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Interval Observer Design for Metzlerian Takagi-Sugeno Systems

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Abstract. The generalized interval observer design conditions for continuous-time Metzlerian Takagi-Sugeno systems are presented in the paper. Attention is focused on the analysis and design guaranteeing the asymptotic convergence of the interval observer error and positivity of interval observer state. The relationship between the nonnegativity of the observer gains and the corresponding positive observer state attractiveness is also shown. The method presented extends and generalizes the results that recently appeared in the literature.

Keywords. interval observers, Metzlerian Takagi-Sugeno fuzzy systems, diagonal stabilization, convex optimization, linear matrix inequalities

1. Introduction

the type of fuzzy inference systems The study of state observers is a topic of great importance in effective control concepts and system fault diagnosis applications. Because the Takagi-Sugeno (T-S) fuzzy approach addresses fuzzy implication of local dynamics by linear models [1], excellent unifications of state observer design for sector-bounded nonlinear systems are related with T-S fuzzy models [2], [3]. Since using Takagi-Sugeno method the consequents are the crisp functions of inputs, nonlinear control theories prefer T-S fuzzy approach with relation to state-space representation of systems, although there exist other fuzzy inference systems [4], [5]. The design algorithm turns out to be feasible utilizing the linear matrix inequalities (LMI) technique [6], [7].

Restricting attention to positive linear systems, maintaining given features whenever the system states are nonnegative [8], [9], concepts in this field prioritize in analysis and design the theory of matrices of Metzler structure [10], [11]. In order to reflect large number of constraints, an excellent unification is an LMI-based design strategy for linear positive (Metzlerian) systems [12].

Based on nominal system models, the state observers in general asymptotically estimate unknown system state. Counterpart of this approach is outlined in [13] to provide, for given system matrix bounds, the system state estimation in projected intervals. In addition to the above, reference [14] presents a cooperative observer error approach for exact analysis of interval observers with grasp of the Metzler matrix details. Interval observer algorithms, with LMI projection of interval bounds, for uncertain Metzlerian systems are analyzed in [15].

Adapting the above results in synthesis of T-S fuzzy observers and interval observers for uncertain Metzlerian systems and reflecting [16], the paper extend the latter approaches to design interval observer for Metzlerian Takagi-Sugeno systems. Preferring LMI formulation for solving the problem, the interval observer stability proofs use standard arguments and reflect the concept of diagonal stabilization that is in keeping with the structure of Metzler matrices. Working along these lines to establish the observer design conditions, the newly presented theoretical aspect gives relationships between system parametric constraints, the LMIs feasibility and the observer state attractiveness guaranty. Because only a set of LMIs defines conditions, practical aspects are standard.

The paper is organized as follows. In Sec. 2 the essential properties of Metzlerian Takagi-Sugeno fuzzy models are adduced and Sec. 3 outlines problems with Metzlerian observer design for given class of strictly Metzlerian T-S systems. The set of LMIs, describing the design conditions for Metzlerian Takagi-Sugeno interval observer, and its solution is the theme of Sec. 4 and an example to implement the algorithm is included into Sec. 5. Within the underlying concept, Sec. 6 draws conclusions and the topics of the research activity in the future.

Throughout the paper, the following notations are used: x^{T}, X^{T} denotes the transpose of the vector x, and the matrix X, respectively, $\operatorname{diag}[\cdot]$ marks a (block) diagonal matrix, for a square symmetric matrix $X \prec 0$ means that X is negative definite matrix, the symbol I_n indicates the n-th order unit matrix, $\mathbb{R}_+$, $\mathbb{R}_+^n$ qualify the sets of nonnegative real numbers and n-dimensional real vectors, $\mathbb{R}_+^{n\times n}$ refers to the set of nonnegative real matrices and $\mathbb{R}_{-+}^{n\times n}$ covers the set of Metzler matrices.

2. Metzlerian Takagi-Sugeno Fuzzy Model

The used systems are multi-input and multi-output (MIMO) Metzlerian continuous-time dynamic systems, represented in T-S form as

$$\dot{q}(t) = \sum_{i=1}^{s} \mathrm{h}_i(\vartheta(t))(A_i q(t) + B_i u(t)), \tag{1}$$

$$y(t) = Cq(t), \tag{2}$$

where $q(t) \in \mathbb{R}_+^n$, $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}_+^m$, are vectors of the state, input, and output variables and $A \in \mathbb{M}_{-+}^{n\times n}$, $B \in \mathbb{R}_+^{n\times r}$, $C \in \mathbb{R}_+^{m\times n}$. Moreover, $\mathrm{h}_i(\theta(t))$ is averaging weight for the i-th rule, representing the normalized grade of membership, where

$$0 \le \mathrm{h}_i(\vartheta(t)) \le 1, \quad \sum_{i=1}^{s} \mathrm{h}_i(\vartheta(t)) = 1 \quad for\ all\ i \in \langle 1, s\rangle, \tag{3}$$

while s is the number fuzzy rules (linear sub-models) and

$$\vartheta(t) = \left[\, \theta_1(t)\ \theta_2(t) \cdots \theta_q(t) \,\right] \tag{4}$$

is q-dimensional vector of premise variables. More details can be found, e.g., in [6], [12].

Within the above model, there are considered nonnegative matrices $B_i \in \mathbb{R}_+^{n\times r}$, $C \in \mathbb{R}_+^n$ and a strictly Metzler $A_i \in \mathbb{R}_{-+}^{n\times n}$, where strictly Metzler A_i means that all its off diagonal elements are greater then zero and all its diagonal elements are negative. Such above given system is noted as strictly Metzlerian T-S system. In general, a Metzler matrix is so confronted with n^2 boundaries implying from the structural constraints

$$a_{lh} < 0,\ l = h, \quad a_{lh} > 0, l \neq h, \quad \forall l, h \in \langle 1, n\rangle. \tag{5}$$

This just means in consequence that continuous-time strictly Metzlerian systems are diagonally stabilizable. To respect diagonal structures the following are reflected.

Definition 1. *[17] A square matrix $L_p \in \mathbb{R}^{n\times n}$ is permutation matrix if exactly one element in each column and each row is equal to 1 and all others are equal to 0.*

Definition 2. *[17] Let $L \in \mathbb{R}_{+}^{n\times n}$ be a permutation matrix. L is called circulant if*

$$L = \begin{bmatrix} 0^{\mathrm{T}} & 1 \\ I_{n-1} & 0 \end{bmatrix}. \tag{6}$$

Moreover, the following remark can be easily checked applying the circulant permutation matrix representstion.

Remark 1. *If $X \in \mathbb{R}^{n\times n}$ is a diagonal matrix then, consequently,*

$$L^{\mathrm{T}} diag \begin{bmatrix} x_1 \; x_2 \; \cdots \; x_n \end{bmatrix} L = diag \begin{bmatrix} x_2 \; \cdots \; x_n \; x_1 \end{bmatrix}. \tag{7}$$

In the case when only a matrix $A_i \in \mathbb{R}_{-+}^{n\times n}$ is analyzed and structural constraints (65) are represented as a set of n LMIs, then

Lemma 1. *[12] $A_i \in \mathbb{R}_{-+}^{n\times n}$ is Metzler and Hurwitz if and only if there exists a positive definite diagonal matrix $P \in \mathbb{R}_{+}^{n\times n}$ such that*

$$P \succ 0, \tag{8}$$

$$PA_i(p,p)_\Delta \prec 0, \tag{9}$$

$$PL^h A_i(p+h,p)_\Delta L^{h\mathrm{T}} \succ 0, \tag{10}$$

$$A_i^{\mathrm{T}} P + PA_i \prec 0, \tag{11}$$

for $i = 1,2,\ldots,s$, $h = 1,2,\ldots,n-1$, $p = 1,2,\ldots,n$, where

$$A_i(p+h,p)_\Delta = diag \begin{bmatrix} a_{i,1+h,1} \; \cdots \; a_{i,n,n-h} \;\; a_{i,1,n-h+1} \; \cdots \; a_{ihn} \end{bmatrix}, \tag{12}$$

$$\Delta = (1 \leftrightarrow n)/n. \tag{13}$$

Note, with the matrix P thus defined, the set of LMIs (9), (10) reflects (65) and the Lyapunov inequality (11) guaranties that A_i is Hurwitz.

3. Strictly Metzlerian Takagi-Sugeno Fuzzy Observer

The state estimation assumes the observer to strictly Metzlerian Takagi-Sugeno fuzzy system (1), (2) in the form

$$\dot{q}_e(t) = \sum_{i=1}^{s} h_i(\theta(t))(A_i q_e(t) + B_i u(t) + J_i(y(t) - y_e(t))), \tag{14}$$

$$y_e(t) = Cq_e(t), \tag{15}$$

where $q_e(t) \in \mathbb{R}_{+}^{n}$ is the estimation of the system state vector, $J_i \in \mathbb{R}_{+}^{n\times m}$, $i = 1,2,\ldots,s$ is the set of the positive observer gain matrices and forms of the parameters are

$$A_i = \begin{bmatrix} a_{i11} \cdots a_{i1n} \\ \vdots \\ a_{in1} \cdots a_{inn} \end{bmatrix}, C = \begin{bmatrix} c_1^{\mathrm{T}} \\ \vdots \\ c_m^{\mathrm{T}} \end{bmatrix}, c_k^{\mathrm{T}} = \begin{bmatrix} c_{k1} \cdots c_{kn} \end{bmatrix}, J_i = \begin{bmatrix} j_{i1} \cdots j_{im} \end{bmatrix}, j_{ik} = \begin{bmatrix} j_{i1k} \\ \vdots \\ j_{ink} \end{bmatrix}, \tag{16}$$

$$J_{dik} = \mathrm{diag} \begin{bmatrix} j_{i1k} \; \cdots \; j_{ink} \end{bmatrix}, \; C_{dk} = \mathrm{diag} \begin{bmatrix} c_{1k} \; \cdots \; c_{nk} \end{bmatrix}. \tag{17}$$

Then the structural constraints problem for strictly Metzler $A_{ei} \in \mathbb{R}_{-+}^{n\times n}$ is formable as

$$A_{ei} = A_i - J_i C = A_i - \sum_{k=1}^{m} j_{ik} c_k^{\mathrm{T}} = A_i - \sum_{k=1}^{m} J_{dik} l l^{\mathrm{T}} C_{dk}\,, \tag{18}$$

where

$$l^{\mathrm{T}} = \begin{bmatrix} 1 \; 1 \; \cdots \; 1 \end{bmatrix}, \tag{19}$$

$$a_{elh} - \sum_{k=1}^{m} j_{ihk} c_{kl} < 0,\; h = l, \quad a_{ilh} - \sum_{k=1}^{m} j_{ihk} c_{kl} > 0.\; h \neq l, \quad \forall\, h,l \in \langle 1,\ldots,n \rangle\,. \tag{20}$$

Having in mind constraints (20) it is not hard to establish the following:

Theorem 1. *The observer (14), (15) is stable if there exist positive definite diagonal matrices* $P, R_{ik} \in \mathbb{R}_+^{n\times n}$ *such that*

$$P \succ 0\,, \quad R_i \succ 0\,, \tag{21}$$

$$PA_i(p,p)_\Delta - \sum_{k=1}^{m} R_{ik} C_{dk} \prec 0\,, \tag{22}$$

$$PL^h A_i(p+h,p)_\Delta L^{h\mathrm{T}} - \sum_{k=1}^{m} R_{ik} L^h C_{dk} L^{h\mathrm{T}} \succ 0\,, \tag{23}$$

$$PA_i + A_i^{\mathrm{T}} P - \sum_{k=1}^{m} R_{ik} l l^{\mathrm{T}} C_{dk} - \sum_{k=1}^{m} C_{dk} l l^{\mathrm{T}} R_{ik} \prec 0\,. \tag{24}$$

for $i = 1,2,\ldots,s,\; p = 1,2,\ldots,n,\; h = 1,2,\ldots,n-1,\; k = 1,2,\ldots,m.$

When the above conditions hold, the set of J_{ik} *is given by*

$$J_{dik} = P^{-1} R_{ik}, \quad j_{ik} = J_{dik} l, \quad J_i = \begin{bmatrix} j_{i1} \cdots j_{im} \end{bmatrix}. \tag{25}$$

Proof. Writing A_{ei} from (18) as follows

$$A_{ei} = \begin{bmatrix} a_{i11} & a_{i12} & \cdots & a_{i1n} \\ a_{i21} & a_{i22} & \cdots & a_{i2n} \\ & & \ddots & \\ a_{in1} & a_{in2} & \cdots & a_{inn} \end{bmatrix} - \sum_{k=1}^{m} \begin{bmatrix} j_{i1k} \\ j_{i2k} \\ \vdots \\ j_{ink} \end{bmatrix} \begin{bmatrix} c_{k1} \; c_{k2} \cdots c_{kn} \end{bmatrix}, \tag{26}$$

it can see that the diagonal elements of (26) satisfy the first set of conditions from (20) if

$$A_i(p,p)_\Delta - \sum_{k=1}^{m} J_{idk} C_{dk} \prec 0\,, \tag{27}$$

where

$$A_i(p,p)_\Delta = \mathrm{diag}\begin{bmatrix} a_{i11} \; a_{i22} \cdots a_{inn} \end{bmatrix}. \tag{28}$$

Therefore, multiplying the left side by a positive definite diagonal matrix $P \in \mathbb{R}_+^{n\times n}$ then (27) implies

$$PA_i(p,p)_\Delta - \sum_{k=1}^{m} PJ_{idk} C_{dk} \prec 0\,, \tag{29}$$

and with the notation

$$R_{ik} = PJ_{idk} \tag{30}$$

(29) implies (22) and (30) forces (25).

Rewriting (26) in the circular shifted structures to cover other sets of algebraic constraints then

$$L^{h\mathrm{T}}A_{ei} = \begin{bmatrix} a_{i,h+1,1} & a_{i,h+1,2} & \cdots & a_{i,h+1,n} \\ & & \vdots & \\ a_{in1} & a_{in2} & \cdots & a_{inn} \\ a_{i11} & a_{i12} & \cdots & a_{i1n} \\ & & \vdots & \\ a_{ih1} & a_{ih2} & \cdots & a_{ihn} \end{bmatrix} - \sum_{k=1}^{m} \begin{bmatrix} j_{i,h+1,k} \\ \vdots \\ j_{ink} \\ j_{i1k} \\ \vdots \\ j_{ihk} \end{bmatrix} \begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kn} \end{bmatrix} \tag{31}$$

and it can see that the diagonal elements of (31) force the set (18) for fixed h if

$$A_i(p+h,p)_\Delta - \sum_{k=1}^{m} J_{dikch}C_{dk} \succ 0\,, \tag{32}$$

where $A_i(p+h,p)_\Delta$ is defined in (12) and J_{dikch} is derived from the diagonal matrix J_{idk} by h circular shifts of its diagonal elements applying circulant L from (6).

Since, it yields for $h = 1,2,\ldots n-1$,

$$J_{idk} = L^h J_{dikch} T^{h\mathrm{T}}, \tag{33}$$

pre-multiplying the left side by T^h and post-multiplying the right side by $T^{h\mathrm{T}}$ then (32) can be represented as

$$T^h A_i(p+h,p)_\Delta T^{h\mathrm{T}} - \sum_{k=1}^{m} T^h J_{dkch} T^{h\mathrm{T}} T^h C_{dk} T^{h\mathrm{T}} \succ 0\,, \tag{34}$$

$$T^h A(i+h,i)_\Delta T^{hT} - \sum_{k=1}^{m} J_{dk} T^h C_{dk} T^{hT} \succ 0\,, \tag{35}$$

respectively. Thus, multiplying the left side by positive definite diagonal matrix $P \in \mathbb{R}_+^{n\times n}$ and using (33) then (35) implies (23).

Introducing the error in system state observations as

$$e(t) = q(t) - q_e(t) \tag{36}$$

and performing the time derivative then, exploiting (1) and (14), it is obtained

$$\dot{e}(t) = \sum_{i=1}^{s} h_i(\theta(t))(A_i(q(t) - q_e(t)) - J_i(y(t) - y_e(t)))\,, \tag{37}$$

which can be written using (2), (18) as follows

$$\dot{e}(t) = \sum_{i=1}^{s} h_i(\theta(t))A_{ei}e(t)\,. \tag{38}$$

Defining the Lyapunov function

$$v(e(t)) = e^{\mathrm{T}}(t)Pe(t) > 0\,, \tag{39}$$

where $P \in \mathbb{R}_+^{n\times n}$ is positive definite diagonal matrix, then (39) implies

$$\dot{v}(e(t)) = \dot{e}(t)Pe(t) + e^{\mathrm{T}}(t)P\dot{e}(t)\,. \tag{40}$$

Substituting (38) into (40) gives

$$\dot{v}(e(t)) = e^{\mathrm{T}}(t)\sum_{i=1}^{s} h_i(\theta(t))(PA_{ei} + A_{ei}^{\mathrm{T}}P)e(t)\,, \tag{41}$$

which results with (18) to

$$P(A_i - \sum_{k=1}^{m} J_{dik}ll^{\mathrm{T}}C_{dk}) + (A_i - \sum_{k=1}^{m} J_{dik}ll^{\mathrm{T}}C_{dk})^{\mathrm{T}}P \prec 0 \;\; \forall\, i\,. \tag{42}$$

Therefore, using (30), then (42) implies (24). This concludes the proof. □

4. Metzlerian Takagi-Sugeno Fuzzy Interval Observer

Consider (1), (2), where (A_i, C) and $q(0)$ are unknown but bounded and for all $i \in \langle 1, s \rangle$ the known constant bounds satisfy elementwise

$$\underline{q}(0) \leq q(0) \leq \overline{q}(0)\,, \quad \underline{A}_i \leq A_i \leq \overline{A}_i\,, \quad \underline{C} \leq C \leq \overline{C}\,. \tag{43}$$

The aforementioned problem can be turned to construction of the (strictly) Metzlerian T-S fuzzy interval observer, defined as the couple of the algorithms

$$\dot{\overline{q}}_e(t) = \sum_{i=1}^{s} h_i(\vartheta(t))(\overline{A}_i \overline{q}_e(t) + B_i u(t) + J_i \underline{C}(q(t) - \overline{q}_e(t))\,, \tag{44}$$

$$\dot{\underline{q}}_e(t) = \sum_{i=1}^{s} h_i(\vartheta(t))(\underline{A}_i \underline{q}_e(t) + B_i u(t) + J_i \overline{C}(q(t) - \underline{q}_e(t))\,, \tag{45}$$

where the design objective constraints can be stated as

$$0 \leq \underline{q}_e(t) \leq q(t) \leq \overline{q}_e(t) \tag{46}$$

for all $t \geq 0$ if $\overline{q}_e(0) = \overline{q}(0)$, $\underline{q}_e(0) = \underline{q}(0)$. It is the problem that is reformulated in the following definition.

Definition 3. *The set of equations (44), (45) give stable observer for uncertain Metzlerian Takagi-Sugeno fuzzy plant (1), (2) if both the lower estimation error $\underline{e}(t)$ and the upper estimation error $\overline{e}(t)$ converge to the equilibrium.*

Assumption 1. *Supposing that it is possible to force equilibrium convergence of errors*

$$\overline{e}(t) = q(t) - \overline{q}_e(t)\,, \quad \underline{e}(t) = q(t) - \underline{q}_e(t)\,. \tag{47}$$

If (1), (2), (44), (45) are rearranged as

$$\dot{\overline{e}}(t) = \sum_{i=1}^{s} h_i(\vartheta(t))(\overline{A}_i - J_i \underline{C})\overline{e}(t) = \sum_{i=1}^{s} h_i(\vartheta(t))\overline{A}_{ei}\overline{e}(t)\,, \tag{48}$$

$$\dot{\underline{e}}(t) = \sum_{i=1}^{s} h_i(\vartheta(t))(\underline{A}_i - J_i \overline{C})\underline{e}(t) = \sum_{i=1}^{s} h_i(\vartheta(t))\underline{A}_{ei}\underline{e}(t) \tag{49}$$

trajectories (48), (49) are asymptotically stable if for given set ($\underline{A}_i$, $\overline{A}_i$, $\underline{C}$, $\overline{C}$, $i \in \langle 1, s \rangle$) satisfying (43), and nonnegative $\overline{e}(0)$, $\underline{e}(0)$, all matrices $\underline{A}_{ei}$, $\overline{A}_{ei}$ are Metzler and Hurwitz.

Corollary 1. *By performing an inner adjustment for (49)*

$$\dot{q}(t) - \dot{\underline{q}}_e(t) = \sum_{i=1}^{s} h_i(\vartheta(t))\underline{A}_{ei}(q(t) - \underline{q}_e(t))\,, \tag{50}$$

it then follows from (50)

$$\dot{\underline{q}}_e(t) = (\dot{q}(t) - \sum_{i=1}^{s} h_i(\vartheta(t))\underline{A}_{ei}q(t)) + \sum_{i=1}^{s} h_i(\vartheta(t))\underline{A}_{ei}\underline{q}_e(t)) \tag{51}$$

and considering the autonomous part of (1)

$$\begin{aligned} \dot{\underline{q}}_e(t) &= \sum_{i=1}^{s} h_i(\vartheta(t))(A_i - (\underline{A}_i - J_i \overline{C}))q(t)) + \underline{A}_{ei}\underline{q}_e(t) \\ &= \sum_{i=1}^{s} h_i(\vartheta(t))\underline{A}_{ei}\underline{q}_e(t) + \sum_{i=1}^{s} h_i(\vartheta(t))J_i \overline{C} q(t) + \sum_{i=1}^{s} h_i(\vartheta(t))(A_i - \underline{A}_i)q(t)\,. \end{aligned} \tag{52}$$

Thus, for $\overline{C} \in \mathbb{R}_+^{m \times n}$, $q(t) \in \mathbb{R}_+^n$ the lower state estimate is nonnegative if $J_i \in \mathbb{R}_+^{n \times m}$ is nonnegative and all $\underline{A}_{ei}$ are (strictly) Metzler and Hurwitz.

With the above facts in mind, Theorem 1 is adapted to obtain $\overline{A}_{ei}$, $\underline{A}_{ei} \in \mathbb{R}^{n\times n}_{-+}$.

Theorem 2. *Using algorithms (44), (45) in state estimation of uncertain strictly Metzlerian Takagi-Sugeno fuzzy system (1), (2), then matrices* $\overline{A}_{ei}$, $\underline{A}_{ei} \in \mathbb{R}^{n\times n}_{-+}$ *for all* $i \in \langle 1,s\rangle$ *are strictly Metzler and Hurwitz if for given strictly Metzler matrices* $\overline{A}_i, \underline{A}_i \in \mathbb{R}^{n\times n}_{-+}$, $i \in \langle 1,s\rangle$ *and non-negative matrices* $\overline{C}, \underline{C} \in \mathbb{R}^{m\times n}_{+}$ *there exist positive definite diagonal matrices* $P, R_{ik} \in \mathbb{R}^{n\times n}_{+}$ *such that*

$$P \succ 0, \qquad R_{ik} \succ 0, \tag{53}$$

$$P\overline{A}_i(p,p)_\Delta - \sum_{k=1}^{m} R_{ik}\underline{C}_{dk} \prec 0, \tag{54}$$

$$P\underline{A}_i(p,p)_\Delta - \sum_{k=1}^{m} R_{ik}\overline{C}_{dk} \prec 0, \tag{55}$$

$$PL^h\overline{A}_i(p+h,p)_\Delta L^{h\mathrm{T}} - \sum_{k=1}^{m} R_{ik}L^h\underline{C}_{dk}L^{h\mathrm{T}} \succ 0, \tag{56}$$

$$PL^h\underline{A}_i(p+h,p)_\Delta L^{h\mathrm{T}} - \sum_{k=1}^{m} R_{ik}L^h\overline{C}_{dk}L^{h\mathrm{T}} \succ 0, \tag{57}$$

$$P\overline{A}_i + \overline{A}_i^{\mathrm{T}}P - \sum_{k=1}^{m} R_{ik}ll^{\mathrm{T}}\underline{C}_{dk} - \sum_{k=1}^{m} \underline{C}_{dk}ll^{\mathrm{T}}R_{ik} \prec 0, \tag{58}$$

$$P\underline{A}_i + \underline{A}_i^{\mathrm{T}}P - \sum_{k=1}^{m} R_{ik}ll^{\mathrm{T}}\overline{C}_{dk} - \sum_{k=1}^{m} \overline{C}_{dk}ll^{\mathrm{T}}R_{ik} \prec 0, \tag{59}$$

for $i = 1,2,\ldots,s$, $h = 1,2,\ldots,n-1$, $p = 1,2,\ldots,n$, *where* L, l^{T} *are predefined and*

$$\overline{A}_i(p+h,p)_\Lambda = diag\left[\overline{a}_{i,1+h,1} \cdots \overline{a}_{i,n,n-h}\ \overline{a}_{i,1,n-h+1} \cdots \overline{a}_{ihn}\right], \tag{60}$$

$$\underline{A}_i(p+h,p)_\Delta = diag\left[\underline{a}_{i,1+h,1} \cdots \underline{a}_{i,n,n-h}\ \underline{a}_{i,1,n-h+1} \cdots \underline{a}_{ihn}\right], \tag{61}$$

$$\overline{C}_{dk} = diag\left[\overline{c}_{k1}\ \overline{c}_{k2} \ldots \overline{c}_{kn}\right], \qquad \underline{C}_{dk} = diag\left[\underline{c}_{k1}\ \underline{c}_{k2} \cdots \underline{c}_{kn}\right]. \tag{62}$$

When these conditions are successfully met, (25) defines the rule to compute a set of strictly positive gain matrices $J_i \in \mathbb{R}^{n\times m}_{+}$.

Proof. Adequately adapting (26) then $\underline{A}_{ei}$ takes its open structure

$$\underline{A}_{ei} = \begin{bmatrix} \underline{a}_{i11}\ \underline{a}_{i12} \cdots \underline{a}_{i1n} \\ \underline{a}_{i21}\ \underline{a}_{i22} \cdots \underline{a}_{i2n} \\ \ddots \\ \underline{a}_{in1}\ \underline{a}_{in2} \cdots \underline{a}_{inn} \end{bmatrix} - \sum_{k=1}^{r} \begin{bmatrix} j_{i1k} \\ j_{i2k} \\ \vdots \\ j_{ink} \end{bmatrix} \left[\overline{c}_{k1}\ \overline{c}_{k2} \cdots \overline{c}_{kn}\right] \tag{63}$$

and the constraints on the diagonal elements of (63), if $J_i \in \mathbb{R}^{n\times m}_{+}$ is strictly positive for all i, are by definition

$$\underline{A}_i(p,p)_\Delta - \sum_{k=1}^{r} J_{dik}\overline{C}_{dk} \prec 0, \tag{64}$$

where J_{dik} is defined in (17) and (61) for $h = 0$ implies

$$\underline{A}_i(p,p)_\Delta = \mathrm{diag}\left[\underline{a}_{i11}\ \underline{a}_{i22} \cdots \underline{a}_{inn}\right]. \tag{65}$$

Multiplying the left side by a positive definite diagonal matrix $P \in \mathbb{R}^{n\times n}_{+}$ and using (30) then (64) implies (55). Quite analogously, it can be applied to $\overline{A}_{ei}$ and derived (54).

The condition (31) can be reformulated to $\underline{A}_{ei}$ in the analogous way as

$$L^{h\mathrm{T}}\underline{A}_{ei} = \begin{bmatrix} \underline{a}_{i,h+1,1} & \underline{a}_{i,h+1,2} & \cdots & \underline{a}_{i,h+1,n} \\ & & \vdots & \\ \underline{a}_{in1} & \underline{a}_{in2} & \cdots & \underline{a}_{inn} \\ \underline{a}_{i11} & \underline{a}_{i12} & \cdots & \underline{a}_{i1n} \\ & & \vdots & \\ \underline{a}_{ih1} & \underline{a}_{ih2} & \cdots & \underline{a}_{ihn} \end{bmatrix} - \sum_{k=1}^{m} \begin{bmatrix} j_{i,h+1,k} \\ \vdots \\ j_{ink} \\ j_{i1k} \\ \vdots \\ j_{ihk} \end{bmatrix} \begin{bmatrix} \overline{c}_{k1} & \overline{c}_{k2} & \cdots & \overline{c}_{kn} \end{bmatrix}, \tag{66}$$

while the diagonal elements of (66) for fixed h implies

$$\underline{A}_i(p+h,p)_\Delta - \sum_{k=1}^{m} J_{dikch}\overline{C}_{dk} \succ 0\,, \tag{67}$$

where $\underline{A}_i(p+h,p)_\Delta$ is defined in (61) and J_{dikch} is related to J_{dik} in (33).

Pre-multiplying the left side by PL^h and post-multiplying the right side by $L^{h\mathrm{T}}$ with $P \in \mathbb{R}_+^{n\times n}$ defined as above, (67) with (33) gives

$$PL^h\underline{A}(p+1,p)_\Delta L^{h\mathrm{T}} - \sum_{k=1}^{r} PJ_{ik}L^h\overline{C}_{dk}L^{h\mathrm{T}} \succ 0 \tag{68}$$

and applying (30) then (68) implies (57).

Constructing a positive Lyapunov function candidate

$$v(\underline{e}(t)) = \underline{e}^\mathrm{T}(t)P\underline{e}(t) > 0\,, \tag{69}$$

using the same P as above, then solving for

$$\dot{v}(\underline{e}(t)) = \underline{e}^\mathrm{T}(t)\sum_{i=1}^{s} h_i(\theta(t))\left(\underline{A}_{ei}^\mathrm{T}P + P\underline{A}_{ei}\right)\underline{e}(t) < 0\,, \tag{70}$$

which corresponds to the conditions writable for $i \in \langle 1,s\rangle$ in the set of LMIs

$$\underline{A}_{ei}^\mathrm{T}P + P\underline{A}_{ei} \prec 0 \tag{71}$$

and results in

$$P\Big(A_i - \sum_{k=1}^{m} J_{dik}ll^\mathrm{T}C_{dk}\Big) + \Big(A_i - \sum_{k=1}^{m} J_{dik}ll^\mathrm{T}C_{dk}\Big)^\mathrm{T} P \prec 0 \;\; \forall\, i\,. \tag{72}$$

Thus, for $k = 1,\ldots,m$, $i = 1,\ldots,s$ and with (30) then (72) implies (59), and analogously, (58), (59) guaranty required stability. This completes the proof. □

To apply for uncertain non-strictly Metzlerian Takagi-Sugeno fuzzy system the principle of structured matrix variables [18] can be adapted, with the following procedure, excluding cases that both the column of $\overline{C}$ and $\underline{C}$ indexed by β are zero column vectors.

Corollary 2. *Let for given $\alpha,\beta \in \langle 1,n\rangle$ the off-diagonal element $\overline{a}_{i\alpha\beta}$ of $\overline{A}_i$ as well as the off-diagonal element $\underline{a}_{i\alpha\beta}$ of $\underline{A}_i$ are zero. Then R_{ik} must be structured so that*

$$R_{ik} = diag\left[r_{ik1} \cdots r_{ik,\alpha-1}\; r_{ik\alpha}\; r_{ik,\alpha+1} \cdots r_{ikn}\right], \tag{73}$$

where

$$r_{ik\alpha} = 0, \quad r_{ik\gamma} > 0\; for\; \gamma \neq \alpha,\; \gamma = 1,\ldots,n\,. \tag{74}$$

If columns $\overline{c}_\beta$, $\underline{c}_\beta$ of $\overline{C}$, $\underline{C}$ are strictly positive, (73), (74) must be satisfied for all $k = 1,\ldots m$, $i = 1,\ldots p$. Otherwise, for i,k related to positive elements in nonnegative $\overline{c}_\beta$, $\underline{c}_\beta$.

The proposed design conditions have no tuning parameters with relation to (58), (59). This problem can be redefined using the approaches proposed in [19].

5. Illustrative Example

The system is represented by the Metzlerian Takagi-Sugeno equations (1), (2), where

$$\underline{A}_1 = \begin{bmatrix} -0.272 & 1.940 & 1.450 \\ 0.058 & -3.960 & 0 \\ 0.100 & 0 & -2.910 \end{bmatrix}, \quad \underline{A}_2 = \begin{bmatrix} -0.272 & 1.940 & 1.450 \\ 0.058 & -3.960 & 0.100 \\ 0.100 & 0 & -2.910 \end{bmatrix},$$

$$\underline{A}_3 = \begin{bmatrix} -0.272 & 1.940 & 1.450 \\ 0.058 & -3.960 & 0 \\ 0.100 & 0.080 & -2.910 \end{bmatrix}, \quad \underline{A}_4 = \begin{bmatrix} -0.272 & 1.940 & 1.450 \\ 0.058 & -3.960 & 0.100 \\ 0.100 & 0.080 & -2.910 \end{bmatrix},$$

$$\overline{A}_1 = \begin{bmatrix} -0.258 & 2.060 & 1.550 \\ 0.142 & -3.640 & 0 \\ 0.200 & 0 & -2.550 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} -0.258 & 2.060 & 1.550 \\ 0.142 & -3.640 & 0.100 \\ 0.200 & 0 & -2.550 \end{bmatrix},$$

$$\overline{A}_3 = \begin{bmatrix} -0.258 & 2.060 & 1.550 \\ 0.142 & -3.640 & 0 \\ 0.200 & 0.080 & -2.550 \end{bmatrix}, \quad \overline{A}_4 = \begin{bmatrix} -0.258 & 2.060 & 1.550 \\ 0.142 & -3.640 & 0.100 \\ 0.200 & 0.080 & -2.550 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.50 & 1.00 \\ 1.00 & 0.90 \\ 0.70 & 1.10 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1.2 & 0 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.5 & 0 \end{bmatrix},$$

while, for all i, $B_i = B$.

Tt is not hard to attest that $\overline{A}_i$, $\underline{A}_i$ are Metzler and Hurwitz for all i, $\underline{A}_i \le \overline{A}_i$, $\underline{C} \le \overline{C}$, B is positive matrix and $\underline{C}$, $\overline{C}$ are nonnegative matrices.

For the vector of premise variables and the sector bounds

$$\vartheta(t) = \begin{bmatrix} \theta_1(t) & \theta_2(t) \end{bmatrix} = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix},$$

$$d_1 = \max(q_1) = 1,\ d_2 = \min(q_1) = 0,\ e_1 = \max(q_2) = 1,\ e_2 = \min(q_2) = 0,$$

the sector functions are given as

$$w_{11}(q_1(t)) = \frac{d_1 - q_1(t)}{d_1 - d_2}, \quad w_{12}(q_1(t)) = \frac{q_1(t) - d_2}{d_1 - d_2} = 1 - w_{11}(q_1(t)),$$

$$w_{21}(q_2(t)) = \frac{e_1 - q_2(t)}{e_1 - e_2}, \quad w_{22}(q_2(t)) = \frac{q_2(t) - e_2}{e_1 - e_2} = 1 - w_{21}(q_2(t))$$

and the set of membership functions is aggregated as

$$h_1(\theta(t)) = w_{12}(q_1(t))w_{22}(q_2(t)), \quad h_2(\theta(t)) = w_{12}(q_1(t))w_{21}(q_2(t)),$$

$$h_3(\theta(t)) = w_{11}(q_1(t))w_{22}(q_{21}(t)), \quad h_4(\theta(t)) = w_{11}(q_2(t))w_{21}(q_2(t)).$$

It is evident that some from the matrices $\underline{A}_i$, $\overline{A}_i$ are not strictly Metzler and a certain structuring of the diagonal matrix variables R_{dk} is necessary. Since the third column of $\underline{C}$ as well as $\overline{C}$ is zero vector, it is not necessary to define zero elements of the structured diagonal matrix variables R_{dk} in the lower right corner position for $k = 1,2$ and so the problem with zero elements $\underline{a}_{i23}$, $\overline{a}_{i23}$ for $i = 1,3$ is solved in general. Conversely, since $\underline{c}_{22}$, $\overline{c}_{22}$ are not equal to zero, it is necessary to choose zero elements $r_{i2} = 0$ in structured diagonal matrix variables R_{i2}, $i = 1,2$. Summarising,

$$R_{ik} = \operatorname{diag}\begin{bmatrix} r_{ik1} & r_{ik2} & r_{ik3} \end{bmatrix} \succ 0 \text{ for } (k = 1,\ i = 1,2,3,4) \text{ and } (k = 2,\ i = 3,4),$$

$$R_{ik} = \operatorname{diag}\begin{bmatrix} r_{ik1} & r_{ik2} & 0 \end{bmatrix} \succeq 0 \text{ for } k = 2,\ i = 1,2.$$

To reflect diagonal LMIs structures, the representations of $\underline{C}$, $\overline{C}$ are given as

$$\underline{C}_{d1} = \text{diag}\left[0.9\ 0\ 0\right],\ \underline{C}_{d2} = \text{diag}\left[0\ 0.2\ 0\right],\ \overline{C}_{d1} = \text{diag}\left[1.1\ 0\ 0\right],\ \overline{C}_{d2} = \text{diag}\left[0\ 0.5\ 0\right]$$

and with modulo n summation operator $\Delta = (1 \leftrightarrow 3)/3$, for example the representations of $\underline{A}_1$ are

$$\underline{A}_1(p,p)_\Delta = \underline{A}_i(p,p)_\Delta = \text{diag}\left[-0.272\ -3.960\ -2.910\right]\ \forall\, i \in \langle 1,4\rangle,$$

$$\underline{A}_1(p+1,p)_\Delta = \text{diag}\left[0.058\ 0\ 1.450\right],\ \underline{A}_1(p+2,p)_\Delta = \text{diag}\left[0.100\ 1.940\ 0\right].$$

The remaining matrices $\underline{A}_i$, $\overline{A}_i$ are parameterized analogously.

By applying Theorem 2 for solving by toolbox SeDuMi [20], the feasible solution is obtained as follows

$$J_1 = \begin{bmatrix} 1.1667 & 0.7376 \\ 0.0210 & 0.6443 \\ 0.0286 & 0 \end{bmatrix},\ J_2 = \begin{bmatrix} 1.1679 & 0.7389 \\ 0.0210 & 0.6449 \\ 0.0286 & 0 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} 1.1812 & 0.7476 \\ 0.0198 & 0.6491 \\ 0.0270 & 0.0181 \end{bmatrix},\ J_4 = \begin{bmatrix} 1.1812 & 0.7476 \\ 0.0198 & 0.6491 \\ 0.0270 & 0.0181 \end{bmatrix},$$

guaranteing Metzler and Hurwitz local system matrices of the interval observer

$$\underline{A}_{e1} = \begin{bmatrix} -1.5553 & 0.8336 & 1.450 \\ 0.0349 & -4.9265 & 0 \\ 0.0686 & 0 & -2.910 \end{bmatrix},\quad \underline{A}_{e2} = \begin{bmatrix} -1.5567 & 0.8317 & 1.450 \\ 0.0349 & -4.9274 & 0.100 \\ 0.0685 & 0 & -2.910 \end{bmatrix},$$

$$\underline{A}_{e3} = \begin{bmatrix} -1.5713 & 0.8186 & 1.450 \\ 0.0362 & -4.9337 & 0 \\ 0.0703 & 0.0528 & -2.910 \end{bmatrix},\quad \underline{A}_{e4} = \begin{bmatrix} -1.5727 & 0.8170 & 1.450 \\ 0.0362 & -4.9339 & 0.100 \\ 0.0702 & 0.0527 & -2.910 \end{bmatrix},$$

$$\overline{A}_{e1} = \begin{bmatrix} -1.3080 & 1.1749 & 1.550 \\ 0.1231 & -4.4132 & 0 \\ 0.1743 & 0 & -2.550 \end{bmatrix},\quad \overline{A}_{e2} = \begin{bmatrix} -1.3091 & 1.1733 & 1.550 \\ 0.1231 & -4.4139 & 0.100 \\ 0.1742 & 0 & -2.550 \end{bmatrix},$$

$$\overline{A}_{e3} = \begin{bmatrix} -1.3080 & 1.1749 & 1.550 \\ 0.1231 & -4.4132 & 0 \\ 0.1743 & 0 & -2.550 \end{bmatrix},\quad \overline{A}_{e4} = \begin{bmatrix} -1.3091 & 1.1733 & 1.550 \\ 0.1231 & -4.4139 & 0.100 \\ 0.1742 & 0 & -2.550 \end{bmatrix}.$$

Having in mind (43) it is not hard to verify that the condition $\underline{A}_{ei} \leq \overline{A}_{ei}$ is satisfied for all $i \in \langle 1,4\rangle$.

Note, feasibility of the presented set of LMIs can be checked also by using the LMI toolbox of MATLAB©.

Based on the structured matrix variable properties, defined in (73), (74), it is verified that conditions (53)-(59) allows the existence of nonnegative J_i for $i \in \langle 1,4\rangle$ such that Lyapunov function (69) establishes asymptotic stability of the interval observer equilibrium. Moreover, set of nonnegative J_i for $i \in \langle 1,4\rangle$ for nonnegative initial state of Metzlerian Takagi-Sugeno system guaranties that the lower observer estimate is nonnegative in the sense of (52).

Involving additional inequality constraints the problem of interval observer design is transformed to equivalent linear time invariant forms and make the design problem standard.

6. Concluding Remarks

The key observation is that it can obtain a finite number of linear matrix inequalities to account in design for Metzler and Hurwitz interval observer system matrices and non-negative interval observer gains. Therefore, to obtain a solution, the design method can be applied yielding feasibility of the set of linear matrix inequalities. Moreover, the condition extensions take into account the fact that certain elements of bounds can to equal zero and so reflect also non strictly Metzler matrix structures. The novelty lies in strictly LMI representation of interval bounds, parametric constraints and stability. The example, demonstrating how one can formulate design task, also indicates that defined LMI design conditions are necessary in synthesis of interval observers for uncertain Metzlerian Takagi-Sugeno multidimensional systems.

Presented version prefers standard LMI numerical procedures to manipulate the interval observer stability and structural properties and is guided in the direction of the second Lyapunov method, which guarantees convergence to equilibria of the estimation errors. It seems to be significant to extend the approach for uncertain Metzlerian Takagi-Sugeno continuous-time systems with external disturbances.

Since interval estimation of switched Takagi-Sugeno systems is connected with Metzler system matrices, further future research is naturally focused on this application field. A similar trend can be expected in the positive control of agent systems in the case when additional criteria are found for design of nonnegative gains for agents whose system matrices are not Metzler. Hence it is apparent that exactly the same methods of solution can carry potentially through to fractional fuzzy inference systems.

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