsf{R}_i$ and

$$\kappa^Z(S,X) = \max\{Z(r) \mid S \text{ violates } r\} + 1$$

where $X \subseteq \mathsf{A}$ is any set s.t. $S \cap X = \emptyset$.

So all attacks in R_0 are tolerated by the set of attacks of AF, while attacks in R_1 are only tolerated if we remove all attacks from R_0. Now we can state when a set of arguments is more plausible than another one, i. e., if the first set violates either no attacks or only attacks which are in lower levels. In a sense these levels represent the impact of each attack in an AF. Hence, it is more important to satisfy a single highly ranked attack than to satisfy multiple lowly ranked attacks.

Example 5. Consider Example 1 again. The Z-attack-Partitioning of R is $(\mathsf{R}_0,\mathsf{R}_1)$ with

$$\mathsf{R}_0 = \{(a,b),(c,d),(d,c)\}$$
$$\mathsf{R}_1 = \{(b,c)\}$$

Table 2 depicts κ^Z_{AF} for AF from Example 1.

Next, we want to prove, that the function κ^Z satisfies an AF if κ^Z is defined.

Theorem 1. If κ^Z is defined, then κ^Z satisfies AF.

Proof. Let $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ be an AF. In order to show that κ^Z satisfies AF, we need to prove, that κ^Z satisfies both principles of an OCF, i. e. acceptance of attacks and possibly reinstating an argument.

Case: accept attack. Let $(a,b) \in \mathsf{R}$ with $a \neq b$ an attack, it has to hold that $\kappa^Z(\{a\},\{b\}) < \kappa^Z(\{a,b\},\emptyset)$. We know that $\kappa^Z(\{a\},\{b\}) = 0$, because $\{a\}$ can only violate the attack (a,a), which can not exist. Hence, it is enough to show, that $\kappa^Z(\{a,b\},\emptyset) > 0$. Since, (a,b) exists, we know that $\{a,b\}$ violates this attack, and therefore $\kappa^Z(\{a,b\},\emptyset) > 0$.

Case: argument possibly reinstated. Let $a \in \mathsf{A}$ be an argument. Assume $\kappa^Z(S \cup a, a^-) > \kappa^Z(S, a \cup a^-)$ for some $S \subseteq \mathsf{A}$ with $S \cap (a^- \cup a^+) = \emptyset$. This is only possible, if $S \cup \{a\}$ violates an attack $r \in \mathsf{R}$ and S does not violate r. So, there is one argument $b \in S$ s.t. $r = (a,b)$ or $r = (b,a)$ and $a \neq b$. Hence, $b \in a^- \cup a^+$. However, because of $S \cap (a^- \cup a^+) = \emptyset$ we know that, there can not be such an argument $b \in S$ with $b \in a^- \cup a^+$. Therefore $\kappa^Z(S \cup a, a^-) > \kappa^Z(S, a \cup a^-)$ is impossible. $\square$

Besides being undefined for AF with self-attacks, κ^Z is also undefined for AFs without a stable extension. Let $\mathsf{AF}_4 = (\{a,b,c\},\{(a,b),(b,c),(c,a)\})$ be an AF. If we try to tolerate (a,b) by $\{(a,b),(b,c),(c,a)\}$, then we know that, we need to verify (a,b) so $a \in S$. However, this also means that $b,c \notin S$, which entails that attack admissibility of c is violated. Similar we can show, that (b,c) and (c,a) cannot be tolerated either. So, we cannot define a Z-attack-Partitioning for AF_4. Next, we show that in general it holds that if an AF does not have a stable extension, then κ^Z is undefined.

Theorem 2. κ^Z is undefined for AF if $\mathrm{st}(\mathsf{AF}) = \emptyset$.

Proof. Let $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ be an AF. We will show the contraposition, so if κ^Z is defined for AF, then $\mathrm{st}(\mathsf{AF}) \neq \emptyset$. Let κ^Z be defined. So we can find a Z-attack-Partitioning $(\mathsf{R}_0,...,\mathsf{R}_n)$. For every attack r in R_0 we know that there is a set S s.t. r is verified, every attack is satisfied and attack admissibility of every argument $a \in \mathsf{A}$ is satisfied. We show that S is stable. First, S has to be conflict-free, otherwise there is an attack, which is violated. Next we show that $S \cup S^+ = \mathsf{A}$, so we need to show, that every argument not in S is attacked by S. Let $b \notin S$, then because attack admissibility of b is satisfied we know that there is an argument $c \in b^-$ with $c \in S$, hence we have found an attacker of b which is part of S. □

Looking at the levels of a Z-attack-Partitioning in detail, we observe, that if an attack (a,b) is in R_0, then a is credulously admissible accepted in AF.

Theorem 3. For any AF $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ and Z-attack-Partitioning $(\mathsf{R}_0,...,\mathsf{R}_n)$. If $(a,b) \in \mathsf{R}_0$, then a is credulous accepted wrt. admissible semantics.

Proof. Let $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ be an AF, $(\mathsf{R}_0,...,\mathsf{R}_n)$ a Z-attack-Partitioning of R and $(a,b) \in \mathsf{R}_0$, then (a,b) is tolerated by R, meaning that there is an $S \subseteq \mathsf{A}$ s.t. (a,b) is verified, each attack in R is satisfied by S and attack admissibility of each argument $c \in \mathsf{A}$ is satisfied. In order to verify (a,b), we know that $a \in S$ and $b \notin S$. Also it has to hold for all $c \in a^-$ that $c \notin S$. So all attackers of a are *out*. Next, we will show that S is admissible. S is conflict-free, otherwise, one attack would be violated. We know for every $d \in S$ that no attacker of d is in S. In order to not violate attack admissibility, we know for every $e \notin S$ that at least one attacker of e has to be in S, meaning that S attacks every argument not contained in S. Hence, for every attacker b of an argument $a \in S$ we have an argument $d \in S$ s.t. d attacks b. So S is admissible, and therefore a is part of some admissible extension of AF making a credulous accepted w.r.t. admissible semantics. □

5. Extension-ranking Semantics based on System Z

First, we recall the definitions from [2] for *extension-ranking semantics*.

5.1. Extension-Ranking Semantics

Extension-ranking semantics defined in [2] are a generalisation of extension-based semantics. Using them, we can state not only that a set of arguments is jointly accepted or not, but also we can say whether a set E_1 is more plausible than a set E_2.

$$\emptyset \cong^{\kappa^Z}_{\mathsf{AF}} \{a\} \cong^{\kappa^Z}_{\mathsf{AF}} \{b\} \cong^{\kappa^Z}_{\mathsf{AF}} \{c\} \cong^{\kappa^Z}_{\mathsf{AF}} \{d\} \cong^{\kappa^Z}_{\mathsf{AF}} \{a,c\} \cong^{\kappa^Z}_{\mathsf{AF}} \{b,d\} \cong^{\kappa^Z}_{\mathsf{AF}} \{a,d\}$$
$$\preceq^{\kappa^Z}_{\mathsf{AF}} \{a,b\} \cong^{\kappa^Z}_{\mathsf{AF}} \{c,d\} \cong^{\kappa^Z}_{\mathsf{AF}} \{a,b,d\} \cong^{\kappa^Z}_{\mathsf{AF}} \{a,c,d\}$$
$$\preceq^{\kappa^Z}_{\mathsf{AF}} \{b,c\} \cong^{\kappa^Z}_{\mathsf{AF}} \{a,b,c\} \cong^{\kappa^Z}_{\mathsf{AF}} \{b,c,d\} \cong^{\kappa^Z}_{\mathsf{AF}} \{a,b,c,d\}$$

Table 3. The ranking for AF based on $\preceq^{\kappa^Z}_{\mathsf{AF}}$.

Definition 12. An *extension ranking* on AF is a preorder[1] over the powerset of arguments 2^{A}. An *extension-ranking semantics* τ is a function that maps each AF to an extension ranking $\preceq^{\tau}_{\mathsf{AF}}$ on AF.

For an extension-ranking semantics τ, an extension ranking $\preceq^{\tau}_{\mathsf{AF}}$, $E, E' \subseteq \mathsf{A}$, and for $E \preceq^{\tau}_{\mathsf{AF}} E'$ we say that E is *at least as plausible as* E' by τ in AF.

Using the OCF κ^Z, we can define an *extension-ranking semantics*. So we can state that a set of arguments E is more plausible than another one E', if the OCF κ^Z returns a lower value for E than for E'.

Definition 13. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an AF and $E, E' \subseteq \mathsf{A}$. Define the *System Z extension-ranking semantics* $\preceq^{\kappa^Z}_{\mathsf{AF}}$ via

$$E \preceq^{\kappa^Z}_{\mathsf{AF}} E' \text{ iff } \kappa^Z(E, \mathsf{A} \setminus E) \leq \kappa^Z(E', \mathsf{A} \setminus E')$$

So E is at least as plausible as E', if E being considered *in* and all arguments not in E being *out*, is more plausible than E' being considered *in* and all arguments not in E' being *out*.

Example 6. Consider again AF from Example 1. Then Table 3 depicts the ranking corresponding to $\preceq^{\kappa^Z}_{\mathsf{AF}}$. We see, that all conflict-free sets are part of the most plausible sets, while sets with conflicts are ranked worse. Also, the number of conflicts is not as important as for the approaches of [2]. In their approaches, it always holds that $\{b,c\}$ is ranked strictly better than $\{b,c,d\}$. While for κ^Z these two sets are ranked equally.

5.2. Study of the System Z Extension-ranking Semantics

Next, we want to evaluate $\preceq^{\kappa^Z}_{\mathsf{AF}}$ based on principles defined by [2].

We begin with σ-*generalisation*, which states that sets of arguments, which satisfies extension semantics σ should also be ranked best by an extension-ranking semantics and every set not satisfying σ should be ranked worse. In Example 6, we can see that $\preceq^{\kappa^Z}_{\mathsf{AF}}$ violates σ-*generalisation* for $\sigma \in \{\mathrm{ad}, \mathrm{co}, \mathrm{pr}, \mathrm{gr}, \mathrm{st}\}$, because the set $\{b,d\}$ is not admissible, however, it is ranked as a most plausible set. Therefore $\preceq^{\kappa^Z}_{\mathsf{AF}}$ cannot satisfy σ-*generalisation* for any admissible based semantics σ.

The next properties (*composition* and *decomposition*) states that unconnected arguments should not influence a ranking.

Theorem 4. $\preceq^{\kappa^Z}_{\mathsf{AF}}$ satisfies *composition*. Where τ satisfies *composition* if for every AF such that $\mathsf{AF} = (\mathsf{A}_1, \mathsf{R}_1) \cup (\mathsf{A}_2, \mathsf{R}_2)$ and $E, E' \subseteq \mathsf{A}_1 \cup \mathsf{A}_2$:

[1] A preorder is a (binary) relation that is *reflexive* ($E \preceq E$ for all E) and *transitive* ($E_1 \preceq E_2$ and $E_2 \preceq E_3$ implies $E_1 \preceq E_3$)

$$\text{if } \left\{ \begin{matrix} E \cap \mathsf{A}_1 \preceq^{\tau}_{\mathsf{AF}_1} E' \cap \mathsf{A}_1 \\ E \cap \mathsf{A}_2 \preceq^{\tau}_{\mathsf{AF}_2} E' \cap \mathsf{A}_2 \end{matrix} \right\} \text{ then } E \preceq^{\tau}_{\mathsf{AF}} E'.$$

Proof. Let $\mathsf{AF} = (\mathsf{A}_1, \mathsf{R}_1) \cup (\mathsf{A}_2, \mathsf{R}_2)$ be an AF and $E, E' \subseteq \mathsf{A}_1 \cup \mathsf{A}_2$. For composition we need to show that, if $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$ and $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$ then $\kappa^Z(E, \mathsf{A} \setminus E) \leq \kappa^Z(E', \mathsf{A} \setminus E')$. By definition of κ^Z we know that $\kappa^Z(E, \mathsf{A} \setminus E)$ is the maximal value between $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E)$ and $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E)$, because if an attack r_1 is violated by E, then r_1 is also violated by either $E \cap \mathsf{A}_1$ or $E \cap \mathsf{A}_2$. Similar holds for $\kappa^Z(E', \mathsf{A} \setminus E')$. So we have to check four possible cases for $max(\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E), \kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E)) \leq max(\kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E'), \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E'))$.

1. $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$
2. $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$
3. $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$
4. $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$

Case 1 and 2 are clear via definition. For case 3 we know that $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \geq \kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E)$ and $\kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E') \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$, but we also know that $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$, which proves case 3. Case 4 is similar to case 3. □

For *decomposition*, we see that $\preceq^{\kappa^Z}_{\mathsf{AF}}$ violates it. Recall that τ satisfies *decomposition* if for every AF such that $\mathsf{AF} = (\mathsf{A}_1, \mathsf{R}_1) \cup (\mathsf{A}_2, \mathsf{R}_2)$ and $E, E' \subseteq \mathsf{A}_1 \cup \mathsf{A}_2$:

$$\text{if } E \preceq^{\tau}_{\mathsf{AF}} E' \text{ then } \left\{ \begin{matrix} E \cap \mathsf{A}_1 \preceq^{\tau}_{\mathsf{AF}_1} E' \cap \mathsf{A}_1 \\ E \cap \mathsf{A}_2 \preceq^{\tau}_{\mathsf{AF}_2} E' \cap \mathsf{A}_2 \end{matrix} \right\}.$$

Example 7. Let $\mathsf{AF}_5 = (\{a,b,c,d,e\}, \{(a,b),(b,c),(d,e)\})$ be an AF. This AF can be split into two disjoint AFs $\mathsf{AF}_{5.1} = (\{a,b,c\}, \{(a,b),(b,c)\})$ and $\mathsf{AF}_{5.2} = (\{d,e\}, \{(d,e)\})$. The Z-attack-Partitioning of R_5 is $\mathsf{R}_{5\,0} = \{(a,b),(d,e)\}$ and $\mathsf{R}_{5\,1} = \{(b,c)\}$. Let $E = \{a,b,d,e\}$ and $E' = \{b,c,d\}$, then $\kappa^Z(E, \mathsf{A}_5 \setminus E) = 1$ and $\kappa^Z(E', \mathsf{A}_5 \setminus E') = 2$. However, we have $\kappa^Z(E \cap \mathsf{A}_{5.2}, \mathsf{A}_{5.2} \setminus E) = 1$ and $\kappa^Z(E' \cap \mathsf{A}_{5.2}, \mathsf{A}_{5.2} \setminus E') = 0$. This shows, that decomposition is violated.

Decomposition is violated, because $\preceq^{\kappa^Z}_{\mathsf{AF}}$ focuses on a global view. Violating (b,c) is worse, than violating any other attack. However, $\preceq^{\kappa^Z}_{\mathsf{AF}}$ satisfies a weak version of decomposition, where instead of satisfying $E \cap \mathsf{A}_1 \preceq^{\tau}_{\mathsf{AF}_1} E' \cap \mathsf{A}_1$ for both disjoint AFs, it is enough if κ^Z_{AF} satisfy this for one AF.

Definition 14 (Weak Decomposition)**.** Let τ be an extension-ranking semantics. τ satisfies *weak decomposition* if for every AF such that $\mathsf{AF} = (\mathsf{A}_1, \mathsf{R}_1) \cup (\mathsf{A}_2, \mathsf{R}_2)$ and $E, E' \subseteq \mathsf{A}_1 \cup \mathsf{A}_2$: if $E \preceq^{\tau}_{\mathsf{AF}} E'$ then $E \cap \mathsf{A}_1 \preceq^{\tau}_{\mathsf{AF}_1} E' \cap \mathsf{A}_1$ or $E \cap \mathsf{A}_2 \preceq^{\tau}_{\mathsf{AF}_2} E' \cap \mathsf{A}_2$.

Theorem 5. $\preceq^{\kappa^Z}_{\mathsf{AF}}$ satisfies *weak decomposition*.

Proof. Let $\mathsf{AF} = (\mathsf{A}_1, \mathsf{R}_1) \cup (\mathsf{A}_2, \mathsf{R}_2)$ be an AF and $E, E' \subseteq \mathsf{A}_1 \cup \mathsf{A}_2$. In order to prove weak decomposition we have to show, that if $\kappa^Z(E, \mathsf{A} \setminus E) \leq \kappa^Z(E', \mathsf{A} \setminus E')$ then $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$ or $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$. By definition we know that $\kappa^Z(E, \mathsf{A} \setminus E) = max(\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E), \kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E))$ and similar for E'. So, we have $max(\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E), \kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E)) \leq max(\kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E'), \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E'))$. Hence, we have four cases to check.

1. $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$
2. $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$
3. $\kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$
4. $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E')$

Case 1 and 2 are clear via definition. For case 3 we know that $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E \cap \mathsf{A}_1, \mathsf{A}_1 \setminus E)$ and therefore also $\kappa^Z(E \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E) \leq \kappa^Z(E' \cap \mathsf{A}_2, \mathsf{A}_2 \setminus E')$. Hence, weak decomposition is satisfied. Case 4 can be proven similar to case 3. □

The final properties we want to recall are the reinstatement ones, which state that if an argument is defended and does not add conflicts into a set, then the addition of this argument into a set should not lower the plausibility, respectively should raise the plausibility of the set.

Theorem 6. $\preceq^{\kappa^Z}_{\mathsf{AF}}$ satisfies *weak reinstatement*. Where τ satisfies *weak reinstatement* iff $a \in F_{\mathsf{AF}}(E)$, $a \notin E$ and $a \notin (E^- \cup E^+)$ implies $E \cup \{a\} \preceq^{\tau}_{\mathsf{AF}} E$.

Proof. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an AF and $E \subseteq \mathsf{A}$. Assume $a \notin E$ and $a \notin (E^- \cup E^+)$. We have to show that $\kappa^Z(E \cup \{a\}, \mathsf{A} \setminus E \cup \{a\}) \leq \kappa^Z(E, \mathsf{A} \setminus E)$. We know that $E \cup \{a\}$ violates the same attacks as E, because E and $\{a\}$ are not in a conflict with each other. This means that $\kappa^Z(E \cup \{a\}, \mathsf{A} \setminus E \cup \{a\})$ can not be greater than $\kappa^Z(E, \mathsf{A} \setminus E)$. □

For *strong reinstatement* i. e., adding an argument into an AF, which is defended by a set and does not create more conflicts, should raise the plausibility, we can look at Example 6. We see, that $\{c\}$ is equally ranked to $\{a, c\}$ despite it holds that $a \in F_{\mathsf{AF}}(\{c\})$, $a \notin \{c\}$ and $a \notin (\{c\}^- \cup \{c\}^+)$. So *strong reinstatement* is violated.

Even though a number of properties are violated by $\preceq^{\kappa^Z}_{\mathsf{AF}}$ this does no lower the impact of this semantics, since $\preceq^{\kappa^Z}_{\mathsf{AF}}$ focuses on a global view. The semantics identifies important attacks in the AF and ensures, that these attacks are satisfied. So it is worse to not satisfy a single highly ranked attacked, than not satisfying multiple lower ranked attacks. Another difference of this semantics to the semantics of Skiba et al. [2] is the fact, that the number of conflicts a set contains in not important just the fact, that the set is not conflict-free is significant.

6. Discussion

In this work, we continue the research of investigating the relationship of conditional logics and abstract argumentation, by using concepts for conditional logics to reason in abstract argumentation. In particular, we defined a formalism of OCFs to rank sets of arguments. It turns out that these preorders are in line with current work about extension-ranking semantics and produce a ranking for the powerset of arguments for an argumentation framework.

One use of conditional logics is belief change. Where preorders are used to update beliefs with information inconsistent with them. There are a number of different works investigating belief change involving preorders over extensions of an argumentation framework [12,13,14]. However, all these works tackle a different problem. To summarise, given an AF and an extension semantics σ, the AF will be changed using a preorder to satisfy new information. This paper talks about using OCFs to reason over sets of argument, while not changing AFs. Weydert [9] investigates a different idea to define extension rankings using conditionals, his definitions could be used to define an extension-ranking semantics similar to Section 5. However, his semantics cannot differentiate conflicting sets. All conflicting sets have the same rank of infinity. A full investigation of the properties of the resulting extension-ranking semantics will be done in future work. A noteworthy mention is that System Z and *rational closure* by Lehmann and Magidor [15] use the same construction. So our work also allows us to draw connections between argumentation and non-monotonic inference. Additionally OCFs with natural numbers and an infinity level are really close to possibilistic logics [16].

As there are more possible OCFs satisfying our proposed principles, we can define more extension-rankings semantics, like for example an extension-ranking semantics based on *c-representations* [17].

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References

[1] Dung PM. On the Acceptability of Arguments and its Fundamental Role in Nonmonotonic Reasoning, Logic Programming and n-Person Games. Artificial Intelligence. 1995.
[2] Skiba K, Rienstra T, Thimm M, Heyninck J, Kern-Isberner G. Ranking Extensions in Abstract Argumentation. In: Proc. of IJCAI'21; 2021. .
[3] Spohn W. Ordinal conditional functions: A dynamic theory of epistemic states. In: Causation in decision, belief change, and statistics. Springer; 1988. .
[4] Goldszmidt M, Pearl J. Qualitative probabilities for default reasoning, belief revision, and causal modeling. Artificial Intelligence. 1996.
[5] Heyninck J. Relations Between Assumption-based Approaches in Nonmonotonic Logics and Formal Argumentation. FLAP. 2019.
[6] Heyninck J, Kern-Isberner G, Thimm M, Skiba K. On the correspondence between abstract dialectical frameworks and nonmonotonic conditional logics. Ann Math Artif Intell. 2021.
[7] Kern-Isberner G, Simari GR. A Default Logical Semantics for Defeasible Argumentation. In: Proc. of FLAIRS'11; 2011. .
[8] Thimm M, Kern-Isberner G. On the Relationship of Defeasible Argumentation and Answer Set Programming. In: Proc. of COMMA'08; 2008. .
[9] Weydert E. A Plausibility Semantics for Abstract Argumentation Frameworks. CoRR. 2014.
[10] Weydert E. On the Plausibility of Abstract Arguments. In: Proc. of ECSQARU'13; 2013. .
[11] Baroni P, Gabbay D, Giacomin M, van der Torre L, editors. Handbook of Formal Argumentation. College Publications; 2018.
[12] Booth R, Kaci S, Rienstra T, van der Torre L. A logical theory about dynamics in abstract argumentation. In: Proc. of SUM'13; 2013. .
[13] Coste-Marquis S, Konieczny S, Mailly JG, Marquis P. On the revision of argumentation systems: Minimal change of arguments statuses. In: Proc. of KR'14; 2014. .
[14] Diller M, Haret A, Linsbichler T, Rümmele S, Woltran S. An extension-based approach to belief revision in abstract argumentation. International Journal of Approximate Reasoning. 2018.
[15] Lehmann D, Magidor M. What does a Conditional Knowledge Base Entail? Artif Intell. 1992.
[16] Dubois D, Prade H. Possibilistic Logic - An Overview. In: Siekmann JH, editor. Computational Logic. Handbook of the History of Logic. Elsevier; 2014. .
[17] Kern-Isberner G. Conditionals in nonmonotonic reasoning and belief revision: considering conditionals as agents. Springer; 2001.