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Some Advances on the Solution of the Generalized Law of Importation for Fuzzy Implication Functions

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Abstract. The law of importation has attracted the interest of many researchers devoted to fuzzy implication functions in the last decades. This property has several important applications, especially in approximate reasoning and image processing. Several generalizations of this property have been proposed. Specifically, one generalization related to the law of migrativity was recently introduced by Baczyński et al. in which two fuzzy implication functions are involved. In this paper, some advances on the solution of this functional equation for the particular case where the involved fuzzy conjunction is a t-norm are presented. Indeed, a complete characterization of all those pairs of fuzzy implication functions with a strict natural fuzzy negation satisfying the generalized law of importation is achieved.

Keywords. Fuzzy implication function, T-norm, Functional equation, Generalized law of importation, Natural negation

1. Introduction

One of the main lines of research in fuzzy logic is the theoretical study of fuzzy implication functions. These operators, which are the generalization of the classical implication to the fuzzy framework, have important applications in approximate reasoning or image processing [2]. For a concrete application, fuzzy implication functions fulfilling some additional algebraical properties are needed and therefore, the characterization of such operators (obtained by solving functional equations) provides a bunch of operators useful for those applications.

One of such additional properties is the *law of importation* with respect to a fuzzy conjunction C. This property is given by

$$I(C(x,y),z) = I(x,I(y,z)) \quad \text{for all} \quad x,y,z \in [0,1], \; (\mathbf{LI}) \tag{1}$$

where I is a fuzzy implication function and C a fuzzy conjunction (for more details see [1,6]). The significance of the law of importation is related to the simplification of the

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process of applying the Compositional Rule of Inference (CRI) of Zadeh reducing its complexity through the so-called Hierarchical CRI [4]. In [3], Baczyński et al. discuss a generalization of the laws of α-migrativity, which leads to a generalized version of the law of importation. In this way, the triple (C,I,J) is said to satisfy the *generalized law of importation* if it satisfies

$$I(C(x,y),z) = I(x,J(y,z)) \quad \text{for all} \quad x,y,z \in [0,1], \qquad \textbf{(GLI)}$$

where I,J are fuzzy implication functions and C a fuzzy conjunction. It is straightforward to check that if $I = J$, then the standard law of importation is retrieved. In [3], an initial study of the functional equation is presented studying in depth its fulfillment when the pairs (I,C) or (J,C) satisfy the law of importation. However, these conditions are not necessary and the characterization of the solutions remains an open problem.

In this paper, we present some advances on the characterization of the solutions for the particular case in which a t-norm is considered as fuzzy conjunction and the two involved fuzzy implication functions have strict natural negations. This is an ongoing study with the final goal of characterizing all pairs of fuzzy implication functions with continuous natural negations which satisfy **(GLI)** with a t-norm.

2. Preliminaries

For the sake of completeness, we recall the definitions of t-norms, fuzzy implication functions and fuzzy negations.

Definition 1 ([1]). *A function $T : [0,1]^2 \to [0,1]$ is called a* triangular norm *(shortly t-norm) if T is associative, commutative, increasing in each variable and has neutral element* 1.

Definition 2 ([1]). *A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be a* fuzzy implication function *if it satisfies:*

(I1) $I(x,z) \geq I(y,z)$ *when* $x \leq y$, *for all* $z \in [0,1]$.
(I2) $I(x,y) \leq I(x,z)$ *when* $y \leq z$, *for all* $x \in [0,1]$.
(I3) $I(0,0) = I(1,1) = 1$ *and* $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the values $I(x,0)$ and $I(1,x)$ are not derived.

Definition 3 ([1]). *A decreasing function $N : [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. Moreover, a fuzzy negation is strict if it is strictly decreasing and continuous.*

The natural negation of a fuzzy implication function will be also useful in our study.

Definition 4 ([1]). *Let I be a fuzzy implication function. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the* natural negation *of I.*

3. The generalised law of importation: a characterization

In this section we will prove the characterization of all pairs of fuzzy implication functions with strict natural negations satisfying the generalized law of importation.

Proposition 1. *Let $I,J:[0,1]^2 \to [0,1]$ be two binary functions with N_I and N_J strict negations and T be a t-norm. If the triple (I,J,T) satisfies $(\mathbf{GLI})$ then for all $x,y \in [0,1]$,*

$$I(x,y) = N_I(T(x,N_J^{-1}(y))), \tag{2}$$

$$J(x,y) = N_J(T(x,N_J^{-1}(y))). \tag{3}$$

Proof. First of all, note that N_J is a strict fuzzy negation and, consequently, $Ran(N_J) = [0,1]$. Then, in order to prove Expression (2), it is enough to check that

$$I(x,N_J(y)) = N_I(T(x,N_J^{-1}(N_J(y)))).$$

Thus,

$$I(x,N_J(y)) = I(x,J(y,0)) = I(T(x,y),0) = N_I(T(x,y)) = N_I(T(x,N_J^{-1}(N_J(y)))).$$

On the other hand, to prove Expression (3), note that

$$I(x,y) = I(T(1,x),y) = I(1,J(x,y)) = N_I(T(1,N_J^{-1}(J(x,y))) = N_I \circ N_J^{-1}(J(x,y))$$

and then $J(x,y) = N_J \circ N_I^{-1}(I(x,y)) = N_J(T(x,N_J^{-1}(y)))$ for all $x,y \in [0,1]$. ■

Next result shows that if we consider two binary functions $I,J:[0,1]^2 \to [0,1]$ whose expressions are given by Formulas (2) and (3) respectively, then the triple (I,J,T) fulfills the generalized law of importation.

Proposition 2. *Let T be a t-norm and let $I,J:[0,1]^2 \to [0,1]$ be two binary functions whose expressions are given by Formulas (2) and (3), respectively, where N_I and N_J are strict negations. Then, the triple (I,J,T) fulfills $(\mathbf{GLI})$.*

Proof. Suppose that binary operations I and J are given by Formulas (2) and (3), respectively. Then, for all $x,y,z \in [0,1]$, we have that

$$\begin{aligned} I(T(x,y),z) &= N_I(T(T(x,y),N_J^{-1}(z))) = N_I(T(x,T(y,N_J^{-1}(z)))) \\ &= N_I(T(x,N_J^{-1}(N_J(T(y,N_J^{-1}(z)))))) = I(x,J(y,z)). \end{aligned}$$

■

Taking into account the two previous results, the next theorem follows.

Theorem 1. *Let T be a t-norm and $I,J:[0,1]^2 \to [0,1]$ be two binary functions with N_I and N_J strict negations. Then, the triple (I,J,T) satisfies $(\mathbf{GLI})$ if and only if,*

$$I(x,y) = N_I(T(x,N_J^{-1}(y))),$$

$$J(x,y) = N_J(T(x,N_J^{-1}(y))).$$

Proof. It follows immediately from Propositions 1 and 2. ■

The following example illustrates the previous Theorem 1.

Example 1. *Let us consider the strict negations $N_J(x) = 1 - x$ and $N_I(x) = 1 - x^2$ for all $x \in [0,1]$ and the product t-norm $T_{\mathbf{P}}(x,y) = xy$ for all $x, y \in [0,1]$. In this case, applying Formulas (2) and (3), we obtain that the only fuzzy implication functions with such natural negations satisfying* **(GLI)** *with $T_{\mathbf{P}}$ are given by*

$$I(x,y) = 1 - x^2 + 2yx^2 - x^2y^2,$$
$$J(x,y) = 1 - x + xy \text{ (the Reichenbach implication).}$$

4. Conclusions and Future Work

In this paper, we have characterized all the couples of fuzzy implication functions (I,J) fulfilling $(\mathbf{GLI})$ with a t-norm T when their natural negations are strict. We have shown that the expressions of these fuzzy implication functions can be obtained from their natural negations N_I and N_J and from the considered t-norm T. As a future work, we wish to continue the study in the case that the natural negations are continuous (but not necessarily strict) and, in a more general framework, we want to consider a conjunctive uninorm instead of a t-norm.

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References

[1] M. Baczyński, B. Jayaram, Fuzzy Implications. Studies in Fuzziness and Soft Computing, vol. 231. Springer, Berlin Heidelberg, 2008.

[2] M. Baczyński, B. Jayaram, S. Massanet, and J. Torrens, "Fuzzy implications: Past, present, and future", in Springer Handbook of Computational Intelligence, J. Kacprzyk and W. Pedrycz, Eds. Springer Berlin Heidelberg, 2015, pp. 183–202.

[3] M. Baczyński, B. Jayaram and R. Mesiar, Fuzzy Implications: alpha migrativity and generalised laws of importation, Information Science. 2020; 531: 87-96.

[4] B. Jayaram, On the law of importation $(x \wedge y) \to z \equiv (x \to (y \to z))$ in fuzzy logic, IEEE Transactions on Fuzzy Systems. 2008; 16: 130–144.

[5] M. Mas, M. Monserrat, J. Torrens, E. Trillas, A survey on fuzzy implication functions,IEEE Transactions on Fuzzy Systems. 2007; 15(6): 1107-1121.

[6] S. Massanet, J. Torrens. Characterization of Fuzzy Implications Functions With a Continuous Natural Negation Satisfying the Law of Importation With a Fixed t-Norm. IEEE Transactions on Fuzzy Systems. 2017;25(1):100-113.

[7] S.Massanet, J.Torrens. The law of Importation versus the exchange principle on fuzzy implications. Fuzzy Sets Systems. 2011; 168 (1): 47-69.