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Minimality of Combined Qualitative Constraint Networks

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Abstract. We are interested in the minimality problem in the context of combinations of qualitative constraint networks (such as temporal sequences, multiscale networks, and loose integrations). For this, we generalize the minimality problem of the classical framework. This brings us to two distinct and complementary notions of minimality. We then study the complexity of the generalized minimality problem. In addition, we identify conditions ensuring that the algebraic closure computes the generalized minimal network. Based on this result, we prove that the topological temporal sequences of constant-size regions over the subclass $\mathcal{S}_{\mathrm{RCC}_8}$ check this property. This contrasts with the sequences of convex relations that do not verify it. Moreover, we study the complexity of the satisfiability decision of relations as well as the enumeration of satisfiable basic relations.

1 Introduction

Representation and reasoning with qualitative information is an essential ability of human intelligence. This capacity is notably formalized, studied, and applied in the context of natural language processing, constraint solving, geographic information systems, computer vision, autonomous robot navigation, intelligent environments, and human-computer interaction [30, 34, 41, 24]. Numerous studies have been carried out on the various reasoning tasks of the classical framework of qualitative formalisms [30, 10, 17]. These reasoning tasks, dealing with qualitative constraint networks, are among others the satisfiability decision [30], the redundancy decision and the prime networks computation [27, 43], networks merging [16, 15], the minimality decision [31] and the minimal network computation [33, 22, 1, 2, 3, 9, 45]. The problem of computing the minimal network is also called the minimal labeling problem or the deductive closure problem. It consists in deducing the maximum of information from a given qualitative constraint network. In some cases, the reasoning operator of qualitative formalisms, called *algebraic closure*, computes the minimal network. This property is interesting because it offers a more efficient calculation procedure than that of the general case. Conditions guaranteeing that the algebraic closure always calculates the minimal network have been identified [33].

Several extensions of qualitative formalisms have been proposed to increase their expressiveness (for reasoning on time *and* space [42, 48, 20, 13, 47, 51, 37, 38, 46, 19, 35, 6], reasoning with different levels of precision [11, 28, 7, 18, 5, 8], or reasoning about information from different qualitative formalisms [14, 23, 50, 21, 49, 36, 4, 26, 32, 25]). However, the vast majority of these extensions are not qualitative formalisms in the classical meaning. The different tasks of reasoning must thus be generalized to each of these extensions. Only the satisfiability problem has been studied, and only for some of these extensions [12, 13, 14, 42, 48, 23, 21, 20, 4, 6, 26].

Recently, a general framework, called the framework of *multi-algebras*, has been proposed [12]. It includes the extensions of the qualitative formalisms whose descriptions are sequences of constraint networks (each network being associated with a different temporality, a different precision or a different classical qualitative formalism). The satisfiability problem in this context has been studied [12]. However, again, none of the other reasoning tasks have been defined and studied. In this paper, we therefore propose to define and study, in the general context of multi-algebras, the problem of computing the minimal network.

We begin by recalling the basics of the classical framework of qualitative formalisms and the associated notion of minimality. We also recall the basics of the multi-algebra framework. We then study the *satisfiability problem of relations* and the *problem of enumerating the satisfiable basic relations*, in the framework of multi-algebras, which has not been done until now. These two problems are related to minimality, as we will see later. In Section 4, we extend the notion of minimality to the framework of multi-algebras. In the next section, we are interested in the generalization of the minimal network computation. We then determine, in Section 6, conditions ensuring that the algebraic closure computes the minimal network. We illustrate this result in Section 7 by identifying subclasses that verify this property. Finally, we conclude and expose the different perspectives.

2 Background

2.1 Classic Framework of Qualitative Formalisms

We begin by recalling the classical framework of qualitative formalisms [30, 10, 17]. A *qualitative formalism* (in the sense of Ligozat and Renz) is based on a finite *non-associative binary relation algebra*, namely a finite set of relations $\mathcal{A}$ associated with several reasoning operators on its relations: the union $\cup$, the intersection $\cap$, the converse $\bar{\cdot}$, and the *(abstract) composition* $\circ$, satisfying some properties. Reasoning operators are used to infer new relations: let $r, r' \in \mathcal{A}$ and x, y, z being variables, $x\ r\ y \implies y\ \overline{r}\ x$, $x\ r\ y \wedge x\ r'\ y \implies x\ (r \cap r')\ y$, and $x\ r\ y \wedge y\ r'\ z \implies x\ (r \circ r')\ z$. A qualitative formalism deals with a set of entities, generally infinite, the *universe* denoted $\mathcal{U}$. It also has an *interpretation function* φ associating with each abstract relation of $\mathcal{A}$ a relation over $\mathcal{U}$. One of the simplest examples of qualitative formalisms is the *point algebra* [44], denoted PA. The relations of PA are $<$, $=$, $>$, $\leq$, $\geq$, $\neq$, $\varnothing$, and $\mathcal{B}_{\mathrm{PA}}$. $\mathcal{B}_{\mathrm{PA}}$ is the *universal relation* (the relation union of all relations of PA). For example, the composition of the relations $\leq$ and $<$ is the relation $<$. The relations of PA are generally interpreted on the points of $\mathbb{R}$ ($\mathcal{U} = \mathbb{R}$), but they can also be interpreted on regions [21]. In

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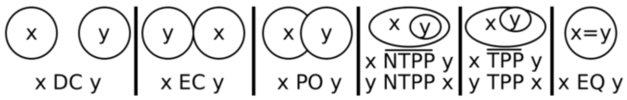

Figure 1. The 8 relations of RCC_8 in the plane.

that case, PA allows one to describe the relations between the sizes of regions (for example, the size of the region x is smaller than the size of the region y). Qualitative formalisms have special relations, called *basic relations*. From these relations and from the union, we can generate all the relations of $\mathcal{A}$. The basic relations of PA are $<, =$, and $>$. RCC_8 is another qualitative formalism [39, 29, 30]. It allows one to express the *topological* relations between regions ($\mathcal{U}$ is the set of regions). Its basic relations: disconnected (DC) ; externally connected (EC) ; partially overlapping (PO) ; tangential proper part (TPP) ; tangential proper part inverse ($\overline{\mathrm{TPP}}$) ; non-tangential proper part (NTPP) ; non-tangential proper part inverse ($\overline{\mathrm{NTPP}}$) ; equal (EQ) , are described in Figure 1. The set of basic relations of a formalism is denoted $\boldsymbol{\mathcal{B}}$.

Descriptions in the context of qualitative formalisms are *(qualitative constraint) networks*. A network is a set of variables $\mathtt{E}$ and a function N, associating with each pair of variables $(x, y) \in \mathtt{E}^2$ such that $x \neq y$ a relation of $\mathcal{A}$, which satisfies $N(x, y) = \overline{N(y, x)}$ for all distinct $x, y \in \mathtt{E}$. We denote $N(x, y)$ more succinctly by N^{xy}. A network is said to be *satisfiable* (or *consistent*) if there exists a *solution* to this network, that is, an assignment for the variables $\{u_x\}_{x \in \mathtt{E}} \subseteq \mathcal{U}$ satisfying the relations of the network, i.e., satisfying $(u_x, u_y) \in \varphi\left(N^{xy}\right)$ for all distinct $x, y \in \mathtt{E}$. A network is said *over* a subset of relations $\mathcal{S} \subseteq \mathcal{A}$ if for all distinct $x, y \in \mathtt{E}$, $N^{xy} \in \mathcal{S}$. A *scenario* is a network over $\boldsymbol{\mathcal{B}}$. A network N is *trivially unsatisfiable* if one of its relations N^{xy} satisfies $N^{xy} = \varnothing$.

The reasoning operator on networks is the *algebraic closure*, which applies the operators of the algebra. It propagates information within networks, i.e., makes inferences, by refining relations. A relation r *refines* a relation r' if $r \subseteq r'$. More generally, N *refines* N', denoted $N \subseteq N'$, if for all distinct $x, y \in \mathtt{E}$, $N^{xy} \subseteq (N')^{xy}$. The algebraic closure applies, until reaching a fixed point, the following operation: $N^{xz} \leftarrow N^{xz} \cap (N^{xy} \circ N^{yz})$ for all distinct $x, y, z \in \mathtt{E}$. The resulting network is said to be *algebraically closed*. An algebraically closed network which is not trivially unsatisfiable is said to be *algebraically consistent*.

Example 1. Consider the following network: $\mathtt{E} = \{x, y, z\}$, $N^{xy} = \mathrm{DC} \cup \mathrm{EC} \cup \mathrm{EQ}$, $N^{yz} = \mathrm{PO} \cup \mathrm{TPP}$, and $N^{xz} = \mathrm{NTPP} \cup \mathrm{EQ}$. Its algebraic closure, denoted $\mathfrak{C}(N)$, which is an algebraically consistent and satisfiable network, satisfies $\mathfrak{C}(N)^{xy} = \mathrm{DC} \cup \mathrm{EC}$, $\mathfrak{C}(N)^{yz} = \mathrm{PO} \cup \mathrm{TPP}$, and $\mathfrak{C}(N)^{xz} = \mathrm{NTPP}$.

By restricting networks to certain subsets of relations $\mathcal{S}$, we get the following property: if the algebraic closure of a network over $\mathcal{S}$ is algebraically consistent, then this network is satisfiable. Such subsets are said to be *algebraically tractable*. PA is fully algebraically tractable. The search for algebraically tractable subsets has focused on particular subsets [30]. A subset is called a *subclass* if it is closed under intersection, composition, and converse. Subclasses containing all basic relations are said *subalgebras*.

2.2 Minimality in the Classical Framework

One of the main problems of qualitative reasoning is the *computation of the minimal network* of a qualitative constraint network N. This consists in determining the smallest network $M \subseteq N$ having the same solutions as N. The network M contains the maximum of information that can be deduced (the number of basic relations contained in each relation is minimal). More formally:

Definition 2. A network M is *minimal* if for all distinct $x, y \in \mathtt{E}$ and all basic relations $b \subseteq M^{xy}$, there exists a satisfiable scenario $S \subseteq M$ such that $S^{xy} = b$.

Example 3. The network N of Ex. 1 is not minimal. However, its algebraic closure is minimal. It is actually the minimal network of N.

Computing the minimal network is polynomially equivalent to the satisfiability decision [2], which is generally an NP-hard problem [30]. To compute the minimal network, a method is the following algorithm: for all distinct variables $x, y \in \mathtt{E}$ and for each basic relation $b \subseteq N^{xy}$, we check that the network T, satisfying $T^{uv} = \begin{cases} b & \text{if } u = x \wedge v = y \\ N^{uv} & \text{otherwise} \end{cases}$ for $u, v \in \mathtt{E}$, is satisfiable. If this is not the case, we remove b from N^{xy}. The resulting network is the minimal network.

However, the minimal network can be computed with this algorithm in polynomial time when the network is over an algebraically tractable subalgebra. The complexity of the computation of the minimal network is then in $O\left(n^5\right)$ with n the number of variables.

Conditions on subalgebras ensuring that the algebraic closure directly calculates the minimal network have been determined [33]. In that case, the complexity of the minimal network computation is that of the algebraic closure (in $O\left(n^3\right)$ [30]). Such subclasses can be used to accelerate the computation of the minimal network in the general case [2]. In the following, we will say that a subset $\mathcal{S}$ is *(algebraically) minimizable* when the algebraic closure of any network over $\mathcal{S}$ is minimal. Two minimizable subclasses have been identified for RCC_8 : $\mathcal{D}^8_{41}$ and $\mathcal{D}^8_{64}$ [27, 33], also denoted $\mathcal{C}_{\mathrm{RCC}_8}$ and $\mathcal{S}_{\mathrm{RCC}_8}$ respectively. Two subclasses have also been identified for PA, denoted $\mathcal{C}_{\mathrm{PA}}$ and $\mathcal{S}_{\mathrm{PA}}$ [1]. In fact, any minimizable subalgebra of PA is included in one of these two subclasses.

2.3 Framework of Multi-algebras

We now recall the basics of the multi-algebra framework [12, 13]. Multi-algebras generalize non-associative binary relation algebras. A *multi-algebra* is a Cartesian product $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ of relation algebras satisfying some properties. We denote $\mathtt{I} = \{1, \ldots, m\}$, the index set of the multi-algebra. Each algebra $\mathcal{A}_i$ can correspond, for example, to a different algebra, to a different time period, and/or to a different precision. The *set of basic relations* of $\mathcal{A}$, denoted $\boldsymbol{\mathcal{B}}$, is $\boldsymbol{\mathcal{B}}_1 \times \cdots \times \boldsymbol{\mathcal{B}}_m$ where $\boldsymbol{\mathcal{B}}_i$ is the set of basic relations of $\mathcal{A}_i$. Multi-algebras are equipped with additional operators $\Rsh_i^j$ from $\mathcal{A}_i$ to $\mathcal{A}_j$ for distinct $i, j \in \mathtt{I}$, called *projections*. Any projection $\Rsh$ satisfies by definition $\Rsh(r \cup r') = (\Rsh r) \cup (\Rsh r')$ and $\Rsh(\overline{r}) = \overline{\Rsh(r)}$. Projections describe, for example, the evolution of relations during a change of qualitative formalism, temporality, or scale of precision. A *relation* of a multi-algebra $\mathcal{A}$ is an element of $\mathcal{A}$, i.e., an m-tuple of classical relations. By adding a semantics, namely a universe $\mathcal{U}$ and a specific interpretation function φ, we obtain a *loosely combined* qualitative formalism, also called *sequential (qualitative) formalism*.

Example 4. Temporalized topology of constant-size regions [13], that we denote $\mathrm{TRCC}_8^{\mathrm{s}}$, is a sequential formalism. It allows one to describe the evolution of the topological relations of constant-size moving regions evolving continuously over time ($\mathcal{U}$ is the set of continuous evolutions of constant-size regions). The Cartesian product of its multi-algebra is $(\mathrm{RCC}_8)^{m-1} \times \mathrm{PA}$, where $m-1$ is the length of the considered temporal sequences of relations. These sequences describe the relations at $m-1$ successive instants $t_1, \ldots, t_{m-1} \in \mathbb{R}$. The component $i \in \{1, \ldots, m-1\}$ of a relation R of $\mathrm{TRCC}_8^{\mathrm{s}}$, i.e., R_i, is the relation of RCC_8 which must be satisfied at the instant t_i. The component m of a relation R of $\mathrm{TRCC}_8^{\mathrm{s}}$, R_m, is the relation of PA, interpreted on the sizes of regions, which must be satisfied at all times. An example of relations of $\mathrm{TRCC}_8^{\mathrm{s}}$, with $m = 4$, is $(\mathrm{TPP} \cup \mathrm{NTPP}, \mathrm{PO} \cup \mathrm{EQ}, \mathrm{EC} \cup \mathrm{DC}, \leq)$. This relation means on the one hand that the first region is first included in the second (R_1 is satisfied at t_1), then they overlap or are equal (R_2 is satisfied at t_2) and finally they are disjoint (R_3 is satisfied at t_3). It means on the other hand that the size of the first is smaller or equal than that of the second (R_4 is satisfied at every moment). Its projections are the operators $\Rsh_i^j$ fully defined by $\Rsh_i^j b = \mathcal{B}_{\mathrm{RCC}_8}$, $\Rsh_i^m b = \Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} b$, and $\Rsh_m^i b' = \Rsh_{\mathrm{PA}}^{\mathrm{RCC}_8} b'$ with $b \in \boldsymbol{\mathcal{B}}_{\mathrm{RCC}_8}, b' \in \boldsymbol{\mathcal{B}}_{\mathrm{PA}}$, $i, j \in \{1, \ldots, m-1\}$, and where $\Rsh_{\mathrm{PA}}^{\mathrm{RCC}_8}$ and $\Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}}$ are defined in Table 1. For example, we have $\Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} \mathrm{TPP} = \{<\}$, because if one region is included tangentially in another, then this region has a size strictly inferior to the other.

b	TPP	NTPP	EQ	PO	EC	DC
$\Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} b$	$<$		$=$	$\mathcal{B}_{\mathrm{PA}}$		

b	$<$	$=$
$\Rsh_{\mathrm{PA}}^{\mathrm{RCC}_8} b$	DC∪EC∪PO∪TPP∪NTPP	DC∪EC∪PO∪EQ

Table 1. Correspondences between topological relations and size relations of regions

A description, in this context, is simply a network over a multi-algebra $\mathcal{A}$. A sequence of (classical) constraint networks is thus represented by a single constraint network whose relations are sequences of relations, i.e., relations of a multi-algebra. The majority of concepts have the same definitions as in the classical framework (satisfiability, solutions, scenarios, subclasses, subalgebras, network refinement $\subseteq$, to be algebraically consistent, to be algebraically tractable, ...). Most other concepts are generalized componentwise (composition, union, intersection, converse, relation refinement $\subseteq$). For example, a relation R *refines* a relation R' if for all $i \in \mathtt{I}$, $R_i \subseteq R'_i$. The composition of R and R', $R \circ R'$, is defined by $(R \circ R')_i = R_i \circ R'_i$ for all $i \in \mathtt{I}$. It is sometimes useful to refer to the " subnetwork " corresponding to the index $i \in \mathtt{I}$ of a network N, denoted N_i, called *slice*. N_i is defined by $(N_i)^{xy} = (N^{xy})_i$ for all distinct $x, y \in \mathtt{E}$. Similarly, the *slice* $i \in \mathtt{I}$ of a subset $\mathcal{S} \subseteq \mathcal{A}$, denoted $\mathcal{S}_i$, is $\{R_i \mid R \in \mathcal{S}\}$.

In the context of multi-algebras, there is a new operator: the *closure under projection* of a relation R, denoted $\Rsh(R)$. It is obtained by sequentially applying the following operation until reaching a fixed point: for all $j \in \mathtt{I}$, $R_j \leftarrow R_j \cap \bigcap_{i \neq j} \Rsh_i^j R_i$. The projection closure refines relations by removing classical basic relations that are impossible to satisfy. The projection closure of the relation $(\mathrm{TPP} \cup \mathrm{NTPP}, \mathrm{PO} \cup \mathrm{EQ}, \mathrm{EC} \cup \mathrm{DC}, \leq)$ is $(\mathrm{TPP} \cup \mathrm{NTPP}, \mathrm{PO}, \mathrm{EC} \cup \mathrm{DC}, <)$. Relations closed under projection, i.e., verifying $\Rsh(R) = R$, are said to be $\Rsh$*-closed*. A subset $\mathcal{S} \subseteq \mathcal{A}$ is said to be $\Rsh$*-closed* if for all $R \in \mathcal{S}$, $\Rsh(R) \in \mathcal{S}$.

The *algebraic closure* is generalized so as to close also by projection. It thus alternates closure under composition (classical algebraic closure) and projection closure (on each relation) until reaching a fixed point. A network N is said to be *algebraically closed* if it is closed under composition, i.e., for all distinct $x, y, z \in \mathtt{E}$, $N^{xz} \subseteq N^{xy} \circ N^{yz}$, and if each of its relations N^{xy} is closed under projection, i.e., for all distinct $i, j \in \mathtt{I}$, $N_j^{xy} \subseteq \Rsh_i^j N_i^{xy}$. A relation R is said to be *trivially unsatisfiable* if there exists $i \in \mathtt{I}$ such that $R_i = \varnothing$. Unlike the classical framework, a non-trivially unsatisfiable relation may be *unsatisfiable*, i.e., $\varphi(R) = \varnothing$. This is the case of $(\mathrm{TPP}, >)$. A relation is $\Rsh$*-consistent* if it is $\Rsh$-closed and not trivially unsatisfiable. A network is said to be *trivially unsatisfiable* if there exists distinct $x, y \in \mathtt{E}$ such that N^{xy} is trivially unsatisfiable.

Note that a list of conditions ensuring algebraic tractability has been identified [12]. One of these conditions is simplicity. A subset $\mathcal{S}$is *simple* if closing under projection then under composition makes algebraically consistent or trivially unsatisfiable all networks over $\mathcal{S}$.

3 Satisfiability and Enumeration of Relations

In this section, we are interested in the following two problems that we call the satisfiability decision of relations and the enumeration of satisfiable basic relations contained in a relation. For this, we focus on particular subsets of relations, that we call Cartesian. A subset $\mathcal{S} \subseteq \mathcal{A}$ is *Cartesian* if $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$.

The *satisfiability problem of relations* is, given a relation $R \in \mathcal{A}$, to decide if R is satisfiable, i.e., if $\varphi(R) \neq \varnothing$. The *problem of enumerating satisfiable basic relations* is, given a relation $R \in \mathcal{A}$, to determine all satisfiable basic relations B such that $B \subseteq R$. To solve the enumeration problem, we can use Algorithm 1. Its complexity is in $\mathcal{O}\left(bm\left(c + m^2\right)\right)$ with b the number of satisfiable basic relations, $m = |\mathtt{I}|$, and c the complexity of the satisfiability decision of relations (the complexity of the projection closure is $\mathcal{O}\left(m^2\right)$).

```
Data: R ∈ A
Result: L = {B ∈ 𝓑 | B ⊆ R ∧ φ(B) ≠ ∅}
Function enumerate(R, i ← 0, L ← ∅)
    E ← R
    foreach b ∈ 𝓑_i such that b ⊆ E_i do
        E_i ← b
        E ← ↱(E)
        if E is satisfiable then
            if i < m then
                enumerate(E, i + 1, L)
            else
                add E to L
    return L
```

Algorithm 1: The algorithm $\mathbf{enumerate}(R)$ enumerates the satisfiable basic relations of R.

On an algebraically tractable Cartesian subalgebra, the enumeration is in $\mathcal{O}\left(bm^3\right)$. Indeed, by definition, for these subclasses, a $\Rsh$-closed relation is satisfiable if and only if it is not trivially unsatisfiable. Finally, we have the following result.

Proposition 5. *The satisfiability decision of relations of the sequential formalisms whose* $\Rsh$*-closed basic relations are satisfiable is an* NP*-complete problem (with respect to* m*).*

Proof. We show that the satisfiability decision problem of quantitative binary constraint networks with finite domain (which is an NP-complete problem [40]) is polynomial-time reducible to the satisfiability decision problem of relations of the sequential formalisms whose $\Rsh$-closed basic relations are satisfiable (which is in NP).

Let N be a quantitative binary constraint network over a non-empty finite domain D and a set of variables $\mathtt{E} = \{x_i\}_{i=1}^{m}$. Each relation N^{xy} satisfies $N^{xy} \subseteq D^2$. We construct the following sequential formalism. Its Cartesian product is $\mathcal{A} = \left(2^D\right)^m$. Thus, $\boldsymbol{\mathcal{B}}_1 = \cdots = \boldsymbol{\mathcal{B}}_m = \{\{d\} \mid d \in D\}$. Its projections are defined by $\{d'\} \subseteq \Rsh_i^j \{d\}$ if and only if $(d, d') \in N^{x_i x_j}$ for $d, d' \in D$ and for distinct $i, j \in \mathtt{I}$. Its converse is defined by $\overline{b} = b$ for all $b \in \boldsymbol{\mathcal{B}}_i$ and $i \in \mathtt{I}$. The equality relation e is D. The composition is defined by $b \circ b' = \varnothing$ for all distinct $b, b' \in \boldsymbol{\mathcal{B}}_i$ and $b \circ b = b$ for all $b \in \boldsymbol{\mathcal{B}}_i$ with $i \in \mathtt{I}$. The union and the intersection on each $\mathcal{A}_i$ are the set operators. Its universe is $\mathcal{U} = \{B \in \boldsymbol{\mathcal{B}} \mid B \text{ is } \Rsh\text{-closed}\}$. Its interpretation function φ is the function defined by $\varphi(R) = \bigcup \{\varphi(B) \mid B \in \boldsymbol{\mathcal{B}} \wedge B \subseteq R\}$ with $\varphi(B) = \{(B, B)\}$ if B is basic and $\Rsh$-closed and with $\varphi(B) = \varnothing$ if B is basic and not $\Rsh$-closed. This is a sequential formalism (the proof is not detailed for space reasons). Basic $\Rsh$-closed relations are satisfiable. Consider the relation $R = (D, \ldots, D) = (\mathcal{B}_1, \ldots, \mathcal{B}_m) \in \mathcal{A}$. We have R is satisfiable if and only if N is satisfiable. Indeed, R is satisfiable if and only if it contains a $\Rsh$-closed basic relation $(\{d_1\}, \ldots, \{d_m\})$ if and only if N has $x_i = d_i$ with $i \in \mathtt{I}$ as solution. Finally, the sizes of R and the operator definitions are polynomial with respect to the size of N. □

4 Minimality in the Framework of Multi-algebras

In this section, we adapt the notion of minimality to the framework of multi-algebras. More precisely, we introduce two different notions of minimality for constraint networks, as well as a notion of minimality for relations.

4.1 Minimal Networks

There are two possibilities to generalize the notion of minimality in the context of multi-algebras. The first, *local minimality* characterizes the fact that every basic relation b of N_i^{xy} is *feasible*. The second, global minimality characterizes the fact that every satisfiable basic relation $B = (b_1, \ldots, b_m)$ of N^{xy} is *feasible*.

Definition 6. Let N be a network over $\mathcal{F}$ a sequential formalism.

- N is *locally minimal* if for all distinct $x, y \in \mathtt{E}$, all $i \in \mathtt{I}$, and all basic relations $b \subseteq N_i^{xy}$, there exists a satisfiable scenario $S \subseteq N$ such that $S_i^{xy} = b$.
- N is *globally minimal* if for all distinct $x, y \in \mathtt{E}$ and all satisfiable basic relations $B \subseteq N^{xy}$, there exists a satisfiable scenario $S \subseteq N$ such that $S^{xy} = B$.
- N is *totally minimal* if it is locally and globally minimal.

Remark 7. A network may be globally minimal without being locally minimal and vice versa.

Example 8. Let N be the following algebraically closed network: $x\,(\mathrm{DC} \cup \mathrm{EC}, \leq)\, y$, $y\,(\mathrm{EC}, =)\, z$, and $x\,(\mathrm{TPP} \cup \mathrm{EQ}, \leq)\, z$. Although N is locally minimal, it is not globally minimal. Indeed, there is no satisfiable scenario $S \subseteq N$ such that $S^{xy} = (\mathrm{DC}, =)$.

Example 9. Let N be the network satisfying $\mathtt{E} = \{x, y, z, w\}$ and:

- $N^{xy} = \left(\mathrm{PO} \cup \mathrm{TPP} \cup \overline{\mathrm{TPP}} \cup \mathrm{EQ}, =\right)$,
- $N^{yz} = \left(\overline{\mathrm{TPP}} \cup \mathrm{EQ}, \mathcal{B}_{\mathrm{PA}}\right)$,
- $N^{xz} = \left(\mathrm{PO} \cup \overline{\mathrm{TPP}}, \mathcal{B}_{\mathrm{PA}}\right)$,
- $N^{wx} = (\mathrm{TPP} \cup \mathrm{EQ}, \mathcal{B}_{\mathrm{PA}})$,
- $N^{wy} = (\mathrm{TPP} \cup \mathrm{EQ}, \mathcal{B}_{\mathrm{PA}})$,
- $N^{wz} = \left(\mathrm{PO} \cup \mathrm{TPP} \cup \overline{\mathrm{TPP}} \cup \mathrm{EQ}, =\right)$.

The algebraic closure of this network, $\mathfrak{C}(N)$, satisfies:

- $\mathfrak{C}(N)^{xy} = (\mathrm{PO} \cup \mathrm{EQ}, =)$,
- $\mathfrak{C}(N)^{yz} = \left(\overline{\mathrm{TPP}} \cup \mathrm{EQ}, \geq\right)$,
- $\mathfrak{C}(N)^{xz} = \left(\mathrm{PO} \cup \overline{\mathrm{TPP}}, \geq\right)$,
- $\mathfrak{C}(N)^{wx} = (\mathrm{TPP} \cup \mathrm{EQ}, \leq)$,
- $\mathfrak{C}(N)^{wy} = (\mathrm{TPP} \cup \mathrm{EQ}, \leq)$,
- $\mathfrak{C}(N)^{wz} = (\mathrm{PO} \cup \mathrm{EQ}, =)$.

$\mathfrak{C}(N)$ is neither locally nor globally minimal. Indeed, there is no satisfiable scenario $S \subseteq \mathfrak{C}(N)$ such that $S_1^{yz} = \mathrm{EQ}$ or $S^{yz} = (\mathrm{EQ}, =)$. To replace $\mathfrak{C}(N)^{yz}$ by $(\mathrm{EQ}, =)$ and to apply the algebraic closure gives a trivially unsatisfiable network.

4.2 Minimal Relations

Contrary to the classical framework, the minimality problem arises for relations.

Definition 10. A relation R is *minimal* if for all $i \in \mathtt{I}$ and all basic relations $b \subseteq R_i$, there exists a satisfiable basic relation $B \subseteq R$ such that $B_i = b$.

Example 11. The relation $R = (\mathrm{TPP} \cup \mathrm{EQ}, <)$ is not minimal. The relation $R' = (\mathrm{TPP}, <)$ is minimal.

Remark 12. Minimality of relations is a necessary condition for a network to be locally minimal. However, it is not a prerequisite for being globally minimal.

Example 13. Let N be the network over $\mathcal{C}_{\mathrm{RCC}_8} \times \mathcal{C}_{\mathrm{PA}}$ satisfying $\mathtt{E} = \{x, y, z, w\}$, $N^{xy} = (\mathrm{DC}, =)$, $N^{yz} = (\mathrm{TPP} \cup \mathrm{EQ}, \leq)$, $N^{xz} = (\mathrm{DC} \cup \mathrm{EC}, \leq)$, $N^{wx} = \left(\overline{\mathrm{TPP}} \cup \mathrm{EQ}, \geq\right)$, $N^{wy} = (\mathrm{DC} \cup \mathrm{EC}, \geq)$, and $N^{wz} = (\mathrm{EC}, =)$. N is algebraically consistent. All its relations are minimal. Its slices N_1 and N_2 are minimal. However, the network N is neither locally nor globally minimal. Indeed, there is no satisfiable scenario $S \subseteq N$ such that $S_1^{yz} = \mathrm{EQ}$ or $S^{yz} = (\mathrm{EQ}, =)$. To replace N^{yz} by $(\mathrm{EQ}, =)$ and to apply the algebraic closure gives a trivially unsatisfiable network.

5 Computing Minimal Networks

We are now interested in the means to *compute a globally (resp. locally) minimal network of a given network*, i.e., to determine an *equivalent* network (i.e., having the same solutions) satisfying the corresponding minimality property.

Definition 14. Let N and N' be two networks over a sequential formalism. The networks N and N' are *equivalent* if:

- for any satisfiable scenario $S \subseteq N$, we have $S \subseteq N'$,
- for any satisfiable scenario $S \subseteq N'$, we have $S \subseteq N$.

A network and its algebraic closure are always equivalent. The definition of equivalence for relations is defined in a similar way. Thus, the *computation of the minimal relation* of a relation R is the determination of the relation which is minimal and equivalent to R. In Example 11, R' is the minimal relation of R. Note that there is only one locally minimal network equivalent to a given network.

5.1 From Globally Minimal to Totally Minimal

We begin by focusing on the computation of the locally minimal network of a globally minimal network. If a network is globally minimal, we can compute its locally minimal network (which is in fact totally minimal) by making its relations minimal.

Proposition 15. *Let N be a globally minimal network.*

- *If the relations of N are minimal, then N is totally minimal.*
- *Any equivalent network $N' \subseteq N$ is globally minimal.*
- *Any equivalent network $N' \subseteq N$ whose relations are minimal is totally minimal.*

Proof. Let N be a globally minimal network whose relations are minimal. We show that it is totally minimal. Let $x, y \in \mathtt{E}$, $i \in \mathtt{I}$, and $b \subseteq N_i^{xy}$ a basic relation. Since N^{xy} is minimal, there exists a satisfiable basic relation $B \subseteq N^{xy}$ such that $B_i = b$. Since N is globally minimal, there exists a satisfiable scenario $S \subseteq N$ such that $S^{xy} = B$ and thus such that $S_i^{xy} = b$. Therefore, N is locally minimal and thus totally minimal.

We now show the second property. Let N be a globally minimal network and $N' \subseteq N$ an equivalent network. Let $x, y \in \mathtt{E}$ and $B \subseteq (N')^{xy}$ a satisfiable basic relation. Since $N' \subseteq N$, we have $B \subseteq N^{xy}$. Since N is globally minimal, there exists a satisfiable scenario $S \subseteq N$ such that $S^{xy} = B$. As N and N' are equivalent, $S \subseteq N'$. Thus, N' is globally minimal.

The third property derives from the two previous ones. □

Making the relations of a network minimal is quadratic with respect to the number of variables. To make a relation minimal, one can apply the following algorithm. For all $i \in \mathtt{I}$ and all basic relations $b \subseteq R_i$, if the relation R', defined by $R'_j = \begin{cases} b & \text{if } j = i \\ R_j & \text{otherwise} \end{cases}$ for $j \in \mathtt{I}$, is unsatisfiable, then remove b from R_i (i.e., $R_i \leftarrow R_i \setminus b$). However, making a relation minimal is generally an NP-complete problem with respect to $m = |\mathtt{I}|$ the size of the multi-algebra (remember that m is, for example, the number of precision scales, time periods, and/or combined classical qualitative formalisms that we consider). This is shown by the following proposition together with Proposition 5.

Proposition 16. *The computation problem of the minimal relation is linearly equivalent to the satisfiability decision problem of relations.*

Proof. Let R be a relation of a sequential formalism. We show on the one hand that one can decide if R is satisfiable by computing the minimal relation of R and by using a linear procedure with respect to m. Let M be the minimal relation of R. We check if M is trivially unsatisfiable (which is linear with respect to m). If so, then M and thus R are unsatisfiable. Otherwise, M and thus R are satisfiable (M and R contain the same satisfiable basic relations).

We show on the other hand that one can calculate the minimal relation of R by deciding the satisfiability of subrelations of R whose number is linear with respect to m. For this calculation, we apply the procedure described before this proposition. The satisfiability decision is applied at most $\sum_{i=1}^{m} |\boldsymbol{\mathcal{B}}_i|$ times. The relation computed by this procedure satisfies Definition 10 and contains the same satisfiable basic relations as R. It is thus the minimal relation of R. □

With algebraically tractable Cartesian subalgebras, we can decide if a relation is satisfiable by closing it under projection (a relation is satisfiable if its closure is $\Rsh$-consistent). For these subclasses, making locally minimal a globally minimal network is therefore of complexity $O(n^2 \cdot m^2)$.

5.2 From Locally Minimal to Totally Minimal

We have just seen how to make a globally minimal network totally minimal. How to make a locally minimal network totally minimal? Unfortunately, if a network is locally minimal but not globally minimal, there is no way to make it totally minimal (i.e. to compute an equivalent totally minimal network). Indeed, we cannot remove basic relations from a relation N_i^{xy} without losing satisfiable scenarios. In other words, for some networks, there are no equivalent globally minimal networks. The reason is that the expressiveness of multi-algebras is insufficient in some cases. For example, there is no relation of $\mathrm{RCC}_8 \times \mathrm{PA}$ containing $(\mathrm{EC}, =)$, $(\mathrm{EC}, <)$, and $(\mathrm{DC}, <)$ and not containing $(\mathrm{DC}, =)$. Thus, in the context of Example 8, we cannot express the relation between x and y by a relation of $\mathrm{RCC}_8 \times \mathrm{PA}$ in order to make N globally minimal.

5.3 Computing the Locally Minimal Network

We are now interested in computing the locally minimal network of any network. There are two ways to generalize the procedure for computing the minimal network of the classical framework. First, we can apply the following algorithm: for each $x, y \in \mathtt{E}$, each $i \in \mathtt{I}$ and each basic relation $b \subseteq N_i^{xy}$, if the network T, defined by $T_j^{uv} = \begin{cases} b & \text{if } j = i \wedge u = x \wedge v = y \\ N_j^{uv} & \text{otherwise} \end{cases}$ for distinct $u, v \in \mathtt{E}$ and $j \in \mathtt{I}$, is unsatisfiable, then remove b from N_i^{xy}. Thus, as in the classic setting, we have the following result.

Proposition 17. *The computation problem of the locally minimal network is polynomially equivalent to the satisfiability decision problem of networks.*

Proof. Let N be a network over a sequential formalism ($m = |\mathtt{I}|$, $n = |\mathtt{E}|$). On the one hand, the satisfiability of N can be decided by computing its locally minimal network M and by checking if M is trivially unsatisfiable (which is polynomial with respect to m and n).

We show on the other hand that one can compute the locally minimal network of N by deciding the satisfiability of " subnetworks " of N whose number is polynomial with respect to m and n. For this calculation, we apply the procedure described before this proposition. The satisfiability decision is applied at most $\frac{n(n-1)}{2} \sum_{i=1}^{m} |\boldsymbol{\mathcal{B}}_i|$ times. The network computed by this procedure satisfies Definition 6 and contains the same satisfiable scenarios as N. It is thus the locally minimal network of N. □

With algebraically tractable Cartesian subalgebras, satisfiability can be decided by applying the algebraic closure. With these subclasses, we can compute the locally minimal network by applying the algebraic closure about $\frac{n \cdot (n+1)}{2} \cdot m$ times, with n the number of variables and m the size of the multi-algebra. The complexity of this procedure is thus in $O(m^2 \cdot n^5 + m^3 \cdot n^4)$ (the complexity of the algebraic closure is in $O(m \cdot n^3 + m^2 \cdot n^2)$).

The second possible generalization for computing the locally minimal network of a network N, that we denote by M, is the following method. For each $x, y \in \mathtt{E}$ and each satisfiable basic relation $B \subseteq N^{xy}$, if the network T, defined by $T^{uv} = \begin{cases} B & \text{if } u = x \wedge v = y \\ N^{uv} & \text{otherwise} \end{cases}$ for $u, v \in \mathtt{E}$, is satisfiable, then add B to the relation M^{xy} of the network M, initially defined by $M_i^{xy} = \varnothing$ for all $x, y \in \mathtt{E}$ and $i \in \mathtt{I}$. With algebraically tractable subalgebras, this procedure is polynomial with respect to n (by using the algebraic closure to decide satisfiability).

The first technique seems more efficient. However, perhaps the second method performs better when the number of satisfiable basic relations within each relation N^{xy} is small (smaller than $\sum_i |N_i^{xy}|$?) and that the enumeration of these relations is fast (see Section 3).

Remark 18. If it is possible to make a network globally minimal, applying any of these two methods makes the network globally minimal and therefore totally minimal.

6 Cubic Subclasses for Minimality

In this section, we are interested in the subclasses on which the algebraic closure calculates the locally minimal (resp. globally minimal) network and in the conditions for obtaining these properties.

6.1 Algebraically Minimizable Subclasses

We formally define such subclasses, which we call *algebraically minimizable* subclasses.

Definition 19. Let $\mathcal{S}$ be a subset of a sequential formalism $\mathcal{F}$.

The set $\mathcal{S}$ is *(algebraically) locally minimizable* (for $\mathcal{F}$) if the algebraic closure of any network over $\mathcal{S}$ is locally minimal.

The set $\mathcal{S}$ is *(algebraically) globally minimizable* (for $\mathcal{F}$) if the algebraic closure of any network over $\mathcal{S}$ is globally minimal.

The set $\mathcal{S}$ is *(algebraically) totally minimizable* (for $\mathcal{F}$) if the algebraic closure of any network over $\mathcal{S}$ is totally minimal.

We now introduce our minimality result, which identifies conditions ensuring that a subclass is algebraically locally and globally minimizable. It describes in particular that a combination of minimizable classical subalgebras is totally minimizable when it is simple and that its $\Rsh$-closed relations are minimal.

Theorem 20. *Let $\mathcal{S}$ be a subalgebra of a sequential formalism satisfying the following conditions:*

- (C_0) *: algebraically closed scenarios are satisfiable ;*
- (M_1) *: each $\mathcal{S}_i$ is minimizable ;*
- (C_2) *: $\mathcal{S}$ is simple.*

We have the following implications:

- *If (M_3), i.e., all $\Rsh$-consistent relations of $\mathcal{S}$ is satisfiable, then algebraically closed networks over $\mathcal{S}$ are globally minimal.*
  - *If in addition $\mathcal{S}$ is $\Rsh$-closed, then $\mathcal{S}$ is globally minimizable.*
- *If (M_3'), i.e., all $\Rsh$-closed relations of $\mathcal{S}$ is minimal, then algebraically closed networks over $\mathcal{S}$ are totally minimal.*
  - *If in addition $\mathcal{S}$ is $\Rsh$-closed then $\mathcal{S}$ is totally minimizable.*

Proof. Let $\mathcal{S}$ be a subalgebra satisfying (C_0), (M_1), and (C_2).

We suppose (M_3) and show that all algebraically closed networks over $\mathcal{S}$ are globally minimal. Let N be an algebraically closed network over $\mathcal{S}$. If N is trivially unsatisfiable, then it is globally minimal. Suppose it is not trivially unsatisfiable. Let $x, y \in \mathtt{E}$ and $B \subseteq N^{xy}$ be a satisfiable basic relation. We show that there exists a satisfiable scenario $S \subseteq N$ such that $S^{xy} = B$. Let N' be the network satisfying $(N')^{uv} = \begin{cases} B & \text{if } u = x \wedge v = y \\ N^{uv} & \text{else} \end{cases}$ for all distinct $u, v \in \mathtt{E}$. Each of its relations is $\Rsh$-consistent. Let $i \in \mathtt{E}$. Since N_i is algebraically consistent and over $\mathcal{S}_i$ (a minimizable subalgebra), the composition closure of N_i' is not trivially unsatisfiable. Thus, the composition closure of N', $\circ N'$, is not trivially unsatisfiable. Since $\mathcal{S}$ is simple, $\circ \Rsh(N') = \circ N'$ is therefore algebraically consistent and is still over $\mathcal{S}$ ($\mathcal{S}$ is a subalgebra). Thus, N' is satisfiable, because $\mathcal{S}$ is algebraically tractable. Indeed, $\mathcal{S}$ satisfies the conditions of the Slicing theorem [12] ((M_1) and (M_3) are the conditions (C_1) and (C_3) with H the identity function as refinement). Therefore, there exists a satisfiable scenario $S \subseteq N' \subseteq N$ such that $S^{xy} = B$.

If in addition $\mathcal{S}$ is $\Rsh$-closed, the algebraic closure of any network over $\mathcal{S}$ is over $\mathcal{S}$ and is thus globally minimal. Therefore $\mathcal{S}$ is globally minimizable.

We now suppose (M_3') and show that all algebraically closed networks over $\mathcal{S}$ are totally minimal. Let N be an algebraically closed network over $\mathcal{S}$. (M_3') being a special case of (M_3), N is globally minimal. By (M_3'), its relations are minimal (since they are $\Rsh$-closed). By Proposition 15, N is totally minimal.

If in addition $\mathcal{S}$ is $\Rsh$-closed, the algebraic closure of any network over $\mathcal{S}$ is over $\mathcal{S}$ and is thus totally minimal. Therefore $\mathcal{S}$ is globally minimizable. □

6.2 Weakening and Minimality

In certain cases (as in the next section), some minimizable subclasses do not satisfy a part of the conditions of Theorem 20. In this section, we are interested in a way around this limitation. For this, we modify the projections of multi-algebras by weakening them [13] (which ensures that the modified projections do not produce incorrect inferences). The good weakening allows one, if it exists, to obtain the desired properties and thus to apply Theorem 20. To distinguish the projections from different sequential formalisms, like those of a formalism and those of one of its weakenings, we use the notations " ${}_{\mathcal{F}}\Rsh_i^j$ " and " ${}_{\mathcal{F}}\Rsh$ " (with $\mathcal{F}$ a sequential formalism). Similarly, we denote by $\mathfrak{C}_{\mathcal{F}}$ the algebraic closure of $\mathcal{F}$.

Definition 21. Let $\mathcal{F} = (\mathcal{A}, \mathcal{U}, \varphi)$ be a sequential formalism and $\mathcal{F}' = (\mathcal{A}', \mathcal{U}, \varphi)$ such that $\mathcal{A}'$ is a multi-algebra of the same Cartesian product as $\mathcal{A}$.

$\mathcal{F}'$ is called a *weakening* of the formalism $\mathcal{F}$ if for all distinct $i, j \in \mathtt{I}$ and all $b \in \boldsymbol{\mathcal{B}}_i$:

$$ {}_{\mathcal{F}}\Rsh_i^j\, b \subseteq {}_{\mathcal{F}'}\Rsh_i^j\, b. $$

Example 22. A weakening of TRCC_8^{s} (used in [13]) is obtained by replacing the projections $\Rsh_m^i$ which satisfy $\Rsh_m^i r = \Rsh_{\text{PA}}^{\text{RCC}_8} r$ by the projections $\Rsh_m^i$ satisfying $\Rsh_m^i r = \mathcal{B}_{\text{RCC}_8}$, for each $i \in \{1, \ldots, m-1\}$ (the other projections remain unchanged).

As the following proposition shows, considering a weakening can be used to obtain a minimizability property.

Proposition 23. *Let $\mathcal{F}'$ be a weakening of a sequential formalism $\mathcal{F}$ and $\mathcal{S}$ be a subset of $\mathcal{F}$.*

If $\mathcal{S}$ is globally minimizable for $\mathcal{F}'$ then $\mathcal{S}$ is globally minimizable for $\mathcal{F}$.

If in addition $\mathcal{S}$ is a ${}_{\mathcal{F}}\Rsh$-closed subclass and the ${}_{\mathcal{F}}\Rsh$-closed relations $R \in \mathcal{S}$ are minimal, then $\mathcal{S}$ is totally minimizable for $\mathcal{F}$.

Proof. We first show that if $\mathcal{S}$ is globally minimizable for $\mathcal{F}'$ then $\mathcal{S}$ is globally minimizable for $\mathcal{F}$. Let N be a network over $\mathcal{S}$. We show that $\mathfrak{C}_{\mathcal{F}}(N)$ is globally minimal, i.e., for all $x, y \in \mathtt{E}$ and all satisfiable basic relations $B \subseteq \mathfrak{C}_{\mathcal{F}}(N)^{xy}$, there exists a satisfiable scenario $S \subseteq \mathfrak{C}_{\mathcal{F}}(N)$ such that $S^{xy} = B$. Let $x, y \in \mathtt{E}$ and $B \subseteq \mathfrak{C}_{\mathcal{F}}(N)^{xy}$ be a satisfiable basic relation. Since $\mathfrak{C}_{\mathcal{F}}(N) \subseteq \mathfrak{C}_{\mathcal{F}'}(N)$, we have

$B \subseteq \mathfrak{C}_{\mathcal{F}'}(N)^{xy}$. As $\mathcal{S}$ is globally minimizable for $\mathcal{F}'$ by assumption, there exists a satisfiable scenario $S \subseteq \mathfrak{C}_{\mathcal{F}'}(N)$ such that $S^{xy} = B$. However, $\mathfrak{C}_{\mathcal{F}}(N)$ and $\mathfrak{C}_{\mathcal{F}'}(N)$ contain the same satisfiable scenarios (they are equivalent to N). Thus, $S \subseteq \mathfrak{C}_{\mathcal{F}}(N)$. Therefore, $\mathfrak{C}_{\mathcal{F}}(N)$ is globally minimal. In conclusion, $\mathcal{S}$ is globally minimizable for $\mathcal{F}$.

We now suppose in addition that $\mathcal{S}$ is a ${}_{\mathcal{F}}\!\Rsh$-closed subclass and that its ${}_{\mathcal{F}}\!\Rsh$-closed relations are minimal and we show that $\mathcal{S}$ is totally minimizable for $\mathcal{F}$. Let N be a network over $\mathcal{S}$. The network $\mathfrak{C}_{\mathcal{F}}(N)$ is globally minimal. The relations $\mathfrak{C}_{\mathcal{F}}(N)^{xy}$ belong to $\mathcal{S}$ ($\mathcal{S}$ is a ${}_{\mathcal{F}}\!\Rsh$-closed subclass) and are ${}_{\mathcal{F}}\!\Rsh$-closed. They are therefore minimal. By Proposition 15, $\mathfrak{C}_{\mathcal{F}}(N)$ is totally minimal. Thus, $\mathcal{S}$ is totally minimizable for $\mathcal{F}$. □

7 Illustration

We now illustrate our minimality result, by using notably the weakening technique. Specifically, we identify minimizable subclasses in the context of topological temporal sequences with size preservation (see Example 4). We begin by proving the hypotheses of the minimality theorem.

Proposition 24. *The $\Rsh$-closed relations of* $\mathrm{TRCC}_8^{\mathrm{s}}$ *are minimal.*

Proof. Let R be a $\Rsh$-closed relation of $\mathrm{TRCC}_8^{\mathrm{s}}$ ($R \in \mathrm{RCC}_8^{m-1} \times \mathrm{PA}$). We show that R is minimal. If R is trivially unsatisfiable, then R is minimal. We suppose that R is not trivially unsatisfiable. Let $i \in \mathtt{I}$ and $b \subseteq R_i$ be a basic relation. If $i = m$ ($b \in \mathrm{PA}$), then for all $j \in \{1, \ldots, m-1\}$, there exists a basic relation $b_j \subseteq R_j$ such that $b \subseteq \Rsh_j^m b_j = \Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} b_j$ and $b_j \subseteq \Rsh_m^j b$ (since R is $\Rsh$-consistent (we have $b \subseteq R_m \subseteq \Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} R_j$) and because any projection satisfies $\Rsh(r \cup r') = (\Rsh r) \cup (\Rsh r')$). Since $\Rsh_k^l r = \mathcal{B}_{\mathrm{RCC}_8}$ for all distinct $k, l \in \{1, \ldots, m-1\}$ and $r \in \mathrm{RCC}_8$, the basic relation $(b_1, \ldots, b_{m-1}, b)$, which is included in R, is $\Rsh$-consistent and therefore satisfiable [13]. If $i \in \{1, \ldots, m-1\}$, there exists a basic relation $b_{\mathrm{PA}} \subseteq R_m$ such that $b_{\mathrm{PA}} \subseteq \Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} b$ and $b \subseteq \Rsh_{\mathrm{PA}}^{\mathrm{RCC}_8} b_{\mathrm{PA}}$ (since R is $\Rsh$-consistent). For all $j \in \{1, \ldots, m-1\}$, distinct from i, there exists a basic relation $b_j \subseteq R_j$ such that $b_j \subseteq \Rsh_{\mathrm{PA}}^{\mathrm{RCC}_8} b_{\mathrm{PA}}$ and $b_{\mathrm{PA}} \subseteq \Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} b_j$ (since R is $\Rsh$-consistent). The basic relation $(b_1, \ldots, b_{i-1}, b, b_{i+1}, \ldots, b_{m-1}, b_{\mathrm{PA}})$, which is included in R, is $\Rsh$-consistent and therefore satisfiable [13]. Thus, R is minimal. □

Lemma 25. *We have the following two properties:*

- $\forall r \in \mathrm{PA}$, $\Rsh_{\mathrm{PA}}^{\mathrm{RCC}_8} r \in \mathcal{S}_{\mathrm{RCC}_8}$,
- $\forall r \in \mathcal{S}_{\mathrm{RCC}_8}$, $\Rsh_{\mathrm{RCC}_8}^{\mathrm{PA}} r \in \{<, =, >, \mathcal{B}_{\mathrm{PA}}\}$.

We now apply the minimality theorem, and we obtain the following result.

Proposition 26. *The subclasses $\mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{S}_{\mathrm{PA}}$ and $\mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{C}_{\mathrm{PA}}$ are totally minimizable.*

Proof. Each subalgebra $\mathcal{S} \in \{\mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{S}_{\mathrm{PA}}, \mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{C}_{\mathrm{PA}}\}$ satisfies the following properties for the weakening obtained by weakening only the projections from PA to RCC_8 which satisfies $\Rsh_m^i r = \mathcal{B}_{\mathrm{RCC}_8}$ for all $r \in \mathrm{PA}$ and $i \in \{1, \ldots, m-1\}$:

- (C_0) : all algebraically closed scenarios are satisfiable [13] ;
- (M_1) : each $\mathcal{S}_i$ is minimizable [33] ;
- (C_2) : $\mathcal{S}$ is simple ([13] for $\mathcal{S} = \mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{S}_{\mathrm{PA}}$, the proof is analogous for $\mathcal{S} = \mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{C}_{\mathrm{PA}}$) ;
- (M_3) : each $\Rsh$-consistent relation of $\mathcal{S}$ is satisfiable (see [13]) ;
- $\mathcal{S}$ is $\Rsh$-closed (by Lemma 25).

By Theorem 20, $\mathcal{S}$ is thus globally minimizable for this weakening.

We now consider these subclasses without weakening. The $\Rsh$-closed relations of $\mathcal{S}$ are minimal (Proposition 24) and $\mathcal{S}$ is $\Rsh$-closed (by Lemma 25). Thus, by Proposition 23, $\mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{S}_{\mathrm{PA}}$ and $\mathcal{S}_{\mathrm{RCC}_8}^{m-1} \times \mathcal{C}_{\mathrm{PA}}$ are totally minimizable. □

We thus have two cases of inheritance of *minimizability* by combination. What about other subclasses of $\mathrm{TRCC}_8^{\mathrm{s}}$? The question remains open, except for Cartesian subalgebras satisfying $\mathcal{C}_{\mathrm{RCC}_8} \subseteq \mathcal{S}_i$ for some $i \in \mathtt{I}$. These subclasses are not minimizable. We can build an algebraically closed network which is neither locally nor globally minimal, on the basis of Example 9. The combination of minimizable subclasses is therefore not always minimizable.

8 Conclusion

We have been interested in the following problems: the computation of the minimal network, the computation of the minimal relation, the satisfiability decision of relations and the enumeration of satisfiable basic relations, within the framework of sequential formalisms. For each of these problems, we have proposed algorithms and studied their complexities. In particular, we have shown that the satisfiability decision of relations is NP-complete (when $\Rsh$-closed basic relations are satisfiable) and that the computation problem is polynomially equivalent to the satisfiability problem (for networks and for relations).

To define minimality in the context of sequential formalisms, we have introduced two notions of minimality for networks (local minimality and global minimality) as well as minimality for relations. With regard to these definitions, we have shown that the minimality of relations joint to global minimality ensures local minimality.

Moreover, we have shown that the combination of minimizable subclasses is not necessarily minimizable (the case of convex relations in the context of $\mathrm{TRCC}_8^{\mathrm{s}}$). We have, however, identified conditions ensuring that such combinations are minimizable. This result has been applied to identify totally minimizable subclasses in the context of $\mathrm{TRCC}_8^{\mathrm{s}}$.

There are several directions for future work. On the one hand, the identification of the minimizable subclasses of $\mathrm{TRCC}_8^{\mathrm{s}}$ is to be completed (which requires the identification of all the minimizable subclasses of RCC_8). On the other hand, the minimality theorem opens the possibility of identifying minimizable subclasses in the context of multi-scale reasoning, loose integrations [50] and other temporal sequences [13]. Concerning the algorithmic aspect, an experimental comparison of the two alternative algorithms for computing the locally minimal network should be carried out. Finally, the different works on minimality of the classical framework [2, 3, 22, 45, 1] should be generalizable to the framework of sequential formalisms (in particular the use of minimizable subclasses to accelerate the computation of the minimal network [2]).

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