

# Solving a Fuzzy Linear Equation with a Variable, Using the Expected Interval of a Fuzzy Number

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**Abstract.** In this work, a fuzzy linear equation  $AX + B = 0$ , is solved, where  $A$ ,  $B$  y  $C$  are triangular diffuse numbers, could also be trapezoidal. For this type of equations there are several solution methods, the classic method that does not always obtain solutions, the most used is the method of alpha cuts and arithmetic intervals that although it always finds a solution, as a value is taken closer to zero (more inaccurate), the solution satisfies less to the equation. The new method using the expected interval, allows to obtain a smaller support set where the solutions come closer to satisfying the equation, also allows to find a single interval where the best solutions for decision making are expected to be found. It is recommended to study the incorporation of the concept of the expected interval in the methods to solve systems of fuzzy linear equations

**Keywords.** Fuzzy number, expected interval, fuzzy arithmetic, alpha cut, fuzzy equations.

## 1. Introduction

The fuzzy logic created by Lofti Zadeh in 1965 is an adequate means to represent and model a type of uncertainty such as imprecision, which appears in various reality problems [1].

Fuzzy linear equations are applied in various areas such as optimization, control systems, physics, economy [2-5]. In this work, we focus on the case of the linear equation of the form;

$$\tilde{A}\tilde{X} + \tilde{B} = \tilde{C} \quad (1)$$

where  $\tilde{A}$ ,  $\tilde{B}$  y  $\tilde{C}$  are inaccurate data; since the imprecision is represented with fuzzy triangular numbers, the equation is called the fuzzy linear equation (FLE).

The problem of solving a FLE is not trivial, even for cases such as  $\tilde{A}\tilde{X} = \tilde{B}$ , where  $\tilde{A}$  y  $\tilde{B}$  are triangular fuzzy numbers (TFN), i.e., for some TFN  $\tilde{A}$  y  $\tilde{B}$ , you can't find a fuzzy set  $\tilde{X}$ , such that, using fuzzy arithmetic,  $\tilde{A}\tilde{X}$  is exactly equal to  $\tilde{B}$  [6].

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In [7], a strategy is proposed for the solution of a FLE: First, to seek the solution with the classic method  $\tilde{X}_c$ , because it is the most accurate solution, but it does not always exist; second, if there was no classic solution, look for the solution by the principle of extension  $\tilde{X}_e$  whose construction process is quite complicated (very little used); and finally the solution by the method of alpha cuts and interval arithmetic  $\tilde{X}_a$  that gets an approximate solution, but with the condition that always exists. The relationship is true

$$\tilde{X}_c \leq \tilde{X}_e \leq \tilde{X}_a \quad (2)$$

While the alpha-cuts and arithmetic interval method always gets a solution for Eq. (1), of the form  $x(\alpha) = [x_1(\alpha), x_2(\alpha)]$ ,  $\alpha \in [0, 1]$ ; has a problem, according to the ratio (2), the solution support set is much wider than the other methods, i.e. for the alpha cut  $\alpha=1$ , you get the most accurate (exact) solution, which completely satisfies the equation; as the equation's tends to 0 (increases inaccuracy) the solution of the equation tends to satisfy every single thing much less to the equation.

That is why, in this work it is proposed to incorporate the concept of the expected interval to the method of alpha cuts and interval arithmetic to obtain a solution with a smaller support set and find a single interval where the best solutions for decision making are found.

The method of alpha cuts and interval arithmetic is the basis of several current methods for solving systems of fuzzy linear equations [8-10].

## 2. Material and Methods

### 2.1. Fuzzy Numbers

**Definition 1:** A generalized LR fuzzy number  $\tilde{A}$  with the membership function  $\mu_{\tilde{A}}(x)$ ,  $x \in \mathbb{R}$  can be defined as [11].

$$\mu_{\tilde{A}}(x) = \begin{cases} L(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ R(x), & c \leq x \leq d, \\ 0, & \text{otherwise} \end{cases}$$

Where  $L(x)$  is the left membership function that is an increasing function on  $[a, b]$  and  $R(x)$  is the right membership function that is a decreasing function on  $[c, d]$  such that  $L(a) = R(d) = 0$  and that  $L(b) = R(c) = 1$ .

**Definition 2:** A triangular fuzzy number (TFN) (see Figure 1) is represented by  $\tilde{A} = (a_1, a_2, a_3)$ .

**Definition 3:** A TFN  $\tilde{A} = (a_1, a_2, a_3)$  is positive if and only if  $a_1 > 0$ .

**Definition 4:** Given a fuzzy set  $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$ , the support of  $\tilde{A}$  is defined as the ordinary set.

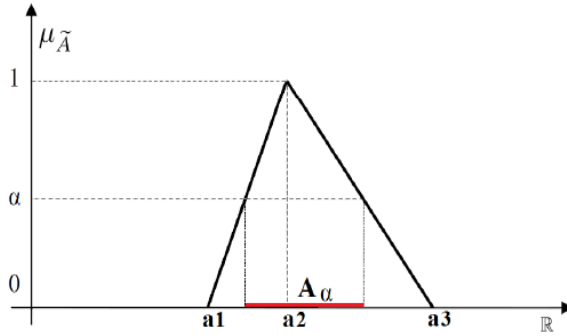
$$Sop(\tilde{A}) = \{x \in \mathbb{R}, \mu_{\tilde{A}}(x) > 0\}$$

**Definition 5:** The *alpha cut* set of a fuzzy set  $\tilde{A}$  is given by:

$$A_\alpha = \{x \in \mathbb{R} / \mu_{\tilde{A}}(x) \geq \alpha\}, \quad \alpha \in [0, 1]$$

For a fuzzy set with a triangular type membership function (see Figure 1) the alpha cut is given by:  $A_\alpha = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]$

The alpha cut is perhaps the most important concept of fuzzy sets, because by adjusting the value  $\alpha$ , the range or set of values that satisfy a certain degree of belonging, the level of satisfaction, precision of the result or robustness of the model can be determined [12].



**Figure 1.** Triangular fuzzy Number.

**Theorem 1:** (Representation theorem [13]) If  $\tilde{A}$  is a fuzzy set and  $A_\alpha$  its alpha cuts,  $\alpha \in [0, 1]$ , it is verified that:

$$\tilde{A} = \bigcup_{\alpha \in [0, 1]} \alpha A_\alpha$$

This formal notation should be understood as the equality between the membership functions of both sets, where

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & \text{si } x \text{ solo si } x \in A_\alpha \\ 0 & \text{otherwise} \end{cases}$$

On the other hand,

$$\mu_{\tilde{A}}(x) = \sup_{\alpha \in [0, 1]} \min(\alpha, \mu_{A_\alpha}(x))$$

## 2.2. Interval arithmetic operations

**Definition 6:** The four basic arithmetic operations in closed intervals are: addition, difference, multiplication and division, they are defined as follows [14]:

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]$$

$$\frac{[a, b]}{[c, d]} = [\min\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}, \max\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}] , \text{ provided that } 0 \notin [c, d]$$

## 2.3. Expected Interval of a Fuzzy Number

Given a fuzzy number  $\tilde{N}$ , if we denote by  $[N_{\alpha}^L, N_{\alpha}^R]$  with  $\alpha \in [0, 1]$  the sets of level  $\alpha$  or  $\alpha$ -cuts, we define the expected interval of  $\tilde{N}$  by [15].

$$EI(\tilde{N}) = \left[ \int_0^1 N_{\alpha}^L d\alpha, \int_0^1 N_{\alpha}^R d\alpha \right]$$

The expected interval of a fuzzy number is an interval that concentrates the best information of a fuzzy number.

## 2.4. Fuzzy Linear Equation

The imprecision of the parameters of the equation, we will represent it with TFN of the form  $\tilde{A} = (a_1, a_2, a_3)$ , that is, the value of the parameter with the most possibility of occurrence is  $a_2$  but it can occur between  $a_1$  and  $a_3$ .

Given the TFN  $\tilde{A} = (a_1, a_2, a_3)$ ,  $\tilde{B} = (b_1, b_2, b_3)$  and  $\tilde{C} = (c_1, c_2, c_3)$ , then the FLE is defined by:

$$\tilde{A}\tilde{X} + \tilde{B} = \tilde{C} \quad (3)$$

One of the methods proposed to solve this equation was that of alpha cut and interval arithmetic [7]. To this method we incorporate the concept of the expected interval.

## 2.5. Alpha Cut Method and Interval Arithmetic + Expected Interval

This method consists of obtaining the solution from a classical linear equation, that is, without imprecision, the alpha cuts of the fuzzy numbers representing the imprecise parameters are then replaced, then simplified by interval arithmetic [2].

Given the equation (3) and assuming that  $0 \notin \text{Sop}(\tilde{A})$ , to find the solution, follow these steps:

**Step 1:** Determine the cut of alpha coefficients imprecise entities:  $\tilde{A}$ ,  $\tilde{B}$  y  $\tilde{C}$ .

**Step 2:** Replace the alpha cuts in the equation:  $\tilde{x} = \frac{\tilde{c}-\tilde{b}}{\tilde{a}}$

$$x(\alpha) = \frac{c[\alpha] - b[\alpha]}{a[\alpha]} = \frac{[c_1(\alpha), c_2(\alpha)] - [b_1(\alpha), b_2(\alpha)]}{[a_1(\alpha), a_2(\alpha)]}$$

**Step 3:** Simplify using the definitions of interval arithmetic.

The interval difference is made

$$x(\alpha) = \frac{[c_1(\alpha) - b_2(\alpha), c_2(\alpha) - b_1(\alpha)]}{[a_1(\alpha), a_2(\alpha)]}$$

And then the interval division.

$$x1(\alpha) = \min \left\{ \frac{c_1(\alpha) - b_2(\alpha)}{a_1(\alpha)}, \frac{c_1(\alpha) - b_2(\alpha)}{a_2(\alpha)}, \frac{c_2(\alpha) - b_1(\alpha)}{a_1(\alpha)}, \frac{c_2(\alpha) - b_1(\alpha)}{a_2(\alpha)} \right\}$$

$$x2(\alpha) = \max \left\{ \frac{c_1(\alpha) - b_2(\alpha)}{a_1(\alpha)}, \frac{c_1(\alpha) - b_2(\alpha)}{a_2(\alpha)}, \frac{c_2(\alpha) - b_1(\alpha)}{a_1(\alpha)}, \frac{c_2(\alpha) - b_1(\alpha)}{a_2(\alpha)} \right\}$$

$$\forall \alpha \in [0,1]$$

The solution of (3) with a degree of precision  $\alpha \in [0,1]$ , is an interval  $x(\alpha) = [x_1(\alpha), x_2(\alpha)]$ ; these intervals are the alpha sets of the fuzzy number (not necessarily a TFN) that will be the solution of the (3) [14].

According to Theorem 1 (see Subsection 2.1), the number that is the solution of the FLE is given by:

$$\tilde{S} = \bigcup_{\alpha \in [0,1]} \alpha \mu_{x(\alpha)} \alpha$$

where  $\mu_{x(\alpha)} \alpha$  is the membership function of  $x(\alpha)$ .

The most used values are  $\alpha = \{0.0, 0.1, 0.2, \dots, 0.9, 1.0\}$ ;  $\alpha = 0$  gives the most robust (most imprecise) solution;  $\alpha = 1$  gives the most precise (exact) solution.

The concept of the expected interval is incorporated into the alpha cut method.

**Step 4:** Calculate the expected interval of  $x(\alpha)$

$$IE(\tilde{S}) = \left[ \int_0^1 x_1(\alpha) d\alpha, \int_0^1 x_2(\alpha) d\alpha, \right]$$

In this interval, we find most solutions of the FLE. This interval corresponds to a unique  $\alpha \in [0,1]$ .

### 3. Results and Discussion

3.1. Case study: let be the linear equation (3).

Given de TFN  $\tilde{A} = (1,2,3)$ ,  $\tilde{B} = (5,6,7)$ ,  $\tilde{C} = (7,11,15)$ . To find the solution, we apply steps 1 to 4 of Section 2.5.

**Step 1:** We find the alpha cut of the triangular fuzzy numbers:

$$A[\alpha] = [1 + \alpha, 3 - \alpha], \quad B[\alpha] = [5 + \alpha, 7 - \alpha], \quad C[\alpha] = [7 + 4\alpha, 15 - 4\alpha]$$

**Step 2:** We replace the alpha cut in the solution:

$$x(\alpha) = \frac{c[\alpha] - b[\alpha]}{a[\alpha]} = \frac{[7 + 4\alpha, 15 - 4\alpha] - [5 + \alpha, 7 - \alpha]}{[1 + \alpha, 3 - \alpha]}$$

**Step 3:** Apply the interval difference in the numerator

$$x(\alpha) = \frac{[(7 + 4\alpha) - (5 + \alpha), (15 - 4\alpha) - (7 - \alpha)]}{[1 + \alpha, 3 - \alpha]} = \frac{[5\alpha, 10 - 5\alpha]}{[1 + \alpha, 3 - \alpha]}$$

Interval division is performed

$$x(\alpha) = [\min\{\frac{5\alpha}{1+\alpha}, \frac{5\alpha}{3-\alpha}, \frac{10-5\alpha}{1+\alpha}, \frac{10-5\alpha}{3-\alpha}\}, \max\{\frac{5\alpha}{1+\alpha}, \frac{5\alpha}{3-\alpha}, \frac{10-5\alpha}{1+\alpha}, \frac{10-5\alpha}{3-\alpha}\}]$$

$$x(\alpha) = [\frac{5\alpha}{3-\alpha}, \frac{10-5\alpha}{1+\alpha}] \quad , \quad \alpha \in [0,1]$$

The solution to the fuzzy linear equation is the fuzzy number,

$$\tilde{S} = \bigcup_{\alpha \in [0,1]} \alpha x(\alpha) = \bigcup_{\alpha \in [0,1]} \alpha \begin{cases} 1 & x \in [\frac{5\alpha}{3-\alpha}, \frac{10-5\alpha}{1+\alpha}] \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{S} = \bigcup_{\alpha \in [0,1]} \begin{cases} \alpha & x \in [\frac{5\alpha}{3-\alpha}, \frac{10-5\alpha}{1+\alpha}] \\ 0 & \text{in another case} \end{cases}$$

The graph of the fuzzy number  $\tilde{S} = \tilde{S}_L \cup \tilde{S}_R$  where

$$\tilde{S}_L = \left\{ \left( \frac{5\alpha}{3-\alpha}, \alpha \right), \alpha \in [0,1] \right\}, \quad \tilde{S}_R = \left\{ \left( \frac{10-5\alpha}{1+\alpha}, \alpha \right), \alpha \in [0,1] \right\}$$

The intersection point of the curves  $\tilde{S}_L$  y  $\tilde{S}_R$  occurs at:  $\frac{5\alpha}{3-\alpha} = \frac{10-5\alpha}{1+\alpha}$

Solving for  $\alpha$  you have  $\alpha = 1$ , which corresponds to a value of  $x = \frac{5(1)}{3-1} = 2.5$

The graph of the fuzzy set  $\tilde{S}$  can be seen in Figure 2. The fuzzy number  $\tilde{S}$  in Cartesian form is given by

$$\tilde{S} = \begin{cases} \frac{3x}{5+x}, & 0 < x \leq 2.5 \\ \frac{10-x}{5+x}, & 2.5 \leq x < 10 \\ 0, & x \leq 0 \text{ and } x \geq 10 \end{cases}$$

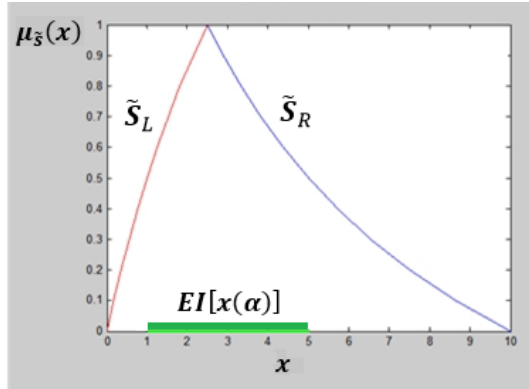


Figure 2. Solution of the first-degree linear fuzzy equation.

The fuzzy set is a fuzzy number (See Definition 1). The solution of the FLE will most likely occur at 2.5, but it can occur between 0 and 10.

The solution of the FLE with different degrees of precision is shown in Table 1.

The most robust (most imprecise) solution is the interval  $[0, 10]$ , obtained with  $\alpha = 0$ . The most precise (exact) solution is the interval  $[2.50, 2.50]$ , obtained with  $\alpha = 1$ ; this interval is the real number  $x = 2.5$ . For  $\alpha = 0.5$ , the solution is the interval  $[1, 5]$ . In other words, you have different alternatives to choose the solution interval with the precision you want.

**Table 1.** Alfa cuts for the solution of the fuzzy linear equation of the first degree

Precision degree $\alpha$	Alpha Cuts
0.0	$[0, 10]$
0.1	$[0.17, 8.64]$
0.2	$[0.36, 7.50]$
0.3	$[0.56, 6.54]$
0.4	$[0.77, 5.71]$
0.5	$[1.00, 5.00]$
0.6	$[1.25, 4.38]$
0.7	$[1.52, 3.82]$
0.8	$[1.82, 3.33]$
0.9	$[2.14, 2.89]$
1.0	$[2.50, 2.50]$

**Step 4:** Evaluate the expected interval of  $\tilde{S}$

$$EI[\tilde{S}] = \left[ \int_0^1 \frac{5\alpha}{3-\alpha} d\alpha, \int_0^1 \frac{10-5\alpha}{1+\alpha} d\alpha \right] = [1.081, 5.317]$$

The solution intervals with different degrees of precision  $\alpha \in [0,1]$  are infinite, if an interval where the majority of solutions, of the fuzzy linear equation possibly occur is desired, this is the expected interval  $[1,081, 5,317]$  (see Figure 2).

### 3.2. Comparison of Solutions

For  $\alpha = 1$  (total precision)

The solution to (3) with the method of alpha cutting and interval arithmetic is the TFN  $\tilde{X} = (2.5, 2.5, 2.5)$ , replacing in (3) and performing interval operations you get  $11 = 11$ , i.e. the equation is satisfied.

For  $\alpha = 0$  (maximum inaccuracy)

The solution to equation (3) with the method of alpha cut and interval arithmetic is the fuzzy number  $\tilde{X} = (0, 2.5, 10)$ ; replacing in (3) and performing interval operations gets  $[5, 37] \neq [7, 15]$ , that is, the equation is not satisfied, and what is worse, the interval of the first member is very different from that of the second member.

The solution to (3) incorporating the concept of the expected interval to the method of alpha cutting and interval arithmetic, is the fuzzy number  $(1,081, 2.5, 5.317)$ ; replacing in (3) and performing interval operations results in  $[6.0811, 22.951] \neq [7, 15]$ , which also does not satisfy the equation, but is closer to equality than the solution obtained with the alpha cuts and interval arithmetic method.

It is noted that for  $\alpha = 0$ , the expected interval  $IE(\tilde{S}) = [1.081, 5.317] \subset [0,10]$ , and decreases the inaccuracy, which is the difficulty of the solution by arithmetic interval according to the ratio (2).

## 4. Conclusions

The new method of solving a fuzzy linear equation of a variable using the expected interval, allows to obtain a smaller support set where the solutions come closer to satisfying the fuzzy linear equation, also allows to find a single interval where the best solutions for decision making are expected to be found.

This method can be applied to problems where inaccuracy can be represented by triangular fuzzy or trapezoidal numbers.

It is recommended to study the incorporation of the concept of the expected interval in the methods to solve systems of fuzzy linear equations.

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