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On Minimality and Consistency Tolerance in Logical Argumentation Frameworks

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Abstract. We examine different methods of handling argument consistency and minimality in logical argumentation frameworks, showing that both properties may (and sometimes even should) be omitted from the definition of arguments. In process, we consider the adequacy of attack rules to the underlying logics.

Keywords. logical argumentation, forms of arguments and attacks, consistency preservation, assumption minimization

1. Motivation

A standard way of viewing an argument A in logical (or, deductive) argumentation frameworks is as a pair $A = \langle \mathsf{S}, \psi \rangle$, where ψ (the *conclusion* of A) is a formula that follows, according to the underlying (base) logic, from the set of formulas S (called the *support* of A). Earlier works on the subject concentrated on classical logic (CL) as the base logic, and since the latter is trivialized in the presence of inconsistency, it was usual to assume that S is consistent. In order to keep the support as relevant as possible to the conclusion, S was kept minimal with respect to the subset relation (see [5]). These considerations lead to the following definition of what we call *classical-con-min arguments*.

Definition 1 A CL-*con-min argument* is a pair $A = \langle \mathsf{S}, \psi \rangle$, where S is a CL-consistent and $\subseteq$-minimal finite set of formulas that entails, according to CL, the formula ψ.[2]

Definition 1 is at the heart of many approaches to logic-based argumentation.[3] However, as noted e.g. in [3], the consistency and minimality requirements on the supports of the arguments cause some complications in the construction and the identification of valid arguments, and so they may be lifted. Moreover, in some reasoning contexts non-classical logics may better serve as the underlying logics of the intended argumentation frameworks, and in some cases (e.g., agent-based systems or deontic systems) the standard propositional language should be extended (e.g., with modal operators), which again means that in those cases classical logic is not adequate. Indeed, many approaches

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[2] In order words, if F denotes the falsity operator and $\vdash_{\mathsf{CL}}$ is the consequence relation of classical logic, then S is a finite set of formulas such that $\mathsf{S} \nvdash_{\mathsf{CL}} \mathsf{F}$, $\mathsf{S} \vdash_{\mathsf{CL}} \psi$, and there is no $\mathsf{S}' \subsetneq \mathsf{S}$ such that $\mathsf{S}' \vdash_{\mathsf{CL}} \psi$.

[3] For more details and references see, e.g., [6,7,13].

to structured argumentation like those that are based on ASPIC systems [16], deductive variation of assumption-based argumentation frameworks [15], sequent-based argumentation [3], and so forth, do not assume anymore that the underlying logic is necessarily classical. Concerning consistency and minimality, in some of these alternatives the handling of these properties is done on the level of the argumentation frameworks themselves, by means of appropriate attack rules, or by posing some restrictions on their applications (see for instance [14]).

In this paper we examine relations between these two approaches for handling minimality and consistency, namely: the one that enforces these properties already on the level of arguments and the other that takes care of them by appropriate attack rules. For the latter we then show how the suitability of the attack rules is affected by the base logic.

2. Preliminaries

For defining logical argumentation frameworks, and arguments in particular, one first has to specify what the underlying logic is.

Definition 2 A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\mathscr{L}$ is a propositional language, and $\vdash$ is a (Tarskian, [21]) *consequence relation* for a language $\mathscr{L}$, that is: a binary relation between sets of formulas and formulas in $\mathscr{L}$, satisfying the following conditions: if $\psi \in \mathsf{S}$ then $\mathsf{S} \vdash \psi$ (*reflexivity*); if $\mathsf{S} \vdash \psi$ and $\mathsf{S} \subseteq \mathsf{S}'$ then $\mathsf{S}' \vdash \psi$ (*monotonicity*); and if $\mathsf{S} \vdash \psi$ and $\mathsf{S}', \psi \vdash \phi$ then $\mathsf{S}, \mathsf{S}' \vdash \phi$ (*transitivity*).

In the sequel we shall assume that the language $\mathscr{L}$ contains at least the following (primitive or defined) connectives and constant:

- $\vdash$-*negation* $\neg$, satisfying: $p \not\vdash \neg p$ and $\neg p \not\vdash p$ (for every atomic p),
- $\vdash$-*conjunction* $\wedge$, satisfying: $\mathsf{S} \vdash \psi \wedge \phi$ iff $\mathsf{S} \vdash \psi$ and $\mathsf{S} \vdash \phi$,
- $\vdash$-*disjunction* $\vee$, satisfying: $\mathsf{S}, \phi \vee \psi \vdash \sigma$ iff $\mathsf{S}, \phi \vdash \sigma$ and $\mathsf{S}, \psi \vdash \sigma$,
- $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.[4]

In some cases we shall assume the availability of a *(deductive)* $\vdash$-*implication* satisfying: $\mathsf{S}, \phi \vdash \psi$ iff $\mathsf{S} \vdash \phi \supset \psi$. Then we shall abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$. For a finite set of formulas S we denote by $\bigwedge \mathsf{S}$ (respectively, by $\bigvee \mathsf{S}$) the conjunction (respectively, the disjunction) of all the formulas in S. We shall also denote by $\wp(\mathsf{S})$ (by $\wp_{\mathsf{fin}}(\mathsf{S})$) the set of the (finite) subsets of S. We say that S is $\vdash$-*consistent* if $\mathsf{S} \not\vdash \mathsf{F}$.

The next definition is a generalization of Definition 1 to every propositional logic, and it avoids the consistency and minimality requirements.

Definition 3 Given a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, an $\mathfrak{L}$-*argument* (an *argument* for short) is a pair $A = \langle \mathsf{S}, \psi \rangle$, where S (the support of A) is a finite set of $\mathscr{L}$-formulas and ψ (the conclusion of A) is an $\mathscr{L}$-formula, such that $\mathsf{S} \vdash \psi$. We denote: $\mathsf{Supp}(\langle \mathsf{S}, \psi \rangle) = \mathsf{S}$ and $\mathsf{Conc}(\langle \mathsf{S}, \psi \rangle) = \psi$. Arguments of the form $\langle \emptyset, \psi \rangle$ are called tautological.

Attacks and counter-attacks between arguments are described by the rules in Table 1 (see, e.g., [3,13,20] for further rules).

[4]In particular, F is not a standard atomic formula, since $\mathsf{F} \vdash \neg \mathsf{F}$.

Rule Name	Acronym	Attacking	Attacked	Attack Conditions
Defeat	Def	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \mathsf{S}_2 \cup \mathsf{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathsf{S}_2$
Full Defeat	FullDef	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \mathsf{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathsf{S}_2$
Direct Defeat	DirDef	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathsf{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \varphi$
Undercut	Ucut	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \mathsf{S}_2 \cup \mathsf{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathsf{S}_2, \ \neg \bigwedge \mathsf{S}_2 \vdash \psi_1$
Full Undercut	FullUcut	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \mathsf{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathsf{S}_2, \ \neg \bigwedge \mathsf{S}_2 \vdash \psi_1$
Direct Undercut	DirUcut	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathsf{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \varphi, \quad \neg \varphi \vdash \psi_1$
Consistency Undercut	ConUcut	$\langle \emptyset, \neg \bigwedge \mathsf{S}_2 \rangle$	$\langle \mathsf{S}_2 \cup \mathsf{S}_2', \psi_2 \rangle$	
Rebuttal	Reb	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \mathsf{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \psi_2, \quad \neg \psi_2 \vdash \psi_1$
Defeating Rebuttal	DefReb	$\langle \mathsf{S}_1, \psi_1 \rangle$	$\langle \mathsf{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \psi_2$

Table 1. Some attack rules. The support sets of the attacked arguments are assumed to be nonempty (to avoid attacks on tautological arguments).

Logical argumentation frameworks are now defined as follows:

Definition 4 Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic and $\mathscr{A}$ a set of attack rules with respect to $\mathfrak{L}$. Let also S be a set of $\mathscr{L}$-formulas. The *(logical) argumentation framework* for S, induced by $\mathfrak{L}$ and $\mathscr{A}$, is the pair $\mathscr{AF}_{\mathfrak{L},\mathscr{A}}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A}) \rangle$,[5] where $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ is the set of the $\mathfrak{L}$-arguments whose supports are subsets of S, and $\mathit{Attack}(\mathscr{A})$ is a relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$, defined by $(A_1, A_2) \in \mathit{Attack}(\mathscr{A})$ iff there is some $\mathscr{R} \in \mathscr{A}$ such that A_1 $\mathscr{R}$-attacks A_2 (that is, the pair (A_1, A_2) is an instance of the relation $\mathscr{R}$).

The Dung-style semantics [12] of an argumentation framework and the corresponding entailment relations are defined in the next two definitions.

Definition 5 Let $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A}) \rangle$ be a logical argumentation framework, and let $\mathscr{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. Below, maximality and minimality are taken with respect to the subset relation.

- We say that $\mathscr{E}$ *attacks* an argument A, if there is an argument $B \in \mathscr{E}$ that attacks A (that is, $(B, A) \in \mathit{Attack}$). The set of arguments that are attacked by $\mathscr{E}$ is denoted $\mathscr{E}^+$. We say that $\mathscr{E}$ *defends* A, if $\mathscr{E}$ attacks every argument that attacks A.
- The set $\mathscr{E}$ is called *conflict-free* with respect to $\mathscr{AF}(\mathsf{S})$, if it does not attack any of its elements (i.e., $\mathscr{E}^+ \cap \mathscr{E} = \emptyset$). A set that is maximally conflict-free with respect to $\mathscr{AF}(\mathsf{S})$ is called a *naive extension* of $\mathscr{AF}(\mathsf{S})$. A conflict-free set $\mathscr{E}$ such that $\mathscr{E} \cup \mathscr{E}^+ = \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ is a *stable extension* of $\mathscr{AF}(\mathsf{S})$.
- An *admissible extension* of $\mathscr{AF}(\mathsf{S})$ is a subset of $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ that is conflict-free with respect to $\mathscr{AF}(\mathsf{S})$ and defends all of its elements. A maximally admissible extension of $\mathscr{AF}(\mathsf{S})$ is called a *preferred extension* of $\mathscr{AF}(\mathsf{S})$.
- A *complete extension* of $\mathscr{AF}(\mathsf{S})$ is an admissible extension of $\mathscr{AF}(\mathsf{S})$ that contains all the arguments that it defends. The minimally complete extension of $\mathscr{AF}(\mathsf{S})$ is called the *grounded extension* of $\mathscr{AF}(\mathsf{S})$.[6]

[5] In what follows we shall usually omit the subscripts and write just $\mathscr{AF}(\mathsf{S})$ for $\langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A}) \rangle$.

[6] As is shown in [12, Theorem 25], the grounded extension of $\mathscr{AF}(\mathsf{S})$ is unique. Also, in the same paper it is shown that preferred extensions are maximally complete and that every stable extension is also preferred. For some other facts and definitions of other extensions, see e.g. [4].

We denote by $\mathsf{Adm}(\mathscr{AF}(\mathsf{S}))$ [respectively, by $\mathsf{Cmp}(\mathscr{AF}(\mathsf{S}))$, $\mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$, $\mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$, $\mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$] the set of all the admissible [respectively, the complete, grounded, preferred, stable] extensions of $\mathscr{AF}(\mathsf{S})$.

Definition 6 Let $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A})\rangle$ be a logical argumentation framework, and let $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$. We denote:

- $\mathsf{S} \mid\!\sim^{\mathfrak{L},\mathscr{A}}_{\cup\mathsf{Sem}} \psi$ if there is an argument $\langle \Gamma, \psi\rangle \in \bigcup \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$,
- $\mathsf{S} \mid\!\sim^{\mathfrak{L},\mathscr{A}}_{\cap\mathsf{Sem}} \psi$ if there is an argument $\langle \Gamma, \psi\rangle \in \bigcap \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$,
- $\mathsf{S} \mid\!\sim^{\mathfrak{L},\mathscr{A}}_{\Cap\mathsf{Sem}} \psi$ if for every $\mathscr{E} \in \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$ there is an argument $\langle \Gamma_{\mathscr{E}}, \psi\rangle \in \mathscr{E}$.

In what follows, when the framework is clear from the context, we shall sometimes write $\mathscr{AF}(\mathsf{S}) \mid\!\sim_{\cup\mathsf{Sem}} \psi$ instead of $\mathsf{S} \mid\!\sim^{\mathfrak{L},\mathscr{A}}_{\cup\mathsf{Sem}} \psi$ (and similarly for the other two entailments).

3. Consistency Preservation in Logical Frameworks

In this section we relate the two methods of maintaining inconsistency in logical argumentation frameworks: by posing the consistency restriction of the supports on the arguments (cf. Definition 1) and by using appropriate attack relations between arguments.

Definition 7 Recall from Definition 5, that $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})^{+}$ is the set of arguments that are attacked by some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. In what follows we shall also denote this set by S^{+}.

Example 1 The set $\emptyset^{+}$ consists of the arguments that are attacked by tautological arguments (i.e., by those whose support set is empty).

Definition 8 A set of attack is $\emptyset$-*normal* if it excludes attacks on tautological arguments.

Example 2 By their definitions, all the rules in Table 1 are $\emptyset$-normal, since they exclude attacks on arguments with empty support sets (as is indicated in the caption of Table 1). In [direct] undercut and [direct] defeat, this also follows from the attack conditions and in consistency undercut this follows from the form of the attacking and the attacked arguments.

Proposition 1 *Let* $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A})\rangle$ *be a logical argumentation framework for* S*, based on a logic* $\mathfrak{L} = \langle \mathscr{L}, \vdash\rangle$ *and a set* $\mathscr{A}$ *of* $\emptyset$*-normal attack rules. For* $\mathscr{E} \subseteq \emptyset^{+}$ *and* $\mathscr{A}^{\star} \subseteq \mathscr{A}$ *such that* $\mathit{Attack}(\mathscr{A}^{\star}) \subseteq (\mathsf{Arg}_{\mathscr{L}}(\emptyset) \times \mathscr{E})$*, we let* $\mathscr{AF}^{\star}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}, \mathit{Attack}(\mathscr{A} \setminus \mathscr{A}^{\star})\rangle$*. Then* $\mathsf{Sem}(\mathscr{AF}(\mathsf{S})) = \mathsf{Sem}(\mathscr{AF}^{\star}(\mathsf{S}))$ *for every* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$.

Note 1 Intuitively, the set $\mathscr{E}$ in Proposition 1 consists of the 'contradictory' S-based arguments (cf. Example 1) and $\mathscr{A}^{\star}$ consists of the rules that allow to attack the elements in $\mathscr{E}$. What Proposition 1 says, then, is that if 'contradictory' arguments are not allowed (as in Definition 1) then attack rules in the style of $\mathscr{A}^{\star}$ may be avoided, and vice-versa: in case that no restrictions are posed on the arguments' supports (as in Definition 3) then $\mathscr{A}^{\star}$-type attack rules are needed.

Proof. We consider $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Prf}, \mathsf{Stb}\}$, leaving the other cases to the reader.

• Consider $\mathsf{Sem} = \mathsf{Adm}$. Let $\mathscr{H} \in \mathsf{Adm}(\mathscr{AF}(\mathsf{S}))$. We first observe that $\mathscr{H} \subseteq \mathsf{Arg}_{\mathscr{L}}(\mathsf{S}) \setminus \mathscr{E}$. Indeed, if there were an argument $A \in \emptyset^+$ in $\mathscr{H}$, there would be an argument $B \in \mathsf{Arg}_{\mathscr{L}}(\emptyset)$ $\mathscr{A}$-attacking A,[7] and by the $\emptyset$-normality of $\mathscr{A}$ there would not be an attacker of A in $\mathscr{H}$, contradicting the admissibility of $\mathscr{H}$ in $\mathscr{AF}(\mathsf{S})$.

Clearly, $\mathscr{H}$ is conflict-free in $\mathscr{AF}^\star(\mathsf{S})$. Suppose now that there is some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$ that $(\mathscr{A} \setminus \mathscr{A}^\star)$-attacks some $B \in \mathscr{H}$. Since $\mathscr{H} \in \mathsf{Adm}(\mathscr{AF}(\mathsf{S}))$, there is a $C \in \mathscr{H}$ that $\mathscr{A}$-attacks A. Since $\mathscr{A}$ is $\emptyset$-normal, A has a non-empty support. Since $Attack(\mathscr{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathscr{E})$ and $A \notin \mathscr{E}$, A also $(\mathscr{A} \setminus \mathscr{A}^\star)$-attacks A. This shows that $\mathscr{H} \in \mathsf{Adm}(\mathscr{AF}^\star(\mathsf{S}))$.

Let now $\mathscr{H} \in \mathsf{Adm}(\mathscr{AF}^\star(\mathsf{S}))$. Clearly, $\mathscr{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. Assume for a contradiction that there are $A, B \in \mathscr{H}$ such that A $\mathscr{A}$-attacks B. By the admissibility of $\mathscr{H}$ in $\mathscr{AF}^\star(\mathsf{S})$, A does not $(\mathscr{A} \setminus \mathscr{A}^\star)$-attack B. Thus, A $\mathscr{A}^\star$-attacks B. However, then $B \in \mathscr{E}$, since $Attack(\mathscr{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathscr{E})$. This is a contradiction to $\mathscr{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$. Thus, $\mathscr{H}$ is conflict-free in $\mathscr{AF}_{\mathfrak{L}}(\mathsf{S})$.

Suppose now that some $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ $\mathscr{A}$-attacks some $A \in \mathscr{H}$. If it is an $(\mathscr{A} \setminus \mathscr{A}^\star)$-attack, by the admissibility of $\mathscr{H}$ in $\mathscr{AF}^\star(\mathsf{S})$ there is a $C \in \mathscr{H}$ that $\mathscr{A}$-attacks B. Assume it is an $\mathscr{A}^\star$-attack. Then $A \in \mathscr{E}$, since $\mathrm{Attack}(\mathscr{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathscr{E})$. This is a contradiction to $\mathscr{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$. Thus, $\mathscr{H} \in \mathsf{Adm}(\mathscr{AF}(\mathsf{S}))$.

• Consider $\mathsf{Sem} = \mathsf{Prf}$. This follows immediately from the fact that $\mathsf{Adm}(\mathscr{AF}(\mathsf{S})) = \mathsf{Adm}(\mathscr{AF}^\star(\mathsf{S}))$, since preferred extensions are the maximally admissible ones.

• Consider $\mathsf{Sem} = \mathsf{Stb}$. Let $\mathscr{H} \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$. Assume that $\mathscr{H} \cap \mathscr{E} \neq \emptyset$. Let $A \in \mathscr{H} \cap \mathscr{E}$. Then there is a $B \in \mathsf{Arg}_{\mathfrak{L}}(\emptyset)$ that $(\mathscr{A} \setminus \mathscr{A}^\star)$-attacks A. Since $\mathscr{A}$ is $\emptyset$-normal, there is no $C \in \mathscr{H}$ that $\mathscr{A}$-attacks B. By the stability of $\mathscr{H}$, $B \in \mathscr{H}$, which contradicts the conflict-freeness of $\mathscr{H}$. Thus, $\mathscr{H} \cap \mathscr{E} = \emptyset$ and so $\mathscr{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$.

Clearly, $\mathscr{H}$ is $(\mathscr{A} \setminus \mathscr{A}^\star)$-conflict-free since it is $\mathscr{A}$-conflict-free. Suppose that $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus (\mathscr{E} \cup \mathscr{H})$. Then $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{H}$ and so there is a $B \in \mathscr{H}$ that $\mathscr{A}$-attacks A. Since $\mathrm{Attack}(\mathscr{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathscr{E})$ and $A \notin \mathscr{E}$, B also $(\mathscr{A} \setminus \mathscr{A}^\star)$-attacks A. Thus, $\mathscr{H} \in \mathsf{Stb}(\mathscr{AF}^\star(\mathsf{S}))$.

Suppose now that $\mathscr{H} \in \mathsf{Stb}(\mathscr{AF}^\star(\mathsf{S}))$. Assume for a contradiction that $\mathscr{H}$ is not conflict-free in $\mathscr{AF}(\mathsf{S})$. Thus, there are $A, B \in \mathscr{H}$ such that A $\mathscr{A}$-attacks B. Since $\mathscr{H}$ is conflict-free in $\mathscr{AF}^\star(\mathsf{S})$, A does not $(\mathscr{A} \setminus \mathscr{A}^\star)$-attack B, and so it $\mathscr{A}^\star$-attacks B. Since $\mathrm{Attack}(\mathscr{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathscr{E})$, $B \in \mathscr{E}$, which contradicts the fact that $\mathscr{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$. Thus, $\mathscr{H}$ is conflict-free in $\mathscr{AF}(\mathsf{S})$.

Suppose now that $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{H}$. If $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$, there is an argument $A \in \mathscr{H}$ that $\mathscr{A}$-attacks B. Otherwise, $B \in \mathscr{E}$, thus there is an $A \in \mathsf{Arg}_{\mathfrak{L}}(\emptyset)$ that $\mathscr{A}$-attacks B. Since $\mathscr{A}$ is $\emptyset$-normal, $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathscr{E}$ and, since $\mathscr{H}$ is stable in $\mathscr{AF}^\star(\mathsf{S})$, $A \in \mathscr{H}$. Altogether, this shows that $\mathscr{H} \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$. □

As a particular case of Proposition 1, we have the following corollary:

Corollary 1 *Let $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), Attack(\mathscr{A}) \rangle$ be a logical argumentation framework for S, based on a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ and a set $\mathscr{A}$ of $\emptyset$-normal attack rules that contains ConUcut. Let also $\mathscr{AF}^{\mathsf{con}}(\mathsf{S}) = \langle \mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathsf{S}), Attack(\mathscr{A}^\star) \rangle$ be a logical argumenta-*

[7]We say that A $\mathscr{A}$-attacks B iff there is some $\mathscr{R} \in \mathscr{A}$ such that A $\mathscr{R}$-attacks B.

tion framework in which $\mathscr{A}^\star = \mathscr{A} - \{\text{ConUcut}\}$ *and* $\mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathsf{S})$ *is the subset of* $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ *that consists only of* $\vdash_{\mathfrak{L}}$*-consistent arguments (i.e, whose supports are* $\vdash_{\mathfrak{L}}$*-consistent). Then* $\mathsf{Sem}(\mathscr{AF}(\mathsf{S})) = \mathsf{Sem}(\mathscr{AF}^{\mathsf{con}}(\mathsf{S}))$ *for every* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$.

Proof. Follows from Proposition 1 since *Attack*(ConUcut) $\subseteq \mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathsf{Arg}^{\mathsf{incon}}_{\mathfrak{L}}(\mathsf{S})$, where $\mathsf{Arg}^{\mathsf{incon}}_{\mathfrak{L}}(\mathsf{S}) = \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathsf{S})$. □

Note 2 The use of ConUcut for attacking arguments that are based on inconsistent supports goes beyond the standard interpretation of inconsistency as in classical logic. For instance, according to logics of formal inconsistency (LFIs, see [9,10]) $\mathsf{S}_1 = \{\psi, \neg\psi\}$ is not considered inconsistent, but rather $\mathsf{S}_2 = \{\psi, \neg\psi, \circ\psi\}$ (where $\circ$ is the consistency operator, thus $\circ\psi$ is intuitively understood as a claim that 'ψ is consistent'). Indeed, when an LFI is the base logic, an argument whose support is S_1 is not ConUcut-attacked, while an argument whose support set contains S_2 *is* ConUcut-attacked (by $\langle \emptyset, \neg(\psi \wedge \neg\psi \wedge \circ\psi)\rangle$). We shall return to this issue in Section 5.

We show that Proposition 1 and Corollary 1 crucially depend on $\mathscr{A}$ being $\emptyset$-normal:

Example 3 We consider classical logic with the premises $\mathsf{S} = \{p \wedge \neg p, s\}$ and with a more radical form of Rebuttal that does not follow the restriction that only arguments with non-empty support may be attacked. Although $\langle p \wedge \neg p, \neg s\rangle$ is ConUcut-attacked by $\langle \emptyset, \neg(p \wedge \neg p)\rangle$, the latter is Rebut-attacked by $\langle p \wedge \neg p, p \wedge \neg p\rangle$ (given our more radical form of Rebuttal). Thus, e.g., the grounded extension will be empty in the presence of Rebuttal, even in the presence of ConUcut. However, after filtering out inconsistent arguments, it is easy to see that $\langle s, s\rangle$ will be an argument in the grounded extension.

Corollary 2 Let $\mathscr{AF}(\mathsf{S})$ and $\mathscr{AF}^\star(\mathsf{S})$ be as in Proposition 1. Then $\mathscr{AF}(\mathsf{S}) \mathrel{|\!\sim}_{\circ\mathsf{Sem}} \psi$ iff $\mathscr{AF}^\star(\mathsf{S}) \mathrel{|\!\sim}_{\circ\mathsf{Sem}} \psi$ for every $\circ \in \{\cup, \cap, \Cap\}$ and $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$.[8]

4. Enforcement of Minimal Support

We now turn to the other condition in Definition 1 – subset minimality of the arguments' supports. Our main result is given in Proposition 2. First, some definitions and lemmas.

Definition 9 Given an argumentation framework $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A})\rangle$, a *support ordering* for $\mathscr{AF}(\mathsf{S})$ is a preorder[9] $\preceq$ on the finite subsets of S.[10]

Definition 10 Given a framework $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}(\mathscr{A})\rangle$, a support ordering $\preceq$ for $\mathscr{AF}(\mathsf{S})$, a set $\mathscr{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$, and an argument $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. We denote:

- $\min_{\preceq}(\mathscr{E}) = \{A \in \mathscr{E} \mid \nexists B \in \mathscr{E} \text{ s.t. } \mathsf{Conc}(B) = \mathsf{Conc}(A) \text{ and } \mathsf{Supp}(B) \prec \mathsf{Supp}(A)\}$,
- $A^{\min}_{\preceq} = \min_{\preceq}(\{B \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{Supp}(A)) \mid \mathsf{Conc}(A) = \mathsf{Conc}(B)\})$, [11]
- $\mathscr{E}^{\min}_{\preceq} = \bigcup\{A^{\min}_{\preceq} \mid A \in \mathscr{E}\}$.

[8] Here we abuse a bit the notations of Definition 6 to emphasize the relations between the frameworks.

[9] I.e., a reflexive and transitive order.

[10] We denote by $\prec$ the strict version of $\preceq$, that is: if $\preceq$ is a preorder on some domain $\mathscr{D}$, then for all $d, d' \in \mathscr{D}$, $d \prec d'$ iff $d \preceq d'$ and $d' \npreceq d$.

[11] To simplify the notation,we write $A^{\min}_{\preceq}$ instead of $A^{\min}_{\mathfrak{L},\preceq}$.

Thus, $\min_{\preceq}(\mathscr{E})$ filters-out from $\mathscr{E}$ all the arguments whose support is not minimal among the arguments in $\mathscr{E}$, and $\mathscr{E}^{\min}_{\preceq}$ removes from the arguments in $\mathscr{E}$ all the redundant formulas in their supports.

Example 4 Let $\mathfrak{L} = \mathsf{CL}$ (classical logic), $\preceq\, = \,\subseteq$, and $\mathscr{E} = \{p,q \Rightarrow p \vee q\}$. Then $\min_{\preceq}(\mathscr{E}) = \mathscr{E}$ and $\mathscr{E}^{\min}_{\preceq} = \{p \Rightarrow p \vee q,\ q \Rightarrow p \vee q\}$. Also, for $\mathscr{E}' = \mathscr{E} \cup \{p \Rightarrow p \vee q\}$, we have that $\min_{\preceq}(\mathscr{E}') = \{p \Rightarrow p \vee q\}$ and $(\mathscr{E}')^{\min}_{\preceq} = \mathscr{E}^{\min}_{\preceq}$.

Example 5 Obviously, the subset relation $\subseteq$ is the most natural support ordering in our context. However, there are other candidates to be a support ordering $\preceq$, among which are the following:

- For $\Delta, \Gamma \in \wp_{\mathsf{fin}}(\mathsf{S})$ we define $\Delta \preceq_{\vdash} \Gamma$ iff $\Gamma \vdash \bigwedge \Delta$.
- Suppose that S is stratified into a partition $\langle \mathsf{S}_1, \ldots, \mathsf{S}_n \rangle$, where intuitively formulas in S_i are considered more reliable than formulas in S_j when $i > j$.[12] We let $\preceq$ be the lexicographic ordering, i.e., for $\Delta = \langle \Delta_1, \ldots, \Delta_n \rangle$ and $\Gamma = \langle \Gamma_1, \ldots, \Gamma_m \rangle$ (with $\Delta_i, \Gamma_i \in \wp_{\mathsf{fin}}(\mathsf{S}_i)$ for each $1 \leq i \leq \max\{n,m\}$), we define: $\Delta \preceq_{\mathsf{lex}} \Gamma$ iff either $\Delta = \Gamma$, or there is an $1 \leq k \leq \min\{n,m\}$ such that $\Delta_i = \Gamma_i$ for all $i < k$ and $\Delta_k \subsetneq \Gamma_k$.

Note 3 In all the cases of the last example it holds that if $A, B \in \mathsf{Arg}(\mathsf{S})$ have the same conclusion and $\mathsf{Supp}(A) \prec \mathsf{Supp}(B)$, it makes sense to consider B argumentatively more vulnerable, since its support gives more points of attack: Either it contains more formulas ($\preceq\, = \,\subseteq$), or because its support contains stronger logical commitments ($\preceq\, = \,\preceq_{\vdash}$), or because its support contains stronger logical commitments relative to their reliability ($\preceq\, = \,\preceq_{\mathsf{lex}}$). In that sense, the demand of $\preceq$-minimal support from arguments means minimal argumentative vulnerability.

Definition 11 A set of attack rules $\mathscr{A}$ is called $\preceq$*-normal*, if for every $\mathscr{R} \in \mathscr{A}$ the following conditions hold:

1. If A $\mathscr{R}$-attacks B and $\mathsf{Supp}(A') \preceq \mathsf{Supp}(A)$ and $\mathsf{Conc}(A) = \mathsf{Conc}(A')$, then A' $\mathscr{R}$-attacks B.
2. If A $\mathscr{R}$-attacks B and $\mathsf{Supp}(B) \preceq \mathsf{Supp}(B')$ and $\mathsf{Conc}(B) = \mathsf{Conc}(B')$, then A $\mathscr{R}$-attacks B'.

Note 4 The two conditions in Definition 11 resemble rules R_1 and R_2 (respectively) in [1, Definition 12], except that [1] refers only to the supports of the attacking and the attacked arguments, and uses only the subset relation. Also, in R_1 of [1] the condition on the supports are reversed (that is, R_1 refers to attacking super-arguments and R_2 refers to attacked super-arguments). In our case the two conditions assure, respectively, that attacks are closed under $\preceq$-stronger attacking rules and $\preceq$-weaker attacked rules.[13]

The proofs of the next lemmas are omitted due to lack of space.

[12] See [8].

[13] An argument A is a *super-argument* of B, if $\mathsf{Supp}(B) \subseteq \mathsf{Supp}(A)$. If $\mathsf{Supp}(B) \preceq \mathsf{Supp}(A)$ and $\mathsf{Conc}(B) = \mathsf{Conc}(A)$, we say that B is *stronger* than A (or that A is *weaker* than B).

Lemma 1 *Given a logical argumentation framework $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), Attack(\mathscr{A})\rangle$ and a support ordering $\preceq$ for $\mathcal{AF}(\mathsf{S})$ such that $\mathscr{A}$ satisfies Item 2 of Definition 11. If $\mathscr{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathsf{S}))$ then $\mathscr{E}^{\min}_{\preceq} = \min_{\preceq}(\mathscr{E})$.*

Let $\mathsf{F}_{\mathcal{AF}(\mathsf{S})} : \wp(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})) \to \wp(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}))$ be a function that relates every $\mathscr{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ with the set of arguments that are defended by $\mathscr{E}$ in $\mathcal{AF}(\mathsf{S})$. Again, when the context disambiguates we will skip the subscript. The next lemma is easily verified.

Lemma 2 *Given a logical argumentation framework $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), Attack(\mathscr{A})\rangle$, a support ordering $\preceq$ for $\mathcal{AF}(\mathsf{S})$ for which $\mathscr{A}$ is $\preceq$-normal, and a set $\mathscr{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$, it holds that:*

1. $\mathsf{F}(\mathscr{E}) \subseteq \mathsf{F}(\mathscr{E}^{\min}_{\preceq})$,
2. *if* $\mathscr{E}^{\min}_{\preceq} \subseteq \mathscr{E}$ *then* $\mathsf{F}(\mathscr{E}) = \mathsf{F}(\mathscr{E}^{\min}_{\preceq})$, *and*
3. *if* $\mathscr{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathsf{S}))$ *then* $\mathsf{F}(\mathscr{E}) = \mathsf{F}(\mathscr{E}^{\min}_{\preceq}) = \mathscr{E}$.

Lemma 3 *Let $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathrm{Attack}(\mathscr{A})\rangle$ be a logical argumentation framework and $\preceq$ a support ordering for $\mathcal{AF}(\mathsf{S})$. Suppose that $\mathscr{A}$ satisfies Item 2 of Definition 11. If $\mathscr{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ defends $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ then $\mathscr{E}$ defends any $B \in A^{\min}_{\preceq}$.*

The next proposition relates the extensions of a logical argumentation framework (with $\preceq$-normal set of attack rules) and the extensions of the corresponding framework, in which the arguments' supports are minimized.

Proposition 2 *Let $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), Attack(\mathscr{A})\rangle$ be a logical argumentation framework for S, $\preceq$ a support ordering for $\mathcal{AF}(\mathsf{S})$, and $\mathscr{A}$ a $\preceq$-normal set of attack rules. We denote:*

$$Attack^{\min}_{\preceq}(\mathscr{A}) = Attack(\mathscr{A}) \cap \big(\min_{\preceq}(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})) \times \min_{\preceq}(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}))\big),$$
$$\mathcal{AF}^{\min}_{\preceq}(\mathsf{S}) = \langle \min_{\preceq}(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})), Attack^{\min}_{\preceq}(\mathscr{A})\rangle.$$

For every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$ we have: $\mathscr{E}' \in \mathsf{Sem}(\mathcal{AF}^{\min}_{\preceq}(\mathsf{S}))$ iff there is $\mathscr{E} \in \mathsf{Sem}(\mathcal{AF}(\mathsf{S}))$ such that $\mathscr{E}' = \mathscr{E}^{\min}_{\preceq}$, iff there is $\mathscr{E} \in \mathsf{Sem}(\mathcal{AF}(\mathsf{S}))$ such that $\mathscr{E}' = \min_{\preceq}(\mathscr{E})$. Moreover, the extensions $\mathscr{E}$ in the second and the third conditions are the same for every $\mathscr{E}'$, namely $\mathscr{E} = \mathsf{F}_{\mathcal{AF}(\mathsf{S})}(\mathscr{E}')$.

Proof. Suppose that $\mathscr{E}' \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathsf{S}))$, and let $\mathscr{E} = \mathsf{F}_{\mathcal{AF}(\mathsf{S})}(\mathscr{E}')$ be the set of all arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ that are defended by $\mathscr{E}'$ in $\mathcal{AF}(\mathsf{S})$. We first show that $\mathscr{E}' \subseteq \mathscr{E}$. Suppose that some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ attacks $B \in \mathscr{E}'$. By the $\preceq$-normality of $\mathscr{A}$, any $A' \in A^{\min}_{\preceq}$ attacks B. Thus, there is a $C \in \mathscr{E}'$ that attacks A' and by the $\preceq$-normality of $\mathscr{A}$ it also attacks A. Thus, $\mathscr{E}'$ defends every $B \in \mathscr{E}'$ and thus $\mathscr{E}' \subseteq \mathscr{E}$. By Lemma 3, $\mathscr{E}' = \mathscr{E}^{\min}_{\preceq}$. By Item 2 of Lemma 2, $\mathsf{F}_{\mathcal{AF}(\mathsf{S})}(\mathscr{E}) = \mathsf{F}_{\mathcal{AF}(\mathsf{S})}(\mathscr{E}') = \mathscr{E}$.

We still have to show that $\mathscr{E}$ is conflict-free. Assume for a contradiction that there are $A, B \in \mathscr{E}$ for which A attacks B. Thus, any $A' \in A^{\min}_{\preceq}$ attacks B by the $\preceq$-normality of $\mathscr{A}$. However, since $\mathscr{E}'$ defends B there is a $C \in \mathscr{E}'$ that attacks A'. Since $A' \in \mathscr{E}'$, this contradicts the conflict-freeness of $\mathscr{E}'$.

Thus, $\mathscr{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathsf{S}))$. By Lemma 1, $\mathscr{E}^{\min}_{\preceq} = \min_{\preceq}(\mathscr{E}) = \mathscr{E}'$.

Suppose now that $\mathscr{E} \in \mathsf{Cmp}(\mathscr{AF}(\mathsf{S}))$. Consider $\mathscr{E}' = \min_{\preceq}(\mathscr{E})$. By Lemma 1, $\mathscr{E}' = \mathscr{E}^{\min}_{\preceq}$. Hence, $\mathscr{E}' \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}))^{\min}_{\preceq}$. Clearly, $\mathscr{E}'$ is conflict-free since $\mathscr{E}$ is conflict-free. By (Item 3 of) Lemma 2, $\mathsf{F}(\mathscr{E}) = \mathsf{F}(\mathscr{E}') = \mathscr{E}'$, and so $\mathsf{F}(\mathscr{E}')^{\min}_{\preceq} = \mathscr{E}'^{\min}_{\preceq} = \mathscr{E}'$. Thus, $\mathscr{E}' \in \mathsf{Cmp}(\mathscr{AF}^{\min}_{\preceq}(\mathsf{S}))$.

Since the grounded (respectively, preferred) semantics concerns $\subseteq$-minimal (respectively, $\subseteq$-maximal) complete extensions, the proof immediately generalizes for these semantics. The proof for the stable semantics is left to the reader. □

By Proposition 2 we get the following corollaries:

Corollary 3 *Let $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), Attack(\mathscr{A})\rangle$ be a logical argumentation framework for S, induced by a logic $\mathfrak{L}$, $\preceq$ a support ordering for $\mathscr{AF}(\mathsf{S})$, and $\mathscr{A}$ a set of $\preceq$-normal attack rules. Let also $\mathscr{AF}^{\min}_{\preceq}(\mathsf{S}) = \langle \min_{\preceq}(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})), Attack^{\min}_{\preceq}(\mathscr{A})\rangle$ be a logical argumentation framework as defined in Proposition 2. Then for every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$ it holds that $\mathsf{Sem}(\mathscr{AF}^{\min}_{\preceq}(\mathsf{S}))$ consists of the Sem-extensions in $\mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$, restricted to the elements in $(\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}))^{\min}_{\preceq}$, namely: $\mathscr{E}^{\min}_{\preceq} \in \mathsf{Sem}(\mathscr{AF}^{\min}_{\preceq}(\mathsf{S}))$ iff there is an extension $\mathscr{E} \in \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$ and $\mathscr{E}^{\min}_{\preceq} = \mathscr{E} \cap (\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}))^{\min}_{\preceq}$.*

Like the case of consistency preservation (cf. Corollary 2), we have:

Corollary 4 Let $\mathscr{AF}(\mathsf{S})$, and $\mathscr{AF}^{\min}_{\preceq}(\mathsf{S})$ be as in Proposition 2. Then for every $\circ \in \{\cup, \cap, \Cap\}$ and $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$ it holds that $\mathscr{AF}(\mathsf{S}) \mid\!\sim_{\circ\mathsf{Sem}} \psi$ iff $\mathscr{AF}^{\min}_{\preceq}(\mathsf{S}) \mid\!\sim_{\circ\mathsf{Sem}} \psi$.[14]

5. Attack Rules, Revisited

The previous sections show that the handling of inconsistency and minimality in logical argumentation frameworks may be shifted from arguments to the attack rules. Apart of the obvious advantage of a considerable simplification in the construction and the identification of valid arguments (and so, e.g., proof systems may be incorporated for building arguments from simpler arguments, or for searching for counterarguments given a certain argument; See [3]), we believe that representing these consideration is more appropriate in the rule-based level (Indeed, in real-life arguments are not always based on minimal evidence, avoiding inconsistency sometimes means lose of information, etc).

The use of attack rules for maintaining inconsistency and conflicts among arguments should be taken with care, though, especially when non-classical logics are used as the base logic of the framework. In this section we consider some cases in point.

A. Consistency Undercut Corollary 1 indicates that, among others, ConUcut may replace the support consistency requirement. However, in some base logics the use of ConUcut may not be appropriate or even meaningful. This may happen mainly due to the following reasons:

[14] Again, here we abuse a bit the notations in Definition 6 to emphasize how the argumentation frameworks are related.

- *Problems with the attacking arguments*: Consider, for instance, Kleene's 3-valued logic with the connectives $\neg, \wedge, \vee$ (and their usual 3-valued interpretations). This logic has no valid tautological arguments, because in Kleene's logic no formula follows from the emptyset. This means that Consistency Undercut is not applicable in such a logic.
- *Problems with the attacked arguments*: For instance, in Priest's 3-valued logic LP [17,18] with the connectives $\neg, \wedge, \vee$ every set is satisfiable, thus, again. the use of Consistency Undercut is questionable.

Dunn-Belnap's four-valued logic of first-degree entailment (FDE), combining Kleene's logic and LP, suffers from both problems, namely it does not have tautological arguments and every set is satisfiable. However, if the language of $\neg, \wedge, \vee$ is extended with a proper implication connective ($\supset$, see [2]), both tautological and contradictory (unsatisfiable) arguments may be introduced, in which case it makes sense to incorporate consistency undercut.

B. [Direct, Full] Defeat It may happen that certain attack rules need to be adjusted to specific base logics. We demonstrate this with the logics of formal (in)consistency (LFIs), mentioned in Note 2, and the [direct, full] defeat attack rules (see Table 1). According to these rules, the argument $\langle\{\neg\psi\}, \neg\psi\rangle$ should attack $\langle\{\psi\}, \psi\rangle$. However, for frameworks that are based on LFIs such an attack is more problematic, unless ψ is known to be consistent (i.e., $\circ\psi$ can be inferred).

In the presence of a propositional constant F for falsity, a reformulation of the attack condition of [Full] Defeat could be that $\psi_1, \mathsf{S}_2 \vdash \mathsf{F}$, as indicated in Table 2[15]

Rule Name	Acronym	Attacking	Attacked	Attack Condition
Inconsistency Defeat	IncDef	$\langle \mathsf{S}_1, \psi_1\rangle$	$\langle \mathsf{S}_2 \cup \mathsf{S}_2', \psi_2\rangle$	$\psi_1, \mathsf{S}_2 \vdash \mathsf{F}$
Inconsistency Full Defeat	IncFullDef	$\langle \mathsf{S}_1, \psi_1\rangle$	$\langle \mathsf{S}_2, \psi_2\rangle$	$\psi_1, \mathsf{S}_2 \vdash \mathsf{F}$
Inconsistency Direct Defeat	IncDirDef	$\langle \mathsf{S}_1, \psi_1\rangle$	$\langle \{\varphi\} \cup \mathsf{S}_2', \psi_2\rangle$	$\psi_1, \varphi \vdash \mathsf{F}$

Table 2. Attacks by defeat, revisited (again, we assume that supports of the attacked arguments are nonempty).

Note that the revised conditions in the rules of Table 2 avoid the use of conjunction and are suitable for LFI as well: While according to LFI $\langle\{\neg\psi\}, \neg\psi\rangle$ should *not* attack $\langle\{\psi\}, \psi\rangle$ (although $\neg\psi \vdash \neg\psi$), the argument $\langle\{\neg\psi\}, \neg\psi\rangle$ *can* be used for attacking, by inconsistency [full] defeat, the argument $\langle\{\circ\psi, \psi\}, \psi\rangle$, and the latter attack is perfectly justifiable in the context of any LFI, since the attacked argument is based on the assumption that not only its conclusion ψ holds, but it is also consistent.

One may think of several variations the rules in Table 2, following different intuitions. Below are some options:

Intuition 1: Attack based on a consistency assumption of the attacker.
In this case, e.g., $\langle\{\circ p, p\}, p\rangle$ should attack $\langle\neg p, \neg p\rangle$, but not vice versa.

[15] In logics with a conjunction and where the usual contraposition law holds, or when the negation is defined by $\neg\phi = \phi \supset \mathsf{F}$ for a deductive implication $\supset$, this reformulation is even equivalent to the original one.

Intuition 2: Attack based on a consistency conclusion of the attacker.
According to this intuition, $\langle\{\circ p, p\}, \circ p \wedge p\rangle$ attacks $\langle\neg p, \neg p\rangle$, but not vice versa. Here, $\langle\{\circ p, p\}, p\rangle$ should *not* attack $\langle\neg p, \neg p\rangle$.

Intuition 3: Attack based on a consistency assumption of the attacked argument.
This time $\langle\neg p, \neg p\rangle$ attacks $\langle\{\circ p, p\}, p\rangle$, but not vice versa.

The intuitions above may be captured by extending the conditions of the rules of Table 2. For instance, variations of inconsistency full defeat may be the following:

Variation for Intuition 1: $\langle \mathsf{S}_1, \psi_1\rangle$ attacks $\langle \mathsf{S}_2, \psi_2\rangle$ iff $\psi_1, \mathsf{S}_2 \vdash \mathsf{F}$ and $\mathsf{S}_1 \vdash \circ\bigwedge\mathsf{S}_1$.
Variation for Intuition 2: $\langle \mathsf{S}_1, \psi_1\rangle$ attacks $\langle \mathsf{S}_2, \psi_2\rangle$ iff $\psi_1, \mathsf{S}_2 \vdash \mathsf{F}$ and $\psi_1 \vdash \circ\psi_1$.
Variation for Intuition 3: $\langle \mathsf{S}_1, \psi_1\rangle$ attacks $\langle \mathsf{S}_2, \psi_2\rangle$ iff $\psi_1, \mathsf{S}_2 \vdash \mathsf{F}$ and $\mathsf{S}_2 \vdash \circ\bigwedge\mathsf{S}_2$.

The additional condition in each case above just expresses the consistency assumption of the corresponding intuition. In these conditions $\circ\psi$ is intuitively read by 'ψ is $\vdash$-consistent'. In LFI, $\circ$ is a primitive connective, while in other logics it may serve as a defined connective (e.g., $\neg(\psi \wedge \neg\psi)$).

Note 5 (should minimality be enforced?) The examples in this section provide another reason to avoid the minimality requirement in Definition 1: For instance, the support set of $A = \langle\{\psi, \circ\psi\}, \psi\rangle$ is *not* minimal, as indeed $\circ\psi$ is not necessary for the conclusion of the argument, but it *is* necessary for enabling the above attack variation for Intuition 1, of A on $B = \langle\{\neg\psi\}, \neg\psi\rangle$ (thus refuting the latter).[16]

C. [Direct, Full] Undercut and [Defeating] Rebuttal When the conditions in terms of negation are traded by consistency requirements, undercut rules coincide with the corresponding defeat rules. Regarding the rebuttal rules, conditions in the spirit of the previous section could be that the conclusions of the attacking and the attacked arguments are mutually inconsistent, that is: $\psi_1, \psi_2 \vdash \mathsf{F}$. Again, variations of the rules may involve extra conditions, expressing e.g. further consistency assumptions.

6. Conclusion and Further Work

We have shown that logical argumentation frameworks need not be artificially restricted to arguments with minimal supports and that inconsistent arguments may not be filtered out, even in cases that the underlying logic is not trivialized in the presence of inconsistency. Moreover, we have considered some cases in which the attack rules are not faithful to the consistent and/or minimized support assumption, and some reformulations in terms of related conditions are introduced.

The interplay between the nature of the underlying logic and the formulation of the attack rules has already been considered in the literature (see, e,.g., [11] and [19]). The rewriting of the attack rules in Section 5 imply that attacks may reflect considerations that are not encoded by the pure logical consequences depicted by the arguments. For instance, the reason for the attack according to Intuition 1 in Section 5 is not sufficiently

[16] According to this attack rule $\langle\{\neg\psi\}, \neg\psi\rangle$ is also attacked by $\langle\{\psi, \circ\psi\}, \psi \wedge \circ\psi\rangle$, which meets the minimality criterion, but the latter assumes the availability of a conjunction, while $\langle\{\psi, \circ\psi\}, \psi\rangle$ holds only by reflexivity and monotonicity.

explicated by the conclusion of the attacking argument, since the consistency constraint is not contained in it. Thus, a logical condition only in terms of entailments by the latter (as expressed by the defeat rules) won't be enough in this case. This brings up a new bunch of questions, such as if (and how) it is possible to reformulate specific attack rules to preserve basic properties, such as support minimization, without violating the intended argumentation semantics. This remains a topic for future work.

References

[1] Leila Amgoud. Postulates for logic-based argumentation systems. *International Journal of Approximate Reasoning*, 55(9):2028–2048, 2014.
[2] Ofer Arieli and Arnon Avron. The value of the four values. *Artificial Intelligence*, 102(1):97–141, 1998.
[3] Ofer Arieli and Christian Straßer. Sequent-based logical argumentation. *Argument & Computation*, 6(1):73–99, 2015.
[4] Pietro Baroni, Martin Caminada, and Massimiliano Giacomin. An introduction to argumentation semantics. *The Knowledge Engineering Review*, 26(4):365–410, 2011.
[5] Philippe Besnard and Anthony Hunter. A logic-based theory of deductive arguments. *Artificial Intelligence*, 128(1–2):203–235, 2001.
[6] Philippe Besnard and Anthony Hunter. Argumentation based on classical logic. In Guillermo Simari and Iyad Rahwan, editors, *Argumentation in Artificial Intelligence*, pages 133–152. Springer, 2009.
[7] Philippe Besnard and Anthony Hunter. A review of argumentation based on deductive arguments. In *Handbook of Formal Argumentation*, pages 437–484. College Publications, 2018.
[8] Gerhard Brewka. Preferred subtheories: An extended logical framework for default reasoning. In Natesa Sridharan, editor, *Proceedings of IJCAI'89*, pages 1043–1048. Morgan Kaufmann, 1989.
[9] Walter Carnielli, Marcelo Coniglio, and Joao Marcos. Logics of formal inconsistency. In *Handbook of Philosophical Logic*, volume 14, pages 1–95. Springer, 2007. Second edition.
[10] Walter Carnielli and Joao Marcos. A taxonomy of C-systems. In *Paraconsistency: The Logical Way to the Inconsistent*, Lecture Notes in Pure and Applied Mathematics, pages 1–94. Marcel Dekker, 2002.
[11] Esther Anna Corsi and Christian Fermüller. Logical argumentation principles, sequents, and nondeterministic matrices. In *Proceedings of LORI'17*, pages 422–437. Springer, 2017.
[12] Phan Minh Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321–357, 1995.
[13] Nikos Gorogiannis and Anthony Hunter. Instantiating abstract argumentation with classical logic arguments: Postulates and properties. *Artificial Intelligence*, 175(9–10):1479–1497, 2011.
[14] Diana Grooters and Henry Prakken. Two aspects of relevance in structured argumentation: Minimality and paraconsistency. *Journal of Artificial Inteligence Research*, 56:197–245, 2016.
[15] Jesse Heyninck and Ofer Arieli. Simple contrapositive assumption-based frameworks. *Journal of Approximate Reasoning*, 121:103–124, 2020.
[16] Sanjay Modgil and Henry Prakken. The ASPIC+ framework for structured argumentation: a tutorial. *Argument & Computation*, 5(1):31–62, 2014.
[17] Graham Priest. Logic of paradox. *Journal of Philosophical Logic*, 8(1):219–241, 1979.
[18] Graham Priest. Reasoning about truth. *Artificial Intelligence*, 39(2):231–244, 1989.
[19] Chenwei Shi, Sonja Smets, and Fernando R. Velázquez-Quesada. Beliefs supported by binary arguments. *Journal of Applied Non-Classical Logics*, 28(2–3):165–188, 2018.
[20] Christian Straßer and Ofer Arieli. Normative reasoning by sequent-based argumentation. *Journal of Logic and Computation*, 29(3):387–415, 2019.
[21] Alfred Tarski. *Introduction to Logic*. Oxford University Press, 1941.