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How Complex Is the Strong Admissibility Semantics for Abstract Dialectical Frameworks?

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Abstract. Abstract dialectical frameworks (ADFs) have been introduced as a formalism for modeling and evaluating argumentation allowing general logical satisfaction conditions. Different criteria used to settle the acceptance of arguments are called semantics. Semantics of ADFs have so far mainly been defined based on the concept of admissibility. Recently, the notion of strong admissibility has been introduced for ADFs. In the current work we study the computational complexity of the following reasoning tasks under strong admissibility semantics. We address 1. the credulous/skeptical decision problem; 2. the verification problem; 3. the strong justification problem; and 4. the problem of finding a smallest witness of strong justification of a queried argument.

Keywords. argumentation, abstract dialectical frameworks, complexity

1. Introduction

Despite the fact that Dung's abstract argumentation frameworks [1] (AFs for short) are widely used and studied within AI, in certain scenarios AFs are too limited to properly model the complex relations between arguments. Thus, several generalizations of AFs have been introduced [2], e.g., SETAFs and Bipolar AFs. Abstract dialectical frameworks (ADFs) [3,4,5] are an expressive generalization of AFs that can represent logical relations among arguments and subsume many popular generalizations of AFs. Semantics of AFs and ADFs single out coherent subsets of arguments that fit together, according to specific criteria [6].

There are several established semantics for AFs and ADFs. In this work we consider strong admissibility semantics and grounded semantics, which are the most skeptical types of semantics. Characteristics of *grounded* semantics for AFs include that 1. each AF has a unique grounded extension; 2. the grounded extension collects all the arguments about which no one doubts their acceptance; 3. the grounded extension is often a subset

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of the set of extensions of other types of AF semantics. Thus, it is important to investigate whether an argument belongs to the grounded extension of a given AF. The notion of *strong admissibility* is introduced for AFs to answer the query 'Why does an argument belong to the grounded extension?'.

While the grounded extension collects all the arguments of a given AF that can be accepted without any doubt, a strongly admissible extension provides a (minimal) justification why specific arguments can be accepted without any doubt, i.e., belong to the grounded extension. Thus, the strong admissibility semantics can be the basis for an algorithm that can be used not only for answering the credulous decision problem but also for human-machine interaction that requires an explainable outcome (cf. [7,8]).

In AFs, strong admissibility semantics were first defined in the work of Baroni and Giacomin [9], and later in [10]. Furthermore, in [11], Caminada and Dunne presented a labelling account of strong admissibility to answer the decision problems of AFs under grounded semantics. Moreover, Caminada showed in [12,10] that strong admissibility plays a crucial role in discussion games for AFs under grounded semantics. This motivated the study of the computational complexity of strong admissibility of AFs in general and in particular of the problem of computing small strongly admissible sets that justify the acceptance of an argument [13,14].

In previous work, we generalized the concept of strong admissibility to ADFs [15]. This concept fulfils properties that are related to those of the strong admissibility semantics for AFs, as follows: 1. Each ADF has at least one strongly admissible interpretation. 2. The set of strongly admissible interpretations of ADFs forms a lattice with as least element the trivial interpretation and as maximum element the grounded interpretation. 3. The strong admissibility semantics can be used to answer whether an argument is justifiable under grounded semantics. 4. The strong admissibility semantics for ADFs is a proper generalization of the strong admissibility semantics for AFs.

Whereas several fundamental properties of strong admissibility semantics for ADFs have been established, the computational complexity under strong admissibility semantics has not previously been studied. The current work closes this gap by studying the complexity of the central reasoning tasks under the strong admissibility semantics of ADFs. The paper is organised as follows: In Section 2 we recall the basic definitions of ADFs and strong admissibility. In Section 3 we provide exact complexity classifications for the different decision problems for strong admissibility semantics. We consider standard decision problems, i.e., the credulous and skeptial decision problems and the verification problem, the strong justification problem, i.e., deciding whether an argument is strongly justified in an interpretation, and the problem of finding a small witness of strong justification of an argument, i.e, whether there exists a strongly admissible interpretation that satisfies a queried argument and is smaller than a given bound. Finally, we conclude in Section 4. [2]

[2]This paper is based on an earlier presentation at the non-archival workshop NMR 2021. Proofs of all theorems are available in the dissertation [16] (Chapter 4), see `https://research.rug.nl/nl/publications/abstract-dialectical-frameworks-semantics-discussion-games-and-va`.

2. Formal Background

We recall the basics of ADFs [5]. Also we recall the definition of strong admissibility for ADFs, presented in [17].

2.1. Abstract Dialectical Frameworks

We summarize key concepts of abstract dialectical frameworks [3,5].

Definition 1. An abstract dialectical framework (ADF) is a tuple $D = (A, L, C)$ where: 1. A is a finite set of arguments (statements, positions); 2. $L \subseteq A \times A$ is a set of links among arguments; 3. $C = \{\varphi_a\}_{a \in A}$ is a collection of propositional formulas over arguments, called acceptance conditions.

An ADF can be represented by a graph in which nodes indicate arguments and links show the relations between arguments. Each argument a in an ADF is labelled by a propositional formula, called acceptance condition, φ_a over $par(a)$, where $par(a) = \{b \mid (b,a) \in L\}$. The acceptance condition of each argument clarifies under which condition the argument can be accepted.

A *three-valued interpretation* v (for D) is a function $v : A \mapsto \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ that maps arguments to one of the three truth values true ($\mathbf{t}$), false ($\mathbf{f}$), or undecided ($\mathbf{u}$). For reasons of brevity, we will shorten the notation of three-valued interpretation $v = \{a_1 \mapsto t_1, \ldots, a_m \mapsto t_m\}$ as follows: $v = \{a_i \mid v(a_i) = \mathbf{t}\} \cup \{\neg a_i \mid v(a_i) = \mathbf{f}\}$. For instance, $v = \{a \mapsto \mathbf{f}, b \mapsto \mathbf{t}\} = \{\neg a, b\}$. Interpretation v is called *trivial*, and v is denoted by $v_{\mathbf{u}}$, if $v(a) = \mathbf{u}$ for each $a \in A$. Furthermore, v is called a *two-valued interpretation* if for each $a \in A$ either $v(a) = \mathbf{t}$ or $v(a) = \mathbf{f}$.

Truth values can be ordered via the information ordering relation $<_i$ given by $\mathbf{u} <_i \mathbf{t}$ and $\mathbf{u} <_i \mathbf{f}$ and no other pair of truth values are related by $<_i$. Relation $\leq_i$ is the reflexive closure of $<_i$. Meet operator $\sqcap_i$ is defined over the truth values such that $\mathbf{t} \sqcap_i \mathbf{t} = \mathbf{t}$ and $\mathbf{f} \sqcap_i \mathbf{f} = \mathbf{f}$, while it returns $\mathbf{u}$ otherwise. The meet of two interpretations v and w is then defined as $(v \sqcap_i w)(a) = v(a) \sqcap_i w(a)$ for all $a \in A$.

Given an interpretation v (for D), the partial valuation of φ_a by v is $v(\varphi_a) = \varphi_a^v = \varphi_a[b/\top : v(b) = \mathbf{t}][b/\bot : v(b) = \mathbf{f}]$, for $b \in par(a)$. Note that in this work we assume that $D = (A, L, C)$ is a finite ADF and v is an interpretation of D. Semantics for ADFs can be defined via the *characteristic operator* Γ_D, presented in Definition 2.

Definition 2. Let D be an ADF and let v be an interpretation of D. Applying Γ_D on v leads to v' such that for each $a \in A$, $v'(a) = \mathbf{t}$ if φ_a^v is irrefutable, $v'(a) = \mathbf{f}$ if φ_a^v is unsatisfiable, and $v'(a) = \mathbf{u}$, otherwise.

Most types of semantics for ADFs are based on the concept of admissibility. An interpretation v for a given ADF F is called *admissible* iff $v \leq_i \Gamma_F(v)$; it is *preferred* iff v is $\leq_i$-maximal admissible; it is the *grounded* interpretation of D iff v is the least fixed point of Γ_D. The set of all σ interpretations for an ADF D is denoted by $\sigma(D)$, where $\sigma \in \{adm, grd, prf\}$ abbreviates the different semantics in the obvious manner. Given an interpretation v and an argument $a \in A$, a is called *acceptable* with respect to v if φ_a^v is irrefutable and a is called *deniable* with respect to v if φ_a^v is unsatisfiable.

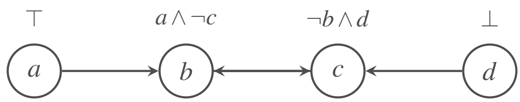

Figure 1. ADF of Example 1

2.2. The Strong Admissibility Semantics for ADFs

In this section, we rephrase the concept of strong admissibility semantics for ADFs from [15], which is defined based on the notion of strongly justifiable arguments (i.e., strongly acceptable/deniable arguments). Below, the interpretation $v_{|_P}$ is equal to $v(p)$ for any $p \in P$, and returns $\mathbf{u}$ otherwise, i.e., $v_{|_P} = v_{\mathbf{u}}|^{p \in P}_{v(p)}$.

Definition 3. Let D be an ADF. Argument a is a *strongly justified* argument in interpretation v with respect to set E if one of the following conditions holds:

- $v(a) = \mathbf{t}$ and there exists a subset P of parents of a excluding E, namely $P \subseteq par(a) \setminus E$, such that (a) a is acceptable with respect to $v_{|_P}$ and (b) all $p \in P$ are strongly justified in v w.r.t. set $E \cup \{p\}$.
- $v(a) = \mathbf{f}$ and there exists a subset P of parents of a excluding E, namely $P \subseteq par(a) \setminus E$, such that (a) a is deniable with respect to $v_{|_P}$ and (b) all $p \in P$ are strongly justified in v w.r.t. set $E \cup \{p\}$.

An argument a is *strongly acceptable*, respectively *strongly deniable*, in v if $v(a) = \mathbf{t}$, resp. $v(a) = \mathbf{f}$, and a is strongly justified in v with respect to set $\{a\}$. We say that a is *strongly justified* in v if it is either strongly acceptable or strongly deniable in v.

Note that in Definition 3, E is used to keep track of the arguments that cannot be used to justify a. We say that *a is not strongly justified in an interpretation v* if there is no set of parents of a that satisfies the conditions of Definition 3 for a. Example 1 presents the notion of strongly justified arguments in an interpretation.

Example 1. *Let D be the ADF depicted in Figure 1. Let $v = \{b, \neg c, \neg d\}$. First, since $\varphi_d^{v_{\mathbf{u}}} \equiv \bot$, it holds that d is strongly deniable in v. We show that c is strongly deniable in v with respect to $E = \{c\}$. Let $P = \{d\}$; it is clear that $\varphi_c^{v_{|_P}}$ is unsatisfiable. That is, c is deniable w.r.t. $v_{|_d}$. Then, since $d \in P$, $v(d) = \mathbf{f}$ and d is strongly justified in v with respect to $E = \{d\}$, c is strongly deniable in v. We show that b is not strongly acceptable in v. Let $P = par(b)$. Since $\varphi_b^{v_{|_P}} \not\equiv \top$, there is no subset of $par(b)$ that satisfies the conditions of Definition 3 for b. Thus, b is not strongly acceptable in v. It will turn out that v is not a strongly admissible interpretation, see Definition 4.*

Definition 4. Let D be an ADF. An interpretation v is a *strongly admissible* interpretation if for each a such that $v(a) = \mathbf{t}/\mathbf{f}$, it holds that a is a strongly justified argument in v. The set of all strongly admissible interpretations of D is denoted by $sadm(D)$.

Consider again the ADF of Example 1. Let $v = \{b, \neg c, \neg d\}$. As shown in Example 1, c and d are strongly justified in v. However, b is not strongly justified in v. Thus, $v \notin$

$sadm(D)$. However, for instance, $v_1 = \{a\}$, $v_2 = \{\neg c, \neg d\}$ and $v_3 = \{a, b, \neg c, \neg d\}$ are strongly admissible interpretations of D. Furthermore, $v_3 \in grd(D)$.

Algorithms in Section 5 of [17] answer the verification problem under strong admissibility semantics and the strong justification problem. To present the algorithms, Definition 28 in [17] introduces a variant of the characteristic operator restricted to a given interpretation v; we rewrite it in Definition 5.

Definition 5. Let D be an ADF and let v, w be interpretations of D. We define $\Gamma^0_{D,v}(w) = w$ and $\Gamma_{D,v}(w) = \Gamma_D(w) \sqcap_i v$, where $\Gamma^j_{D,v}(w) = \Gamma_{D,v}(\Gamma^{j-1}_{D,v}(w))$ for j with $j \geq 1$.

The sequence of interpretations $\Gamma^j_{D,v}(v_{\mathbf{u}})$ as defined in Definition 5 is named the sequence of strongly admissible interpretations constructed based on v in D. Theorems 28 and 29 in [17] show that one can use iterative fixed-point computations of $\Gamma_{D,v}$ operators to decide (a) verification of a given strongly admissible interpretation and (b) whether an argument is strongly acceptable/deniable within a given interpretation. However, because testing whether an argument is acceptable in Γ_D is already NP/coNP-hard [18], these procedures are in P^{NP}. As we will show, both problems allow for algorithms of significantly lower complexity.

3. Computational Complexity

We analyse the complexity under strong admissibility semantics for (a) the standard reasoning tasks of ADFs [18] and (b) two problems specific to strong admissibility semantics: (i) the small witness problem introduced for AFs [14,13] in order to minimize the length of the corresponding discussion games; and (ii) the strong justification problem. For a given ADF D, argument a and the truth value $x \in \{\mathbf{t}, \mathbf{f}\}$, we consider the following problems:

1. *The credulous decision problem*: whether a is credulously justifiable w.r.t. the strong admissibility semantics of D, denoted as $Cred_{sadm}(a, x, D)$, where $Cred_{sadm}(a, x, D) =$ yes if there exists $v \in sadm(D)$ s.t. $v(a) = x$, and it returns *no* otherwise.
2. *The skeptical decision problem*: whether a is skeptically justified w.r.t. the strong admissibility semantics of D, denoted as $Skept_{sadm}(a, x, D)$, where $Skept_{sadm}(a, x, D)$ $=$ yes if for each $v \in sadm(D)$ it holds that $v(a) = x$, and it returns *no* otherwise.
3. *The verification problem*: whether $v \in sadm(D)$ denoted by $Ver_{sadm}(v, D)$, where $Ver_{sadm}(v, D) =$ yes if $v \in sadm(D)$, and it returns *no* otherwise.
4. *The strong justification problem:* The problem whether a given argument a is strongly justified in a given interpretation v, denoted as $StrJust(a, x, v, D)$, where $StrJust(a, x, v, D) =$ yes if a is strongly justified in v, and it returns *no* otherwise.
5. *The small witness problem:* We are interested in computing a strongly admissible interpretation that has the least information of the ancestors of a given argument, namely a, where $v(a) = x$. The decision version of this problem is the k-Witness problem, denoted by $k\text{-}Witness_{sadm}$, indicating whether a given argument is strongly justified in at least one v such that $v \in sadm(D)$ and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| \leq k$. Note that k is part of the input of this problem. This decision problem is presented formally as follows: $k\text{-}Witness_{sadm}(a, x, D) =$ yes if there exists $v \in sadm(D)$ such that $v(a) = x$ and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| \leq k$, and it returns *no* otherwise.

3.1. The Credulous/Skeptical Decision Problems

In this section we show the complexity of deciding whether an argument in question is credulously/skeptically justifiable in at least one/all strongly admissible interpretation(s) of a given ADF. We show that $Cred_{sadm}$ is coNP-complete and $Skept_{sadm}$ is trivial. To this end, we use the fact, presented in [17], that the set of strongly admissible interpretations of a given ADF D forms a lattice with respect to the $\leq_i$-ordering, with the maximum element being $grd(D)$. Thus, any strongly admissible interpretation of D has at most an amount of information equal to $grd(D)$. Thus, answering the credulous decision problem under the strong admissibility semantics coincides with answering the credulous decision problem under the grounded semantics.

Theorem 1. *$Cred_{sadm}$ is coNP-complete.*

Proof. We have that $Cred_{sadm}(a,x,D) = Cred_{grd}(a,x,D)$ and the latter has been shown to be coNP-complete in [19, Proposition 4.1.3.]. □

Concerning skeptical acceptance, notice that the trivial interpretation is the least strongly admissible interpretation in each ADF. Thus, $Skept_{sadm}(a,x,D)$ is trivially *no*.

Theorem 2. *$Skept_{sadm}$ is a trivial problem.*

3.2. The Verification Problem

In this section, we settle the complexity of $Ver_{sadm}(v,D)$. We have mentioned at the end of Section 2.2 that this problem can be solved in P^{NP}; in the sequel, we will show that its complexity is in fact lower. We first sketch a simple translation-based approach that reduces the verification problem of strongly admissible semantics to the verification problem of grounded semantics. In order to reduce $Ver_{sadm}(v,D)$ to $Ver_{grd}(v,D')$, we modify the acceptance conditions φ_a of D to $\varphi'_a = \neg a$ if $v(a) = \mathbf{u}$ and $\varphi'_a = \varphi_a$ otherwise. We then have that $v \in sadm(D)$ iff $v \in grd(D')$, so that we can use the DP procedure for $Ver_{grd}(v,D')$ [19, Theorem 4.1.4]. However, as we will discuss next, $Ver_{sadm}(v,D)$ can even be solved within coNP.

Intuitively, since the grounded interpretation is the maximum element of the lattice of strongly admissible interpretations and the credulous decision problem under grounded semantics is coNP-complete, it seems that the verification problem under the strong admissibility semantics has to be coNP-complete. However, having the positive answer for $Cred_{grd}(a,x,D)$ for each a with $v(a) = \mathbf{t}/\mathbf{f}$ does not lead to the positive answer of $Ver_{sadm}(v,D)$. This is because $v \leq_i grd(D)$ does not imply that v is a strongly admissible interpretation of D (see Example 2 below).

Example 2. *Let $D = (\{a,b\},\{\varphi_a : \top, \varphi_b : a \vee b\})$. The grounded interpretation of D is $\{a \mapsto \mathbf{t}, b \mapsto \mathbf{t}\}$. Furthermore, the interpretation $v = \{a \mapsto \mathbf{u}, b \mapsto \mathbf{t}\}$ is an admissible interpretation of D such that $v \leq_i grd(D)$. However, v is not a strongly admissible interpretation of D. As we know, the answer of $Cred_{grd}(b,\mathbf{t},D)$ is* yes*, but b is not strongly acceptable in v. Thus, the answer to $Ver_{sadm}(v,D)$ is* no.

To show that Ver_{sadm} is coNP-complete, we modify and combine both the fixed-point iteration from [17] and the grounded algorithm from [19]. To this end, we need some auxiliary results that are shown in Lemmas 1 and 2.

Lemma 1. *Given an ADF D with $|A| = n$, the following statements are equivalent:*

1. *v is a strongly admissible interpretation of D;*
2. *$v = \Gamma^n_{D,v}(v_{\mathbf{u}})$;*
3. *for each $w \leq_i v$, it holds that $v = \Gamma^n_{D,v}(w)$.*

In the following, let $v^* = v^{\mathbf{t}} \cup v^{\mathbf{f}}$. The notions of completion of an interpretation and model are presented in Definition 6; they are used in Lemma 2.

Definition 6. Let w be an interpretation. We define the *completion* of w, denoted by $[w]_2$, as follows: $[w]_2 = \{u \mid w \leq_i u \text{ and } u \text{ is a two-valued interpretation}\}$.

Furthermore, a two-valued interpretation u is said to be a *model* of formula φ, if $u(\varphi) = \mathbf{t}$, denoted by $u \models \varphi$.

Lemma 2. *Let D be an ADF and let v be an interpretation of D. Then $v \notin sadm(D)$ iff there exists an interpretation w of D that satisfies all the following conditions:*

1. *$w <_i v$;*
2. *For each $a \in w^{\mathbf{u}} \cap v^{\mathbf{t}}$ there exists $u_a \in [w]_2$ s.t. $u_a \not\models \varphi_a$;*
3. *For each $a \in w^{\mathbf{u}} \cap v^{\mathbf{f}}$ there exists $u_a \in [w]_2$ s.t. $u_a \models \varphi_a$.*

Proof. $\Leftarrow$: Assume that v and w are interpretations of D that satisfy all of the items 1, 2, 3 presented in the lemma. We show that $v \notin sadm(D)$. Toward a contradiction, assume that $v \in sadm(v)$. Let a be an argument such that $a \in w^{\mathbf{u}} \cap v^{\mathbf{t}}$; thus, since w satisfies the conditions of the lemma, it holds that there exists $u_a \in [w]_2$ such that $u_a \not\models \varphi_a$, i.e., $u_a(a) = \mathbf{f}$. Furthermore, since $v(a) = \mathbf{t}$ and $v \in sadm(D)$, for any $j \in [v]_2$ it holds that $j \models \varphi_a$. Since $w <_i v$, it holds that $j \in [w]_2$, i.e., $\Gamma_D(w)(a) = \mathbf{u}$. The proof method for the case that $a \in w^{\mathbf{u}} \cap v^{\mathbf{f}}$ is similar, i.e., if $a \in w^{\mathbf{u}} \cap (v^{\mathbf{t}} \cup v^{\mathbf{f}})$, then $\Gamma_D(w)(a) = \mathbf{u}$. Thus, for $a \in w^{\mathbf{u}} \cap v^*$ we have $\Gamma_{D,v}(w)(a) = (\Gamma_D(w) \sqcap v)(a) = \mathbf{u}$. In other words, $\Gamma_{D,v}(w)(a) \leq_i w$ and thus, by the monotonicity of $\Gamma_{D,v}(w)$, also $\Gamma^n_{D,v}(w)(a) \leq_i w <_i v$. Thus, since $\Gamma^n_{D,v}(w) \not<_i v$, the third item of Lemma 1 does not hold for w with $w <_i v$. Therefore, $v \notin sadm(D)$.
$\Rightarrow$: Assume that $v \notin sadm(D)$. That is, for the fixed point $w = \Gamma^n_{D,v}(v_{\mathbf{u}})$ we have $w <_i v$. Consider $a \in w^{\mathbf{u}} \cap v^{\mathbf{t}}$. Because w is a fixed point, we have that $\Gamma_{D,v}(w)(a) \neq \mathbf{t}$ and thus $\Gamma_D(w) \neq \mathbf{t}$. That is, there is a $u_a \in [w]_2$ such that $u_a \not\models \varphi_a$. Similar reasoning applies to $a \in w^{\mathbf{u}} \cap v^{\mathbf{f}}$. □

Theorem 3 shows that Ver_{sadm} is coNP-complete for ADFs.

Theorem 3. *Ver_{sadm} is coNP-complete for ADFs.*

Proof sketch. We first show that $Ver_{sadm} \in$ coNP for ADFs. Let D be an ADF and let v be an interpretation of D. For membership, consider the co-problem. By Lemma 2, if there exists an interpretation of w that satisfies the condition of Lemma 2, then $v \notin sadm(D)$. Thus, guess an interpretation w, together with an interpretation $u_a \in [w]_2$ for each $a \in v^*$, and check whether they satisfy the conditions of Lemma 2. Note that since $w <_i v$, we have to check the second and the third items of Lemma 2 a total of $|v^* \setminus w^{\mathbf{u}}|$ number of times. That is, this checking has to be done at most $|v^*|$ number of times when w is the trivial interpretation. Thus, this checking step is linear in the size of v^*.

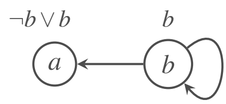

Figure 2. Reduction used in Theorems 3 and 5, for $\psi = \neg b \vee b$.

Therefore, the procedure of guessing of w and checking whether it satisfies $1,2,3$ of Lemma 2 is an NP-problem. Thus, if a w satisfies the items of Lemma 2, then the answer to $Ver_{sadm}(v,D)$ is *no*. Otherwise, if we check all interpretations w such that $w <_i v$ and none of them satisfies the conditions of Lemma 2, then the answer to $Ver_{sadm}(v,D)$ is *yes*. Thus, $Ver_{sadm}(v,D) \in$ coNP.

Now let us show that Ver_{sadm} is coNP-hard. For hardness of Ver_{sadm}, we consider the standard propositional logic problem of VALIDITY. Let ψ be an arbitrary Boolean formula and let $X = atom(\psi)$ be the set of atoms in ψ. Let a be a new atom, i.e., $a \notin X$. Construct ADF $D = (\{X \cup \{a\}\}, L, C)$ where $\varphi_x : x$ for each $x \in X$ and $\varphi_a : \psi$. We show that ψ is valid if and only if $v = v_{\mathbf{u}}|^a_{\mathbf{t}}$ is a strongly admissible interpretation of D. An illustration of the reduction for the formula $\psi = \neg b \vee b$ to the ADF $D = (\{a,b\}, L, \varphi_a : \psi, \varphi_b : b)$ is shown in Figure 2. One can show that ψ is a valid formula iff v is the grounded interpretation of D. □

3.3. Strong Justification of an Argument

Note that it is possible that an interpretation v contains some strongly justified arguments but v is not strongly admissible itself. For instance, in interpretation $v = \{b, \neg c, \neg d\}$, presented in Example 1, arguments c and d are strongly deniable in v, however, argument b is not strongly acceptable in v. Thus, v is not a strongly admissible interpretation of D. However, there exists a strongly admissible interpretation of D in which c and d are strongly acceptable and that has less information than v, namely, $v' = \{c,d\}$. Thus, the problem $StrJust(a,x,v,D)$ of deciding whether an argument is strongly justified in a given interpretation of an ADF is different from the previously discussed decision problems. We show that $StrJust$ is coNP-complete.

Algorithm 2 in [17] presents a direct method of deciding whether a is strongly justified in an interpretation v. That is, a is strongly acceptable/deniable in v iff it is acceptable/deniable by the least fixed point of the operator $\Gamma_{D,v}$, which is equal to $\Gamma^n_{D,v}(v_{\mathbf{u}})$ for sufficiently large n.

However, the repeated evaluation of Γ_D is a costly part of this algorithm and results in a P^{NP} algorithm. We will next discuss a more efficient method to answer this reasoning task. To this end, we translate a given ADF D to ADF D', presented in Definition 7, such that the queried argument is strongly justifiable in a given interpretation of D if and only if it is credulously justifiable in the grounded interpretation of D'. As shown in Proposition 4.1.3 in [19], $Cred_{grd} \in$ coNP. Thus, verifying whether a given argument is strongly justified in an interpretation is a coNP-problem, since the translation can be done in polynomial time with respect to the size of D. In the following, assume that v is an interpretation of D.

Definition 7. Let $D = (A,L,C)$ be an ADF and let v be an interpretation of D. The translation of D under v is $D' = (A', L', C')$ such that $A' = A \cup \{x,y\}$ where $x,y \notin A$.

Furthermore, for each $a \in A'$ we define the acceptance condition of a in D', namely φ'_a, as follows: 1. $\varphi'_x : x$; 2. $\varphi'_y : y$; 3. if $v(a) = \mathbf{u}$, then $\varphi'_a : \neg a$; 4. if $v(a) = \mathbf{t}$, then $\varphi'_a = \varphi_a \vee x$; 5. if $v(a) = \mathbf{f}$, then $\varphi'_a = \varphi_a \wedge y$.

Notice that we introduced arguments x, y to ensure that arguments in v^* are not assigned to the opposite truth value during the iteration of $\Gamma_{D'}$ that leads to $grd(D')$. Theorem 4 shows the correctness of the reduction.

Theorem 4. *Let D be an ADF, let v be an interpretation of D, and let D′ be the translation of D, via Definition 7. Then,* $StrJust(a,x,v,D) = yes$, *iff* $Cred_{grd}(a,x,D') = yes$.

Proof. We assume that $StrJust(a,\mathbf{t},v,D)$ = yes, and we show that $Cred_{grd}(a,\mathbf{t},D')$ = yes. The proof for the case that $StrJust(a,\mathbf{f},v,D)$ = yes is similar. Assume that $v_\mathbf{u}$ is the trivial interpretation of D and $v'_\mathbf{u}$ is the trivial interpretation of D'. Assume that $\Gamma^i_{D,v}(v_\mathbf{u})$ is a sequence of strongly admissible interpretations constructed based on v in D, as in Definition 5. Let w be the limit of the sequence of $\Gamma^i_{D,v}(v_\mathbf{u})$.

$StrJust(a,\mathbf{t},v,D) = yes$ implies that $w(a) = \mathbf{t}$. Since $w \in sadm(D)$, it holds that $g(a) = \mathbf{t}$ where $g \in grd(D)$, i.e., there exists a natural number n such that $\Gamma^n_D(v_\mathbf{u})(a) = \mathbf{t}$. By induction on n, it is easy to show that $\Gamma^n_{D'}(v'_\mathbf{u})(a) = \mathbf{t}$. That is, $g'(a) = \mathbf{t}$ where $g' \in grd(D')$. Thus, $Cred_{grd}(a,\mathbf{t},D')$ = yes.

We assume that $Cred_{grd}(a,x,D') = yes$, and we show that $StrJust(a,x,v,D) = yes$. Assume that a is justified in the grounded interpretation of D', namely w. Thus, there exists a j such that $w = \Gamma^j_{D'}(w_\mathbf{u})$ for $j \geq 0$, where $w_\mathbf{u}$ is the trivial interpretation of D'. By induction we prove the claim that for all i, if $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^i_{D'}(w_\mathbf{u})$, then a is strongly justified in v.

Base case: Assume that $\Gamma^1_{D'}(w_\mathbf{u})(a) \in \{\mathbf{t},\mathbf{f}\}$. By the acceptance conditions of x and y in D', both of them are assigned to $\mathbf{u}$ in w. Then it has to be the case that either $\varphi'_a = \varphi_a \vee x$ or $\varphi'_a = \varphi_a \wedge y$ in D'. Thus, $\Gamma^1_{D'}(w_\mathbf{u})(a) \in \{\mathbf{t},\mathbf{f}\}$ implies that ${\varphi'_a}^{w_\mathbf{u}} \equiv \top/\bot$. Thus, $w(x/y) = \mathbf{u}$, $\varphi'_a = \varphi_a \vee x/\varphi_a \wedge y$ and ${\varphi'_a}^{w_\mathbf{u}} \equiv \top/\bot$ together imply that ${\varphi_a}^{w_\mathbf{u}} \equiv \top/\bot$. Hence, $\varphi_a^{v_\mathbf{u}} \equiv \top/\bot$ where $v_\mathbf{u}$ is the trivial interpretation of D. That is, a is strongly justified in v.

Induction hypothesis: Assume that for all j with $1 \leq j \leq i$, if $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^j_{D'}(w_\mathbf{u})$, then a is strongly justified in v.

Inductive step: We show that if $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^{i+1}_{D'}(w_\mathbf{u})$, then a is strongly justified in v. Because $x/y \mapsto \mathbf{u} \in w$, we have that $\varphi_a^w \equiv \top/\bot$ implies that $\varphi_a^v \equiv \top/\bot$. Furthermore, $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^{i+1}_{D'}(w_\mathbf{u})$ says that there exists a set of parents of a, namely P, where $P \subseteq w^\mathbf{t} \cup w^\mathbf{f}$, such that, $\varphi_a^{w_{|P}} \equiv \top/\bot$. Thus, $\varphi_a^{v_{|P}} \equiv \top/\bot$. By induction hypothesis, each $p \in P$ is strongly justified in v. Thus, a is strongly justified in v. □

We use the auxiliary Theorem 4 to present the main result of this section, i.e., to show that *StrJust* is coNP-complete.

Theorem 5. *Let D be an ADF, let a be an argument, and let v be an interpretation of D. Deciding whether a is strongly justified in v, i.e., whether* $StrJust(a,x,v,D)$, *is* coNP-*complete.*

Proof sketch. First we show that $StrJust(a,x,v,D) \in$ coNP. It is shown in [19, Proposition 4.1.3] that $Cred_{grd}(a,x,D) \in$ coNP. Furthermore, the translation of a given ADF D to D' via Definition 7 can be done in polynomial time. By Theorem 4, it holds that

$Cred_{grd}(a,x,D) = yes$ iff $StrJust(a,x,v,D) = yes$. Thus, deciding whether a given argument is strongly justified in interpretation v is a coNP-problem.

Next we show that $StrJust(a,x,v,D)$ is coNP-hard. Let ψ be any Boolean formula and let $X = atom(\psi)$ be the set of atoms in ψ. Let a be a new variable. Construct $D = (\{X \cup \{a\}\}, L, C)$, s.t. $\varphi_x : x$ for each $x \in X$ and $\varphi_a : \psi$. ADF D can be constructed in polynomial time w.r.t. the size of ψ. One can check that a is strongly acceptable in any v where $v(a) = \mathbf{t}$ iff ψ is a valid formula. An illustration of the reduction for the formula $\psi = \neg b \vee b$ to the ADF $D = (\{a,b\}, L, \varphi_a : \psi, \varphi_b : b)$ is depicted in Figure 2.

For credulous denial of a, it is enough to present the acceptance condition of a equal to the negation of ψ in D, i.e., $\varphi_a : \neg\psi$, and follow a similar method. That is, a is strongly deniable in v, where $v(a) = \mathbf{f}$, iff ψ is a valid formula. □

3.4. Smallest Witness of Strong Justification

Assume that an argument a, its truth value x, and a natural number k are given. We are eager to know whether there exists a strongly admissible interpretation v such that $v(a) = x$ and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| < k$. This reasoning task is denoted by $k\text{-}Witness_{sadm}(a,x,D)$. We show that $k\text{-}Witness_{sadm}$ is Σ_2^{P}-complete. Lemma 3 shows that this problem is a Σ_2^{P}-problem and Lemma 4 indicates the hardness of this reasoning task.

Lemma 3. *Let D be an ADF, let a be an argument, let $x \in \{\mathbf{t},\mathbf{f}\}$, and let k be a natural number. Deciding whether there exists an interpretation v such that $v \in sadm(D)$, $v(a) = x$, and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| < k$ is a Σ_2^{P}-problem, i.e., $k\text{-}Witness_{sadm} \in \Sigma_2^{\mathsf{P}}$.*

Proof. For membership in Σ_2^{P}, non-deterministically guess an interpretation v and verify whether this interpretation satisfies the following items: 1. $v \in sadm(D)$; 2. $v(a) = x$; 3. $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| < k$. If v satisfies all the items, then the answer to the decision problem is *yes*, i.e., $k\text{-}Witness_{sadm}(a,x,D) = yes$. Notice that we have shown in Section 3.2 that testing (1) is coNP-complete and testing (2) and (3) can clearly be done in polynomial time. That is, the algorithm first non-deterministically guesses an interpretation v and then performs checks that are in coNP to verify that v satisfies the requirements of the decision problem. Thus, this gives an $\mathsf{NP}^{\mathsf{coNP}} = \Sigma_2^{\mathsf{P}}$ procedure. □

Lemma 4. *Let D be an ADF, let a be an argument, let $x \in \{\mathbf{t},\mathbf{f}\}$, and let k be a natural number. Deciding whether there exists a strongly admissible interpretation v of D where $v(a) = x$ and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| < k$ is Σ_2^{P}-hard, i.e., $k\text{-}Witness_{sadm}$ is Σ_2^{P}-hard.*

Proof sketch. Consider the following well-known problem on quantified Boolean formulas. Given a formula $\Theta = \exists Y \forall Z\, \theta(Y,Z)$ with atoms $X = Y \cup Z$ (and $Y \cap Z = \emptyset$) and propositional formula θ. Deciding whether Θ is valid is Σ_2^{P}-complete (see e.g. [20]). We can assume that θ is of the form $\psi \wedge \bigwedge_{y \in Y}(y \vee \neg y)$, where ψ is an arbitrary propositional formula over atoms X, and that θ is satisfiable. Moreover, we can assume that the formula θ only uses $\wedge$, $\vee$, $\neg$ operations and that negations only appear in literals. Let $\bar{Y} = \{\bar{y} : y \in Y\}$, i.e., for each $y \in Y$ we introduce a new argument $\bar{y}$.

We construct an ADF $D_\Theta = (A, L, C)$ with $A = Y \cup \bar{Y} \cup Z \cup \{\theta\}$ and $C = \{\varphi_y : \top \mid y \in Y\} \cup \{\varphi_{\bar{y}} : \top \mid y \in Y\} \cup \{\varphi_z : \neg z \mid z \in Z\} \cup \{\varphi_\theta : \theta[\neg y/\bar{y}]\}$.

It is easy to verify that $g \in grd(D_\Theta)$ sets all arguments $Y \cup \bar{Y}$ to $\mathbf{t}$ and all arguments Z to $\mathbf{u}$. Moreover, $g(\theta) \in \{\mathbf{t},\mathbf{u}\}$. An illustration of the reduction for the formula $\theta = ((y_1 \wedge$

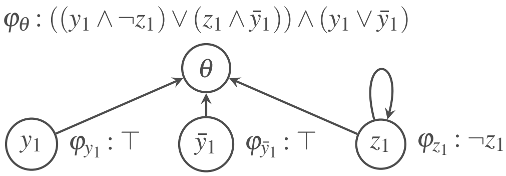

Figure 3. Illustration of the reduction from the proof of Lemma 4 for $\Theta = \exists y_1 \forall z_1 ((y_1 \wedge \neg z_1) \vee (z_1 \wedge \neg y_1)) \wedge (y_1 \vee \neg y_1)$.

$\neg z_1) \vee (z_1 \wedge \neg y_1)) \wedge (y_1 \vee \neg y_1)$ to the ADF $D = (A, L, C)$ is shown in Figure 3, where: $A = \{y_1, \bar{y}_1, z_1, \theta\}$, $\varphi_{y_1} : \top, \varphi_{\bar{y}_1} : \top, \varphi_{z_1} : \neg z$ and $\varphi_\theta : ((y_1 \wedge \neg z_1) \vee (z_1 \wedge \bar{y}_1)) \wedge (y_1 \vee \bar{y}_1)$. One can check that there is an interpretation v with $v \in sadm(D_\Theta)$, $v(\theta) = \mathbf{t}$, and $|S| = |Y| + 1$ where $S = v^{\mathbf{t}} \cup v^{\mathbf{f}}$ iff Θ is a valid formula. □

Theorem 6 is a direct result of Lemmas 3 and 4.

Theorem 6. *k-Witness$_{sadm}$ is Σ_2^{P}-complete.*

4. Conclusion

We studied the computational properties of the strong admissibility semantics of ADFs. When compared to AFs, computational complexity for ADFs typically increases by one step in the polynomial hierarchy for the non-trivial reasoning tasks [21,18]. We have shown that, similarly, ADFs have higher computational complexity under the strong admissibility semantics when compared to AFs.

We next highlight an interesting difference in the complexity landscapes of AFs and ADFs. When relating the complexity of grounded and strong admissibility semantics, we have that for AFs the verification problems can be (log-space) reduced to each other, while for ADFs there is a gap between the coNP-complete Ver_{sadm} problem and the DP-complete Ver_{grd} problem. That is, on the ADF level the step of proving arguments to be **u** in the grounded interpretation adds an NP part to the complexity; a similar effect can be observed for admissible and complete semantics.

Our complexity analysis for ADFs paves the way to investigate the complexity of strong admissibility for generalizations of Dung AFs that form subclasses of ADFs, e.g., different types of bipolar ADFs [3].

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