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Hierarchical Structure in Fuzzy Multi-Information Granular Computing Model

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Abstract. Granularity analysis and measurements for complex data environment are important tools to describe the essential attributes of the granular computing model(GCM). Firstly, based on fuzzy multiple relations, this paper difines a multi-fuzzy granular structure, and studies the hierarchical characteristics and relates mathematical conclusions of the four fuzzy multiple partial-order relations on the structure; Secondly, the measurement method of multi-fuzzy information granularity is proposed, and its properties of measurements are analyzed; Finally, the axiomatic definition of multi-fuzzy information granularity and its properties are discussed.

Keywords. Granular computing; fuzzy multi-information GCM; multi-fuzzy information granularity; partial-order relationship; axiomatic method

1. Introduction

Zadeh proposed the concept of granular computing in 1996[1]. He elaborated that human cognition has three main concepts, namely granulation (decomposing the whole into parts), organization (integrating the parts into the whole), and causation (connections between parts)[1][2]. Since granular computing covers the research of granular theory, methodology, technology and tools, it has become one of the research hotspots in the field of artificial intelligence in recent years, and has been widely used in machine learning, decision analysis, process control, pattern recognition and data mining, etc. Also the ideological of granular computing is structured thinking, peculiaritied by hierarchical modeling, comprehending, processing and learning[3]. Information system is the carrier for us to obtain the most information resources[4]. In granular computing, it is one of the important ways of knowledge representation[5]. The information granularity of an information system reflects the uncertainty measure of its real structure[6]. In recent years, measurement methods have been widely studied. How to analyze the granular structure and measurement of complex information granular computing models are particularly important for the measurement of information uncertainty of complex data in reality. At present, people have proposed a variety of information granularity measurement forms

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in information systems for different data environments[7-12]. Qian et al. mainly studied the information granularity in the information system under the fuzzy binary relation, discussed the three partial-order relations on the fuzzy-binary information granularity, and gave the axiomatic definition of the fuzzy binary information granular[13]. Under the framework of Qian's research on information granularity in information systems, other scholars have expanded their research content. Yang et al. proposed the granularity in the intuitionistic fuzzy information system, gave the granularity and its axiomatic definition, and studied related properties[14]. Based on the interval-value hesitant fuzzy binary relation, Lu et al. gave the concept of interval-valued hesitant fuzzy granularity structure, calculated the interval-valued hesitant fuzzy information granularity; Meanwhile lu et al. put forward the partial-order relationship on interval-valued hesitant fuzzy information granularity[15], and further researched its internal hierarchical granularity structure. In view of the constantly quantified multi-data environment, the research on the hierarchical structure of its internal granularity and its measurement is undoubtedly of great significance to reveal the essence of its internal structure.

Yager first proposed fuzzy multiset, its biggest advantage is to allow the membership of elements in the universe to appear multiple times with the same or different membership values[16]. Kim et al. studied the basic relations and operations of fuzzy multisets, and discussed the application of fuzzy multisets in fuzzy relational database systems [17]. Miyamoto gave the definition of the module of fuzzy multiset, and considered the application of commutativity in fuzzy multisets [18]. El-Azab, M.S. defined fuzzy multi correlation measure and many of its properties are investigated[19]. However, few people have discussed the partial-order relation between the multi-fuzzy information granular structure based on the fuzzy multirelation. And few people use information granularity to study the uncertainty measurement of fuzzy multi-information GCM. Therefore, it is necessary to study the multi-fuzzy granular structure of fuzzy multiple information granularity and information measurement.

The main contributions of this paper are as follow:

- The granular structure under fuzzy multiple environments is defined, and three operations are discussed.
- Four partial-order relations between multi-fuzzy granular structures are established, and their properties are proved mathematically.
- Uncertainty measures of fuzzy multi-information are studied by using information granularity.
- Axiomatization method of fuzzy multi-information granularity presented in this paper has important theoretical significance and application value for multi-information granular computing model.

This paper is organized as follows. Section 2 reviews some basic concepts: fuzzy multisets, the number of membership degrees of elements in the fuzzy multisets, the operations between fuzzy multisets, and proposes a fuzzy multirelation. Section 3 studies the multi-fuzzy granular structure and the three operations. In the section 4, we present for the hierarchical characteristics and related mathematical conclusions of the four fuzzy multiple partial-order relations on the structure. In the Section 5, the multi-fuzzy information granularity and its axiomatization method are defined, and under the four partial-order relations proposed above, the properties of multi-fuzzy information granularity are studied. Finally, section 6 gives the conclusion.

2. Preliminaries

In this section, we will review several basic concepts, such as fuzzy multisets, the number of membership degrees of elements in the fuzzy multisets, the operations between fuzzy multisets, and proposes a fuzzy multirelation.

Fuzzy multisets is an extension of multisets, which is defined as follows:

Definition 2.1[20]: Let $U = \{x_1, x_{2,...,} x_n\}$ be a finite and nonempty set. A fuzzy multisets M on U can be defined as $M = \{d_M(x_1)/x_1, d_M(x_2)/x_2, \ldots, d_M(x_n)/x_n\}$, in which $d_M(x_i) = \left\{d_M^1(x_i), d_M^2(x_i), \cdots, d_M^k(x_i)\right\}$ is a finite multiset of the $[0,1]$, denoting the possible membership degrees of the x_i to the set M, and $d_M^1(x_i) \geq d_M^2(x_i) \geq \cdots \geq d_M^k(x_i)$. In this paper, all fuzzy multisets are defined as $FM(U)$.

Example 1: Consider a fuzzy mltiset $A = \{(x_1, 0.3), (x_1, 0.4), (x_1, 0.5), (x_1, 0.5), (x_2, 0.4), (x_2, 0.6)\}$ of $U = \{x_1, x_2\}$. We may write $A = \{\{0.3, 0.4, 0.5, 0.5\}/x_1, \{0.4, 0.6\}/x_2\}$, in which the multiset of memberships $\{0.3, 0.4, 0.5, 0.5\}$ and $\{0.4, 0.6\}$ correspond to x_1 and x_2.

Definition 2.2: Let $U = \{x_1, x_{2,...,} x_n\}$ be a finite and nonempty set. Let $A, B \in FM(U)$. $A = \{d_A(x_1)/x_1, d_A(x_2)/x_2, \ldots, d_A(x_n)/x_n\}$, $B = \{d_B(x_1)/x_1, d_B(x_2)/x_2, \ldots, d_B(x_n)/x_n\}$, where $d_A(x_i) = \{d_A^1(x_i), d_A^2(x_i), \cdots, d_A^p(x_i)\}$ and $d_B(x_i) = \{d_B^1(x_i), d_B^2(x_i), \cdots, d_B^q(x_i)\}$. The number of values in seqence $d_A(x_i)$ is defined as

$$L(x_i, A) = \max\{s : d_A^s(x_i) \neq 0\}.$$

We can get $L(x_i; A, B) = \max\{L(x_i; A), L(x_i; B)\}$. The resulting length for A and B is defined by

$$L(x_i; A, B) = max\{p, q\}.$$

We sometimes write $L(x_i)$ instead of $L(x_i; A, B)$ when no ambiguity arises.

Definition 2.3[20]: The number of elements, or cardinality of a fuzzy multiset is given by $|M| = \sum_{x_i \in U} \sum_{s=1}^{L(x_i;M)} d_M^s(x_i)$.

Definition 2.4[21]: Let a universe $U = \{x_1, x_{2,...,} x_n\}$ be a finite universe of discourse. Then their basic operations for the membership sequences between A and B are as follows:

$$d_{A\cup B}^s(x_i) = d_A^s(x_i) \vee d_B{}^s(x_i), s = 1, 2, \cdots, L(x_i); \tag{1}$$

$$d_{A\cap B}^s(x_i) = d_A^s(x_i) \wedge d_B^s(x_i), s = 1, 2, \cdots, L(x_i); \tag{2}$$

$$\sim d_A^s(x_i) = 1 - d_A^s(x_i), s = 1, 2, \cdots, L(x_i; A). \tag{3}$$

Note: If $\forall A, B \in FM(U)$, in order to have a correct comparison and define an operation between two fuzzy multisets A and B, the membership sequences $d_A^1(x_i), d_A^2(x_i), \cdots,$

$d_A^p(x_i)$ and $d_B^1(x_i), d_B^2(x_i), \cdots, d_B^q(x_i)$ should have the equal length[20]. If $p < q$, we therefore extend $d_A(x_i)$ with maximum element $d_A^p(x_i)$ until it has same length with $d_B(x_i)$.

Definition 2.5: Let $U = \{x_1, x_2, ..., x_n\}$ be a finite and nonempty set. Then the fuzzy multi-subset defined in the direct product space $U \times U$ is called the fuzzy multirelation $\tilde{R}^*$on U, $\tilde{R}^*$ is given by $\tilde{R}^* = \{(dr(x_{ij})/x_{ij})\,|x_{ij} = (x_i, x_j) \in U \times U, i, j = 1, 2, \cdots, n\}$, where $dr(x_{ij})$ is a subset of the power set $[0,1]$ representing the possible multiple membership degrees of the x_i and x_j.

3. Multi-Fuzzy Granular Structures

In Granular Computing, transformation among granular structures is a significant matter which involves composition, disintegration and transformation.A class of fuzzy information particles produced by fuzzy multiple relations in the universe of discourse is called multi-fuzzy granular structure. The granular structure under multi-fuzzy environment is defined as follow.

Definition 3.1: Suppose $U = \{x_1, x_2, ..., x_n\}$, $\tilde{R}^*$ is a fuzzy multirelation on U. Then the multi-fuzzy granular structure of U is defined as

$$K(\tilde{R}^*) = (S_{\tilde{R}^*}(x_1), S_{\tilde{R}^*}(x_2), \cdots, S_{\tilde{R}^*}(x_n)) \tag{4}$$

where $S_{\tilde{R}^*}(x_i) = (\frac{dr(x_{i1})}{x_i}, \frac{dr(x_{i2})}{x_i}, \cdots, \frac{dr(x_{in})}{x_i})$ is a multi-fuzzy information granule, $dr(x_{ij}) = \{d_r^1(x_{ij}), d_r^2(x_{ij}), \cdots, d_r^s(x_{ij})\}$ denotes a set about the possible multiple degrees of equivalence and between x_i and x_j.

Definition 3.2: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$, in which $dp(x_{ij}) = \{d_p^1(x_{ij}), d_p^2(x_{ij}), \cdots, d_q^s(x_{ij})\}$, $s = L(x_{ij};\ S_{\tilde{P}^*}(x_i))$; $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2), \cdots, S_{\tilde{q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i) = (\frac{dq(x_{i1})}{x_i}, \frac{dq(x_{i2})}{x_i}, \cdots, \frac{dq(x_{in})}{x_i})$, $dq(x_{ij}) = \{d_q^1(x_{ij}), d_q^2(x_{ij}), \cdots, d_q^t(x_{ij})\}$, and $t = L(x_{ij};\ S_{\tilde{Q}^*}(x_i))$. Refer to definition 2.2, the number of values in $dp(x_{ij})$ is defined as

$$L(x_{ij};\ S_{\tilde{P}^*}(x_i)) \ = \ max\left\{m : d_p^m(x_{ij}) \neq 0\right\}.$$

We can get $L(x_{ij}; S_{\tilde{P}^*}(x_i), S_{\tilde{Q}^*}(x_i)) = \max\{L(x_{ij};\ S_{\tilde{P}^*}(x_i)), L(x_{ij}; S_{\tilde{Q}^*}(x_i)\ )\}$. The resuting length for $S_{\tilde{P}^*}(x_i), S_{\tilde{Q}^*}(x_i)$ is defined by

$$L(x_{ij};\ S_{\tilde{P}^*}(x_i),\ S_{\tilde{Q}^*}(x_i)) \ = \ max\{s, t\}.$$

We sometimes write $L(x_{ij})$ instead of $L(x_{ij};\ S_{\tilde{P}^*}(x_i),\ S_{\tilde{Q}^*}(x_i))$ when no ambiguity arises.

Definition 3.3: Let $K(\tilde{R}^*) \in K(U)$,$K(\tilde{R}^*) = (S_{\tilde{R}^*}(x_1), S_{\tilde{R}^*}(x_2), \cdots, S_{\tilde{R}^*}(x_n))$, where $S_{\tilde{R}^*}(x_i) = (\frac{dr(x_{i1})}{x_i}, \frac{dr(x_{i2})}{x_i}, \cdots, \frac{dr(x_{in})}{x_i})$ is a multi-fuzzy information granule, $dr(x_{ij}) = \{d_r^1(x_{ij}), d_r^2(x_{ij}), \cdots, d_r^s(x_{ij})\}$ and $s = L(x_{ij}; S_{\tilde{R}^*}(x_i))$. The cardinality of the multi-fuzzy granule $S_{\tilde{R}^*}(x_i)$ can be calculated with

$$|S_{\tilde{R}^*}(x_i)| = \sum_{j=1}^{n} \left(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{R}^*}(x_i))} d_r^s(x_{ij})}{L(x_{ij};\tilde{R}^*)} \right) \tag{5}$$

which is a natural extension of the cardinality of fuzzy sets.

Given a family of fuzzy multi-granular structures $GCM = (U, \tilde{\mathscr{R}}^*)$, we also denote the multi-fuzzy granular structure induced by $\tilde{P}^* \in \tilde{\mathscr{R}}^*$ by $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, where $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$. Moreover, let $K(U)$ be the collection of all multi-fuzzy granular structures on U.

Example 2: Let $K(\tilde{R}^*) = (S_{\tilde{R}^*}(x_1), S_{\tilde{R}^*}(x_2))$.

$$M(\tilde{R}^*) = \begin{pmatrix} \{0.2, 0.2\} & \{0.3, 0.3, 0.6\} \\ \{0.6, 0.6, 0.8\} & \{0.1, 0.3, 0.3, 0.4\} \end{pmatrix}$$

where $dr(x_{11}) = \{d_r^1(x_{11}), d_r^2(x_{11})\} = \{0.2, 0.2\}$, $dr(x_{12}) = \{d_r^1(x_{12}), d_r^2(x_{12}), d_r^3(x_{12})\} = \{0.3, 0.3, 0.6\}$ and $S_{\tilde{R}^*}(x_1) = (\frac{dr(x_{11})}{x_1}, \frac{dr(x_{12})}{x_1}) = (\frac{\{0.2,0.2\}}{x_1}, \frac{\{0.3,0.3,0.6\}}{x_1})$. According to the definition of cardinality, we can get $|S_{\tilde{R}^*}(x_1)| = 0.2 + 0.4 = 0.6$.

In fact, Qian [13] proposed four operations of the granular structure, proving that the new fuzzy granular structure can be generated by these four operations. Therefore, three operations(intersection operation, union operation and complement operation) for multi-fuzzy granular structures are given below. Before this, We first give the definition of $\sqcap$, $\sqcup$ and $\sim$ in the multi-fuzzy granules.

Definition 3.4: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$, in which $dp(x_{ij}) = \{d_p^1(x_{ij}), d_p^2(x_{ij}), \cdots, d_q^s(x_{ij})\}$, and $s = L(x_{ij};\ S_{\tilde{P}^*}(x_i))$; Also $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2), \cdots, S_{\tilde{q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i) = (\frac{dq(x_{i1})}{x_i}, \frac{dq(x_{i2})}{x_i}, \cdots, \frac{dq(x_{in})}{x_i})$, in which $dq(x_{ij}) = \{d_q^1(x_{ij}), d_q^2(x_{ij}), \cdots, d_q^t(x_{ij})\}$ and $t = L(x_{ij};\ S_{\tilde{Q}^*}(x_i))$. Three operators $\sqcap$, $\sqcup$ and $\sim$ on objects in the multi-fuzzy granules are defined as follow:

$$\begin{aligned} S_{\tilde{P}^*}(x_i) \sqcap S_{\tilde{Q}^*}(x_i) &= \{\frac{dp(x_{i1}) \cap dq(x_{i1})}{x_i}, \frac{dp(x_{i2}) \cap dq(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in}) \cap dq(x_{in})}{x_i}\} \\ &= \{\frac{\{d_{p\cap q}^1(x_{i1}), d_{p\cap q}^2(x_{i1}), ..., d_{p\cap q}^s(x_{i1})\}}{x_i}, \\ &\quad \frac{\{d_{p\cap q}^1(x_{i2}), d_{p\cap q}^2(x_{i2}), ..., d_{p\cap q}^s(x_{i2})\}}{x_i}, \\ &\quad \cdots, \frac{\{d_{p\cap q}^1(x_{in}), d_{p\cap q}^2(x_{in}), ..., d_{p\cap q}^s(x_{in})\}}{x_i}\} \end{aligned} \tag{6}$$

$$
\begin{aligned}
S_{\tilde{P}^*}(x_i) \sqcup S_{\tilde{Q}^*}(x_i) =& \{ \frac{dp(x_{i1}) \cup dq(x_{i1})}{x_i}, \frac{dp(x_{i2}) \cup dq(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in}) \cup dq(x_{in})}{x_i} \} \\
=& \{ \frac{\{d^1_{p\cup q}(x_{i1}), d^2_{p\cup q}(x_{i1}), ..., d^s_{p\cup q}(x_{i1})\}}{x_i}, \\
& \frac{\{d^1_{p\cup q}(x_{i2}), d^2_{p\cup q}(x_{i2}), ..., d^s_{p\cup q}(x_{i2})\}}{x_i}, \\
& \cdots, \frac{\{d^1_{p\cup q}(x_{in}), d^2_{p\cup q}(x_{in}), ..., d^s_{p\cup q}(x_{in})\}}{x_i} \}
\end{aligned}
\tag{7}
$$

$$
\sim S_{\tilde{P}^*}(x_i) = (\frac{\sim dp(x_{i1})}{x_i}, \frac{\sim dp(x_{i2})}{x_i}, \cdots, \frac{\sim dp(x_{in})}{x_i}) \tag{8}
$$

in which, $\sim dp(x_{ij}) = \{(1-d^1_p(x_{ij})), (1-d^2_p(x_{ij})), \cdots, (1-d^s_p(x_{ij}))\}, i, j = 1, 2, \cdots, n.$

Note, suppose $t < s$, we therefore extend $d_q(x_{ij})$ with maximum element $d^t_q(x_{ij})$ until it has same length with $d_p(x_{ij})$.

Definition 3.5: Let $K(U)$ be the collection of all multi-fuzzy granular structures on U, $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$. Three operators $\bigcap$, $\bigcup$ and $\wr$ are defined as follow:

$$
K(\tilde{P}^*) \bigcap K(\tilde{Q}^*) = \{ S_{\tilde{P}^*}(x_i) \sqcap S_{\tilde{Q}^*}(x_i) \} \tag{9}
$$

$$
K(\tilde{P}^*) \bigcup K(\tilde{Q}^*) = \{ S_{\tilde{P}^*}(x_i) \sqcup S_{\tilde{Q}^*}(x_i) \} \tag{10}
$$

$$
\wr K(\tilde{P}^*) = \{ \wr S_{\tilde{P}^*}(x_i) | \wr S_{\tilde{P}^*}(x_i) = \sim S_{\tilde{P}^*}(x_i) \} \tag{11}
$$

Note: The offered three operators can be regarded as intersection operation, union operation and complement operation in-between multi-fuzzy granular structures, which are used to refine and roughen multi-fuzzy granular structures and calculate complement of a multi-fuzzy granular structure, respectively. Next, we investigate some basic mathematical properties of these three operators.

Theorem 3.1: Let $\bigcap$, $\bigcup$ and $\wr$ be three operators on $K(U)$, they have the following algebra properties:

(1) $K(\tilde{P}^*) \bigcap K(\tilde{P}^*) = K(\tilde{P}^*), K(\tilde{P}^*) \bigcup K(\tilde{P}^*) = K(\tilde{P}^*)$;
(2) $K(\tilde{P}^*) \bigcap K(\tilde{Q}^*) = K(\tilde{Q}^*) \bigcap K(\tilde{P}^*), K(\tilde{P}^*) \bigcup K(\tilde{Q}^*) = K(\tilde{Q}^*) \bigcup K(\tilde{P}^*)$;
(3) $K(\tilde{P}^*) \bigcap (K(\tilde{P}^*) \bigcup K(\tilde{Q}^*)) = K(\tilde{P}^*), K(\tilde{P}^*) \bigcup (K(\tilde{P}^*) \bigcap K(\tilde{Q}^*)) = K(\tilde{P}^*)$;
(4) $(K(\tilde{P}^*) \bigcap K(\tilde{Q}^*)) \bigcap K(\tilde{R}^*) = K(\tilde{P}^*) \bigcap (K(\tilde{Q}^*) \bigcap K(\tilde{R}^*))$;
$(K(\tilde{P}^*) \bigcup K(\tilde{Q}^*)) \bigcup K(\tilde{R}^*) = K(\tilde{P}^*) \bigcup (K(\tilde{Q}^*) \bigcup K(\tilde{R}^*))$.

Proof: Item (1)(2)(3) and (4) are straightforward by Definition 3.5.

Theorem 3.2: Let $\bigcap$, $\bigcup$ and $\wr$ be three operators on $K(U)$, then

(1) $\wr(\wr K(\tilde{P}^*)) = K(\tilde{P}^*)$;
(2) $\wr(K(\tilde{P}^*) \bigcap K(\tilde{Q}^*)) = \wr K(\tilde{P}^*) \bigcup \wr K(\tilde{Q}^*)$;

(3) $\wr(K(\tilde{P}^*) \bigcup K(\tilde{Q}^*)) = \wr K(\tilde{P}^*) \bigcap \wr K(\tilde{Q}^*)$.

Proof: According to these operators of Definition 3.3, we can get (1) $\wr K(\tilde{P}^*) = \{\wr S_{\tilde{P}^*}(x_i) | \wr S_{\tilde{P}^*}(x_i) = \sim S_{\tilde{P}^*}(x_i)\}$, where $\sim dp(x_{ij}) = \{(1-d_p^1(x_{ij})), (1-d_p^2(x_{ij})), \cdots, (1-d_p{}^s(x_{ij}))\}$; Then $\sim(\sim dp(x_{ij})) = \{1-(1-d_p^1(x_{ij})), 1-(1-d_p^2(x_{ij})), \cdots, 1-(1-d_p^s(x_{ij}))\}$, $\sim(\sim S_{\tilde{P}^*}(x_i)) = (\frac{\sim(\sim dp(x_{i1}))}{x_i}, \frac{\sim(\sim dp(x_{i2}))}{x_i}, \cdots, \frac{\sim(\sim dp(x_{in}))}{x_i}) = S_{\tilde{P}^*}(x_i), i,j = 1,2,\cdots,n$. Therefore, $\wr(\wr K(\tilde{P}^*)) = K(\tilde{P}^*)$ holds. Items (2) and (3) are straightforward by Definition 3.4.

4. Partial-Order Relations on Multi-Fuzzy Granular Structures

Partial-order relation is one of the important ways to describe the granular structure. In the following, we propose four partial-order relations on multi-fuzzy granular structures.

Definition 4.1: Let $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$, $S_{\tilde{Q}^*}(x_i) = (\frac{dq(x_{i1})}{x_i}, \frac{dq(x_{i2})}{x_i}, \cdots, \frac{dq(x_{in})}{x_i})$. If $dp(x_{ij}) \leq dq(x_{ij})$, then $S_{\tilde{P}^*}(x_i) \subseteq S_{\tilde{Q}^*}(x_i)$.

Definition 4.2: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$, in which $dp(x_{ij}) = \{d_p^1(x_{ij}), d_p^2(x_{ij}), \cdots, d_p^s(x_{ij})\}$, and $s = L(x_{ij};\ S_{\tilde{P}^*}(x_i))$; Also $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2), \cdots, S_{\tilde{q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i) = (\frac{dq(x_{i1})}{x_i}, \frac{dq(x_{i2})}{x_i}, \cdots, \frac{dq(x_{in})}{x_i})$, where $dq(x_{ij}) = \{d_q^1(x_{ij}), d_q^2(x_{ij}), \cdots, d_q^t(x_{ij})\}$, $t = L(x_{ij};\ S_{\tilde{Q}^*}(x_i))$ and $i,j = 1,2,\cdots,n$, a partial-order relation $\preceq_1$ is defined as

$$\begin{aligned} K(\tilde{P}^*) \preceq_1 K(\tilde{Q}^*) &\Leftrightarrow S_{\tilde{P}^*}(x_i) \subseteq S_{\tilde{Q}^*}(x_i) \\ &\Leftrightarrow dp(x_{ij}) \leq dq(x_{ij}) \\ &\Leftrightarrow d_p^1(x_{ij}) \leq d_q^1(x_{ij}), d_p^2(x_{ij}) \leq d_q^2(x_{ij}), \ldots, d_p^s(x_{ij}) \leq d_q^s(x_{ij}). \end{aligned}$$

Clearly, $(K(U), \preceq_1)$ is a poset. Furthermore,

$$\begin{aligned} K(\tilde{P}^*) = K(\tilde{Q}^*) &\Leftrightarrow S_{\tilde{P}^*}(x_i) = S_{\tilde{Q}^*}(x_i) \\ &\Leftrightarrow dp(x_{ij}) = dq(x_{ij}) \\ &\Leftrightarrow d_p^1(x_{ij}) = d_q^1(x_{ij}), d_p^2(x_{ij}) = d_q^2(x_{ij}), \ldots, d_p^s(x_{ij}) = d_q^s(x_{ij}), i,j = \end{aligned}$$

$1,2,\cdots,n$, which can be written as $\tilde{P}^* = \tilde{Q}^*$.

$K(\tilde{P}^*) \prec_1 K(\tilde{Q}^*) \Leftrightarrow K(\tilde{P}^*) \preceq_1 K(\tilde{Q}^*)$ and $K(\tilde{P}^*) \neq K(\tilde{Q}^*)$ is denoted by $\tilde{P}^* \prec_1 \tilde{Q}^*$.

Theorem 4.1: Let $\bigcup$, $\bigcap$ and $\wr$ be three operators on $K(U)$, the following properties hold.

(1) If $K(\tilde{P}^*) \preceq_1 K(\tilde{Q}^*)$, then $\wr K(\tilde{Q}^*) \preceq_1 \wr K(\tilde{P}^*)$;
(2) $K(\tilde{P}^*) \preceq_1 K(\tilde{P}^*) \bigcup K(\tilde{Q}^*), K(\tilde{Q}^*) \preceq_1 K(\tilde{P}^*) \bigcup K(\tilde{Q}^*)$;
(3) $K(\tilde{P}^*) \bigcap K(\tilde{Q}^*) \preceq_1 K(\tilde{P}^*), K(\tilde{P}^*) \bigcap K(\tilde{Q}^*) \preceq_1 K(\tilde{Q}^*)$.

Proof:

$$\begin{aligned} K(\tilde{P}^*) \preceq_1 K(\tilde{Q}^*) &\Rightarrow \forall x_i \in U, S_{\tilde{P}^*}(x_i) \subseteq S_{\tilde{Q}^*}(x_i), i = 1,2,\cdots,n. \\ &\Rightarrow \forall x_i \in U, dp(x_{ij}) \leq dq(x_{ij}), i,j = 1,2,\cdots,n. \\ &\Rightarrow \forall x_i \in U, (1-d_q^1(x_{ij})) \leq (1-d_p^1(x_{ij})), (1-d_q^2(x_{ij})) \leq \\ &\quad (1-d_p^2(x_{ij})), \cdots, (1-d_q^s(x_{ij})) \leq (1-d_p{}^s(x_{ij})), i,j = 1,2,\cdots,n. \\ &\Rightarrow \forall x_i \in U, \wr S_{\tilde{Q}^*}(x_i) \preceq_1 \wr S_{\tilde{P}^*}(x_i). \Rightarrow \wr K(\tilde{Q}^*) \preceq_1 \wr K(\tilde{P}^*). \end{aligned}$$

In the same way, items (2) and (3) are easily obtained by the Definition 4.2.

Example 3: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, $\tilde{P}^*$ and $\tilde{Q}^*$ are fuzzy multirelations on U, where $K(\tilde{P}^*) = (S_{\tilde{p}^*}(x_1), S_{\tilde{p}^*}(x_2))$ and $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2))$,The fuzzy multirelation is denoted by the following maxtrix:

$$M(\tilde{P}^*) = \begin{pmatrix} \{0.1,0.1,0.2,0.2\} & \{0.1,0.1,0.4\} \\ \{0.2,0.5\} & \{0.2,0.5,0.5,0.6,0.7\} \end{pmatrix}$$

$$M(\tilde{Q}^*) = \begin{pmatrix} \{0.3,0.3,0.4\} & \{0.3,0.4,0.4\} \\ \{0.6,0.6,0.7.0.8\} & \{0.3,0.5,0.6,0.7,0.8\} \end{pmatrix}$$

From the values of these knowledge and the definition of $\preceq_1$, we know that $K(\tilde{P}^*)\preceq_1 K(\tilde{Q}^*)$.

Definition 4.3: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$, $dp(x_{ij}) = \{d_p^1(x_{ij}), d_p^2(x_{ij}), \cdots, d_p^s(x_{ij})\}$, $i, j = 1, 2, \cdots, n$ and $s = L(x_{ij};\, S_{\tilde{P}^*}(x_i))$. then the definition of the mean degree of multiple membership is as follows:

$$\breve{d}p(x_{ij}) = \frac{1}{L(x_{ij})} \sum_{s=1}^{L(x_{ij};\, S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij}) \tag{12}$$

Thus, the multi-fuzzy information granule can also be defined as

$$S_{\tilde{P}^*}(x_i) = (\frac{\breve{d}p(x_{i1})}{x_i}, \frac{\breve{d}p(x_{i2})}{x_i}, \cdots, \frac{\breve{d}p(x_{in})}{x_i}) \tag{13}$$

Definition 4.4: Let $S_{\tilde{P}^*}(x_i) = (\frac{\breve{d}p(x_{i1})}{x_i}, \frac{\breve{d}p(x_{i2})}{x_i}, \cdots, \frac{\breve{d}p(x_{in})}{x_i})$ and $S_{\tilde{Q}^*}(x_i) = (\frac{\breve{d}q(x_{i1})}{x_i}, \frac{\breve{d}q(x_{i2})}{x_i}, \cdots, \frac{\breve{d}q(x_{in})}{x_i})$, if $\breve{d}p(x_{ij}) \le \breve{d}q(x_{ij})$, then $S_{\tilde{P}^*}(x_i) \sqsubseteq S_{\tilde{Q}^*}$.

Definition 4.5: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, where $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{\breve{d}p(x_{i1})}{x_i}, \frac{\breve{d}p(x_{i2})}{x_i}, \cdots, \frac{\breve{d}p(x_{in})}{x_i})$, $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2), \cdots, S_{\tilde{q}^*}(x_n))$ and $S_{\tilde{Q}^*}(x_i) = (\frac{\breve{d}q(x_{i1})}{x_i}, \frac{\breve{d}q(x_{i2})}{x_i}, \cdots, \frac{\breve{d}q(x_{in})}{x_i})$, a partial-order relation $\preceq_2$ is defined as

$$K(\tilde{P}^*)\preceq_2 K(\tilde{Q}^*) \Leftrightarrow S_{\tilde{P}^*}(x_i) \sqsubseteq S_{\tilde{Q}^*}(x_i), i = 1, 2, \cdots, n$$
$$\Leftrightarrow \breve{d}p(x_{ij}) \le \breve{d}q(x_{ij}), i, j = 1, 2, \cdots, n,$$

in which $\breve{d}p_{ij} = \frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}$, $\breve{d}q_{ij} = \frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}$, should be written as $\tilde{P}^*\preceq_2\tilde{Q}^*$. Furthermore,

$$K(\tilde{P}^*) \simeq K(\tilde{Q}^*) \Leftrightarrow S_{\tilde{P}^*}(x_i) \simeq S_{\tilde{Q}^*}(x_i), i = 1, 2, \cdots, n.$$
$$\Leftrightarrow \breve{d}p(x_{ij}) = \breve{d}q(x_{ij}), i, j = 1, 2, \cdots, n$$ is written as $\tilde{P}^* \simeq \tilde{Q}^*$.

$K(\tilde{P}^*)\prec_2 K(\tilde{Q}^*) \Leftrightarrow K(\tilde{P}^*)\preceq_2 K(\tilde{Q}^*)$ and $K(\tilde{P}^*) \not\simeq K(\tilde{Q}^*)$ should be written as $\tilde{P}^*\prec_2\tilde{Q}^*$.

Theorem 4.2: Letting $K(U)$ be the collection of all multi-fuzzy granular structures on U, then $(K(U), \preceq_2)$ is a poset.

Proof: Let $K(\tilde{P}^*), K(\tilde{Q}^*), K(\tilde{R}^*) \in K(U)$, $K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{\breve{d}p(x_{i1})}{x_i}, \frac{\breve{d}p(x_{i2})}{x_i}, \cdots, \frac{\breve{d}p(x_{in})}{x_i})$, $\breve{d}p_{ij} = \frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}$; Also $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2), \cdots, S_{\tilde{q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i) = (\frac{\breve{d}q(x_{i1})}{x_i}, \frac{\breve{d}q(x_{i2})}{x_i}, \cdots, \frac{\breve{d}q(x_{in})}{x_i})$, $\breve{d}q_{ij} = \frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}$.

And $K(\tilde{R}^*)=(S_{\tilde{R}^*}(x_1),S_{\tilde{R}^*}(x_2),\cdots,S_{\tilde{R}^*}(x_n))$, $S_{\tilde{R}^*}(x_i)=(\frac{\breve{d}r(x_{i1})}{x_i},\frac{\breve{d}r(x_{i2})}{x_i},\cdots,\frac{\breve{d}r(x_{in})}{x_i})$,$\breve{d}r_{ij}=\frac{\sum_{m=1}^{L(x_{ij};S_{\tilde{R}^*}(x_i))} d_r^m(x_{ij})}{L(x_{ij};S_{\tilde{R}^*}(x_i))}$.

(1) It can be seen from Definition 4.5, for $x_i\in U$, $S_{\tilde{P}^*}(x_i)\sqsubseteq S_{\tilde{P}^*}(x_i)$, $\breve{d}p(x_{ij})\le\breve{d}p(x_{ij})$,$i,j=1,2,\cdots,n$, holds, we can find $\tilde{P}^*\preceq_2\tilde{P}^*$.

(2) Suppose $\tilde{P}^*\preceq_2\tilde{Q}^*$ and $\tilde{Q}^*\preceq_2\tilde{P}^*$, according to Definition 4.5, we can find

$$\begin{aligned}\tilde{P}^*\preceq_2\tilde{Q}^* &\Leftrightarrow S_{\tilde{P}^*}(x_i)\sqsubseteq S_{\tilde{Q}^*}(x_i)\\ &\Leftrightarrow \breve{d}p(x_{ij})\le\breve{d}q(x_{ij});\\ \tilde{Q}^*\preceq_2\tilde{P}^* &\Leftrightarrow S_{\tilde{Q}^*}(x_i)\sqsubseteq S_{\tilde{P}^*}(x_i)\\ &\Leftrightarrow \breve{d}q(x_{ij})\le\breve{d}p(x_{ij}) \text{ for } i=1,2,\cdots,n.\end{aligned}$$

Therefore, we obtain that $\breve{d}p(x_{ij})=\breve{d}q(x_{ij})=\breve{d}p(x_{ij})$, which is $\breve{d}p(x_{ij})=\breve{d}q(x_{ij})$. Hence $\tilde{P}^*\simeq\tilde{Q}^*$.

(3) Suppose $\tilde{P}^*\preceq_2\tilde{Q}^*$, $\tilde{Q}^*\preceq_2\tilde{R}^*$. $\tilde{P}^*\preceq_2\tilde{Q}^*\Leftrightarrow S_{\tilde{P}^*}(x_i)\sqsubseteq S_{\tilde{Q}^*}(x_i)\Leftrightarrow\breve{d}p(x_{ij})\le\breve{d}q(x_{ij})\Leftrightarrow \frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}\le\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_p^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}$, for $i,j=1,2,\cdots,n$. $\tilde{Q}^*\preceq_2\tilde{R}^*\Leftrightarrow S_{\tilde{Q}^*}(x_i)\sqsubseteq S_{\tilde{R}^*}(x_i)\Leftrightarrow\breve{d}p(x_{ij})\le\breve{d}r(x_{ij})\Leftrightarrow\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_p^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}\le\frac{\sum_{m=1}^{L(x_{ij};S_{\tilde{R}^*}(x_i))} d_r^m(x_{ij})}{L(x_{ij};S_{\tilde{R}^*}(x_i))}$, for $i,j=1,2,\cdots,n$. So we can get $\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}\le\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_p^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}\le\frac{\sum_{m=1}^{L(x_{ij};S_{\tilde{R}^*}(x_i))} d_r^m(x_{ij})}{L(x_{ij};S_{\tilde{R}^*}(x_i))}$, thus $\breve{d}p(x_{ij})\le\breve{d}q(x_{ij})\le\breve{d}r(x_{ij})$, $\breve{d}p(x_{ij})\le\breve{d}q(x_{ij})\Leftrightarrow S_{\tilde{P}^*}(x_i)\sqsubseteq S_{\tilde{R}^*}(x_i)$. Hence, $\tilde{P}^*\preceq_2\tilde{R}^*$.

Theorem 4.3: The partial-order relation $\preceq_1$ is a special instance of partial-order relation $\preceq_2$.

Proof: Suppose that $K(\tilde{P}^*),K(\tilde{Q}^*)\in K(U)$ and $K(\tilde{P}^*)\preceq_1K(\tilde{Q}^*)$.According to the Definition 4.2, we can know $S_{\tilde{P}^*}(x_i)\sqsubseteq S_{\tilde{Q}^*}(x_i)$ and $\breve{d}p(x_{ij})\le\breve{d}q(x_{ij})$. So, $\breve{d}q(x_{ij})\le\breve{d}p(x_{ij})$. Hence $S_{\tilde{P}^*}(x_i)\sqsubseteq S_{\tilde{Q}^*}(x_i)$, we obtain that $K(\tilde{P}^*)\preceq_2K(\tilde{Q}^*)$. Therefore, partial-order relation $\preceq_1$ is a special instance of partial-order relation $\preceq_2$.

Definition 4.6: Let $K(\tilde{P}^*),K(\tilde{Q}^*)\in K(U)$, $K(\tilde{P}^*)=(S_{\tilde{P}^*}(x_1),S_{\tilde{p}^*}(x_2),\cdots,S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i)=(\frac{dp(x_{i1})}{x_i},\frac{dp(x_{i2})}{x_i},\cdots,\frac{dp(x_{in})}{x_i})$, in which $dp(x_{ij})=\{d_p^1(x_{ij}),d_p^2(x_{ij}),\cdots,d_p^s(x_{ij})\}$, $s=L(x_{ij};\ S_{\tilde{P}^*}(x_i))$; $K(\tilde{Q}^*)=(S_{\tilde{q}^*}(x_1),S_{\tilde{q}^*}(x_2),\cdots,S_{\tilde{q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i)=(\frac{dq(x_{i1})}{x_i},\frac{dq(x_{i2})}{x_i},\cdots,\frac{dq(x_{in})}{x_i})$, in which $dq(x_{ij})=\{d_q^1(x_{ij}),d_q^2(x_{ij}),\cdots,d_q^t(x_{ij})\}$ and $t=L(x_{ij};\ S_{\tilde{Q}^*}(x_i))$. The partial-order relation $\preceq_3$ is defined as $K(\tilde{P}^*)\preceq_3K(\tilde{Q}^*)\Leftrightarrow|S_{\tilde{P}^*}(x_i)|\le|S_{\tilde{Q}^*}(x_i)|,i=1,2,\cdots,n$, where $|S_{\tilde{P}^*}(x_i)|=\sum_{j=1}^{n}(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))})$ and $|S_{\tilde{Q}^*}(x_i)|=\sum_{j=1}^{n}(\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))})$.

Furthermore, $K(\tilde{P}^*)\cong K(\tilde{Q}^*)\Leftrightarrow|S_{\tilde{P}^*}(x_i)|=|S_{\tilde{Q}^*}(x_i)|,i=1,2,\cdots,n$, is written as $\tilde{P}^*\cong\tilde{Q}^*$. $K(\tilde{P}^*)\prec_3K(\tilde{Q}^*)\Leftrightarrow K(\tilde{P}^*)\preceq_3K(\tilde{Q}^*)$ and $K(\tilde{P}^*)\ncong K(\tilde{Q}^*)$ should be written as $\tilde{P}^*\prec_3\tilde{Q}^*$.

Theorem 4.4: Let $K(U)$ be the collection of all multi-fuzzy granular structures on U, then $(K(U), \preceq_3)$ is a poset.

Proof: Suppose that $K(\tilde{P}^*), K(\tilde{Q}^*), K(\tilde{R}^*) \in K(U)$, $K(\tilde{P}^*) = (S_{\tilde{p}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$; $K(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1), S_{\tilde{q}^*}(x_2), \cdots, S_{\tilde{q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i) = (\frac{dq(x_{i1})}{x_i}, \frac{dq(x_{i2})}{x_i}, \cdots, \frac{dq(x_{in})}{x_i})$; And $K(\tilde{R}^*) = (S_{\tilde{R}^*}(x_1), S_{\tilde{R}^*}(x_2), \cdots, S_{\tilde{R}^*}(x_n))$, $S_{\tilde{R}^*}(x_i) = (\frac{dr(x_{i1})}{x_i}, \frac{dr(x_{i2})}{x_i}, \cdots, \frac{dr(x_{in})}{x_i})$.

(1) According to the definition of the partial-order relation $\preceq_3$, we can find $\tilde{P}^* \preceq_3 \tilde{P}^*$.

(2) Suppose $\tilde{P}^* \preceq_3 \tilde{Q}^*$, $\tilde{Q}^* \preceq_3 \tilde{P}^*$. According the definition of the partial-order relation $\preceq_3$, we can find $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{Q}^*}(x_i)| \Leftrightarrow \sum_{j=1}^{n}(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}) \leq \sum_{j=1}^{n}(\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))})$ and $|S_{\tilde{Q}^*}(x_i)| \leq |S_{\tilde{P}^*}(x_i)| \Leftrightarrow \sum_{j=1}^{n}(\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}) \leq \sum_{j=1}^{n}(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))})$. So, $\sum_{j=1}^{n}(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}) \leq \sum_{j=1}^{n}(\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}) \leq \sum_{j=1}^{n}(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))})$. Therefore, we obtain that $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{Q}^*}(x_i)| \leq |S_{\tilde{P}^*}(x_i)|$, that is $|S_{\tilde{P}^*}(x_i)| = |S_{\tilde{Q}^*}(x_i)|$. Hence, $\tilde{P}^* \cong \tilde{Q}^*$.

(3) Suppose $\tilde{P}^* \preceq_3 \tilde{Q}^*$, $\tilde{Q}^* \preceq_3 \tilde{R}^*$, according the definition of the partial-order relation $\preceq_3$, we can find $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{Q}^*}(x_i)| \Leftrightarrow \sum_{j=1}^{n}(\frac{\sum_{s=1}^{L(x_{ij};S_{\tilde{P}^*}(x_i))} d_p^s(x_{ij})}{L(x_{ij};S_{\tilde{P}^*}(x_i))}) \leq \sum_{j=1}^{n}(\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))})$ and $|S_{\tilde{Q}^*}(x_i)| \leq |S_{\tilde{R}^*}(x_i)| \Leftrightarrow \sum_{j=1}^{n}(\frac{\sum_{t=1}^{L(x_{ij};S_{\tilde{Q}^*}(x_i))} d_q^t(x_{ij})}{L(x_{ij};S_{\tilde{Q}^*}(x_i))}) \leq \sum_{j=1}^{n}(\frac{\sum_{m=1}^{L(x_{ij};S_{\tilde{R}^*}(x_i))} d_r^m(x_{ij})}{L(x_{ij};S_{\tilde{R}^*}(x_i))})$, for $i, j = 1, 2, \cdots, n$. Therefore, we obtain that $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{Q}^*}(x_i)| \leq |S_{\tilde{R}^*}(x_i)|$, that is $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{R}^*}(x_i)|$. Thus, $\tilde{P}^* \preceq_3 \tilde{R}^*$.

Summarizing the (1)-(3), $(K(U), \preceq_3)$ is a poset.

Theorem 4.5: The partial-order relation $\preceq_2$ is a special instance of partial-order relation $\preceq_3$.

Proof: Suppose that $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$ and $K(\tilde{P}^*) \preceq_2 K(\tilde{Q}^*)$.According to the Definition 4.5, we can know $S_{\tilde{P}^*}(x_i) \subseteq S_{\tilde{Q}^*}(x_i)$ and $d_p^1(x_{ij}) \leq d_q^1(x_{ij}), d_p^2(x_{ij}) \leq d_q^2(x_{ij}), \ldots, d_p^s(x_{ij}) \leq d_q^s(x_{ij})$. So, $\breve{d}q(x_{ij}) \leq \breve{d}p(x_{ij})$. Hence $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{Q}^*}(x_i)|$, we obtain that $K(\tilde{P}^*) \preceq_3 K(\tilde{Q}^*)$. Therefore, partial-order relation $\preceq_2$ is a special instance of partial-order relation $\preceq_3$.

Definition 4.8: Let $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, in which $K(\tilde{P}^*) = (S_{\tilde{p}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n))$, $S_{\tilde{P}^*}(x_i) = (\frac{dp(x_{i1})}{x_i}, \frac{dp(x_{i2})}{x_i}, \cdots, \frac{dp(x_{in})}{x_i})$; $K(\tilde{Q}^*) = (S_{\tilde{Q}^*}(x_1), S_{\tilde{Q}^*}(x_2), \cdots, S_{\tilde{Q}^*}(x_n))$, $S_{\tilde{Q}^*}(x_i) = (\frac{dq(x_{i1})}{x_i}, \frac{dq(x_{i2})}{x_i}, \cdots, \frac{dq(x_{in})}{x_i})$. A partial-order relation $\preceq_4$ is defined as

$K(\tilde{P}^*) \preceq_4 K(\tilde{Q}^*) \Leftrightarrow$for $K(\tilde{P}^*)$, there exists a sequence $K'(\tilde{Q}^*)$ of $K(\tilde{Q}^*)$, such that $|S_{\tilde{P}^*}(x_i)| \leq |S_{\tilde{Q}^*}(x_i')|$,$i = 1, 2, \cdots, n$, where $K'(\tilde{Q}^*) = (S_{\tilde{q}^*}(x_1'), S_{\tilde{q}^*}(x_2'), \cdots, S_{\tilde{q}^*}(x_n'))$. Furthermore, $K(\tilde{P}^*) \approx K(\tilde{Q}^*) \Leftrightarrow |S_{\tilde{P}^*}(x_i)| = |S_{\tilde{Q}^*}(x_i')|$, $i = 1, 2, \cdots, n$, which is denoted by $\tilde{P}^* \approx \tilde{Q}^*$. $K(\tilde{P}^*) \prec_4 K(\tilde{Q}^*) \Leftrightarrow K(\tilde{P}^*) \preceq_4 K(\tilde{Q}^*)$ and $K(\tilde{P}^*) \not\approx K(\tilde{Q}^*)$, which is written as $\tilde{P}^* \prec_4 \tilde{Q}^*$.

Therorem 4.7: Let $K(U)$ be the collection of all multi-fuzzy granular structures on U, then $(K(U), \preceq_4)$ is a poset.

Proof: According to the definition of order-relation $\preceq_4$, it can be proved that $(K(U), \preceq_4)$ is a poset.

Therorem 4.6: The partial-order relation $\preceq_3$ is a special instance of partial-order relation $\preceq_4$.

Proof: It is easy to prove from Definition 4.6 and Definition 4.7.

5. Multi-Fuzzy Information Granularity and Its Axiomatic Method

Information granularity can be regarded as a measure of the degree of aggregation (regularity) of knowledge on the domain classification. Fuzzy multi-information granularity is equally important, and can be used to describe the classification ability of multi-fuzzy granular structure. According to Zadeh's research on granular computing, information granularity should be expressed the granulation degree of objects from a hierarchical perspective[2]. That is, information granularity should represent the hierarchical relationship between multi-fuzzy granular structures.

From the viewpoint of sizes of information granules, in the following, we introduce the definition of multi-fuzzy information granularity.

Definition 5.1: Suppose $K(\tilde{R}^*) \in K(U)$, in which $K(\tilde{R}^*) = (S_{\tilde{R}^*}(x_1), S_{\tilde{R}^*}(x_2), \cdots, S_{\tilde{R}^*}(x_n))$. Then, the multiple fuzzy information granularity of $\tilde{R}^*$ is defined as

$$GK(\tilde{R}^*) = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\tilde{R}^*}(x_i)|}{n} \tag{14}$$

where $|S_{\tilde{R}^*}(x_i)|$ is the cardinality of the muti-fuzzy information granule $S_{\tilde{R}^*}(x_i)$. Axiomatic methods and constructive methods play an equal role in mathematical definition. In the following, we give the axiomatic constraint to define a multi-fuzzy information granularity in the context of multi-fuzzy granular structures by employing the partial-order relation $\preceq_3$.

Definition 5.2: Let $K(U)$ be the collection of all multi-fuzzy granular structures on U, if $\forall K(\tilde{P}^*) \in K(U)$, there is a real number $G(\tilde{P}^*)$ with the following properties:

(1) $G(\tilde{P}^*) \geq 0$(nonnegative);
(2) For $\forall K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, if $K(\tilde{P}^*) \cong K(\tilde{Q}^*)$, then $G(\tilde{P}^*) = G(\tilde{Q}^*)$(invariability);
(3) For $\forall K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, for $K(\tilde{P}^*) \prec_3 K(\tilde{Q}^*)$, then $G(\tilde{P}^*) < G(\tilde{Q}^*)$(monotonicity).

then G is called a multi-fuzzy information granularity. Obtained by the above axiomatic method, we come to the following theorem.

Theorem 5.1: Let $\forall K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$, if $K(\tilde{P}^*) \preceq_3 K(\tilde{Q}^*)$, then $G(\tilde{P}^*) \leq G(\tilde{Q}^*)$. This also satisfies the partial-order relation $\preceq_1$, $\preceq_2$ and $\preceq_4$, respectively. It is easy to prove by Definition 5.2 and the definition of partial-order relations $\preceq_1$, $\preceq_2$, $\preceq_3$ and $\preceq_4$. Furthermore, the information granularity GK given by Definition 5.1 satisfies the axiomatic definition methods. Next, we give proof:

(1) Obviously, it is nonnegative;

(2) Letting $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U), K(\tilde{P}^*) = (S_{\tilde{P}^*}(x_1), S_{\tilde{p}^*}(x_2), \cdots, S_{\tilde{p}^*}(x_n)), K(\tilde{Q}^*) = (S_{\tilde{Q}^*}(x_1), S_{\tilde{Q}^*}(x_2), \cdots, S_{\tilde{Q}^*}(x_n))$. According to the definition of partial-order relation $\preceq_3$, if $\tilde{P}^* \cong \tilde{Q}^*$, then $|S_{\tilde{P}^*}(x_i)| = |S_{\tilde{Q}^*}(x_i)|$, $i = 1, 2, \cdots, n$. Therefore

$$GK(\tilde{P}^*) = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\tilde{P}^*}(x_i)|}{n} = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\tilde{Q}^*}(x_i)|}{n} = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\tilde{Q}^*}(x_i)|}{n} = GK(\tilde{Q}^*).$$

(3) Letting $K(\tilde{P}^*), K(\tilde{Q}^*) \in K(U)$ with $\tilde{P}^* \prec_3 \tilde{Q}^*$, such that

$$K(\tilde{P}^*) \prec_3 K(\tilde{Q}^*) \Leftrightarrow |S_{\tilde{P}^*}(x_i)| < |S_{\tilde{Q}^*}(x_i)|, i = 1, 2, \cdots n.$$

$$\Leftrightarrow \sum_{j=1}^{n}\left(\frac{\sum_{s=1}^{L(x_{ij})} d^s_{\tilde{p}^*}(x_i)}{L(x_{ij})}\right) < \sum_{j=1}^{n}\left(\frac{\sum_{s=1}^{L(x_{ij})} d^s_{\tilde{q}^*}(x_{ij})}{L(x_{ij})}\right),$$

$$GK(\tilde{P}^*) = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\tilde{P}^*}(x_i)|}{n} < \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\tilde{Q}^*}(x_i)|}{n} = GK(\tilde{Q}^*).$$

Thus, $GK(\tilde{P}^*) < GK(\tilde{Q}^*)$.Through monotonicity, meaning that relation $\tilde{Q}^*$ is more chaotic than relation $\tilde{P}^*$, so the uncertainty of group $\tilde{Q}^*$ is higher.

In conclusion, the information granularity GK given by Definition 5.1 satisfies the axiomatic definition methods in Definition 5.2.

6. Conclusion

Based on fuzzy multirelations, this paper first proposes a kind of fuzzy information granular structure generated by multi-fuzzy relations in the universe of discourse, namely multi-fuzzy granular structure. Based on this structure, four fuzzy multiple partial-order relations are defined, and the hierarchical characteristics of these four partial order relations and related mathematical conclusions are studied. Secondly, for fuzzy multi-information GCM, a multi-fuzzy information granularity is proposed. Finally, an axiomatic method for defining fuzzy multiple information granularity on the fuzzy multiple granular structure is given, and axiomatic constraints with partial-order relations are established. This paper studies the internal structure nesting relationship and information granularity measurement of fuzzy multi-information GCM, which provides a theoretical basis for the development of artificial intelligence and decision-making theory. In the next step of this paper, the reasoning logic and algorithm design of fuzzy multiple information will be further studied.

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