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On the Complexity of Constructive Control Under Nearly Single-Peaked Preferences

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Abstract. We investigate the complexity of CONSTRUCTIVE CONTROL BY ADDING/DELETING VOTES (CCAV/CCDV) for r-approval, Condorcet, Maximin, and Copeland$^\alpha$ in k-axes and k-candidates partition single-peaked elections. We prove that CCAV and CCDV for most of the voting correspondences mentioned above are NP-hard even if k is a very small constant. Exceptions are CCAV and CCDV for Condorcet and CCAV for r-approval in k-axes single-peaked elections, which we show to be fixed-parameter tractable with respect to k. Additionally, we give a polynomial-time algorithm for recognizing 2-axes elections, resolving an open problem.

1 Introduction

CONSTRUCTIVE CONTROL BY ADDING VOTES (CCAV) and CONSTRUCTIVE CONTROL BY DELETING VOTES (CCDV) are two of the election control problems studied in the pioneering paper by Bartholdi, Tovey, and Trick [4]. These two problems model the applications where an election controller aims to make a distinguished candidate a winner by adding or deleting a limited number of voters. Since their seminal work, the complexity of these problems under a lot of prestigious voting correspondences have been studied, and it turned out that many problems are NP-hard [4, 19, 21, 28]. However, when restricted to single-peaked elections, many of them become polynomial-time solvable [8, 22]. Recall that an election is single-peaked if there is an order of the candidates, the so-called (societal) axis, such that each voter's preference purely increases, or decreases, or first increases and then decreases along this order. A natural question is, as preferences of voters are extended from the single-peaked domain to the general domain with respect to a certain concept of nearly single-peakedness, where does the complexity of these problems change? A large body of results have been reported with respect to some nearly single-peaked domains. This paper aims to extend these study by investigating the complexity of the above two problems under several important voting correspondences when restricted to the k-axes and k-candidates partition single-peaked elections (k-axes and k-CP elections for short respectively). Generally speaking, an election is a k-axes election if there are k axes such that every vote is single-peaked with respect to at least one of the axes. An election is a k-CP election if there is a k-partition $(C_1, \ldots, C_k)$ of the candidates such that the subelection restricted to each C_i is single-peaked. Clearly, 1-axis elections and 1-CP elections are exactly single-peaked elections. The voting correspondences studied in this paper include r-approval, Condorcet, Maximin, and Copeland$^\alpha$, where α is a rational number such that $0 \leq \alpha \leq 1$.

Additionally, we also resolve an open question regarding the complexity of recognizing 2-axes elections by deriving a polynomial-time algorithm.

1.1 Related Work

Our study is clearly related to [8, 22] where many voting problems including particularly CCAV and CCDV for r-approval, Condorcet, Maximin, and Copeland[1] were shown to be polynomial-time solvable when restricted to single-peaked elections. Resolving an open question, Yang [39] recently proved that CCAV and CCDV for Borda are NP-hard even when restricted to single-peaked elections.

Our study is also related to the work of Yang and Guo [41, 42, 43] where CCAV and CCDV for numerous voting correspondences in elections of single-peaked width at most k and k-peaked elections were studied. Generally, an election has single-peaked width k if the candidates can be divided into groups, each of size at most k, such that every vote ranks all candidates in each group consecutively and, moreover, considering every group as a single candidate results in a single-peaked election. An election is k-peaked if there is an axis $\lhd$ such that for every vote π there is a k-partition of $\lhd$ such that π restricted to each component of the partition is single-peaked. Obviously, k-CP elections are a subclass of k-peaked elections. In addition, it is known that any election of single-peaked width k is a k'-CP election for some $k' \leq k$ [17]. However, there are no general relation between k-axes elections and k-CP elections, and between k-axes elections and elections with single-peaked width k [17].

In addition to CCAV and CCDV, many other problems restricted to single-peaked or nearly single-peaked domains have been extensively and intensively studied in the literature in the last decade (see, e.g., [6, 10, 11, 44] for WINNER DETERMINATION, [36] for POSSIBLE/NECESSARY WINNER DETERMINATION, [37] for MANIPULATION, [32] for BRIBERY, and [20, 38] for some other important strategic voting problems). Approval-based multiwinner voting problems restricted to analogs of single-peaked domains have also been investigated from the complexity perspective very recently [15, 29, 33]. Finally, we point out that a parallel line of research on the complexity of voting problems restricted to single-crossing and nearly single-crossing domains has advanced immensely too (see, e.g., [31, 35]). We also refer to the book chapters [16, 25] and references therein for important development on these studies.

A 3-page extended abstract of this paper appeared in the proceedings of AAMAS 2018 [40]. This version provides many proofs and resolves some open questions left in [40].

1.2 Our Contributions

- We study CCAV and CCDV in k-axes and k-CP elections under r-approval, Condorcet, Copeland$^\alpha$, and Maximin.

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- We show that many problems already become NP-hard even when k is a very small constant. However, there are several exceptions. (See Table 1 for the concrete results.) In addition, our results reveal that from the parameterized complexity point of view, CCAV and CCDV for some voting correspondences behave completely differently. For instance, for r-approval, CCAV in k-axes elections is fixed-parameter tractable (FPT) with respect to k, but CCDV is already NP-hard even for $k = 2$, meaning that CCDV restricted to k-axes elections is even para-NP-hard with respect to k. Our results also reveal that when restricted to different domains, the same problem may behave differently. For instance, for Condorcet, we show that both CCAV and CCDV in k-axes elections are FPT with respect to k, but they become para-NP-hard with respect to k when restricted to k-CP elections. Finally, we point out that our study also leads to numerous dichotomy results for CCAV and CCDV with respect to the values of k.
- We study the complexity of determining whether an election is a k-axes election. It is known that for $k = 1$, the problem is polynomial-time solvable [5, 12, 18]. Erdélyi, Lackner, and Pfandler [17] proved that the problem is NP-hard for every $k \geq 3$. We complement these results by showing that determining whether an election is a 2-axes election is polynomial-time solvable, filling the last complexity gap of the problem with respect to k.

2 Preliminaries

In this section, we give the notions used in the paper. For a positive integer i, we use $[i] = \{j \in \mathbb{N} : 1 \leq j \leq i\}$ to denote the set of all positive integers no greater than i.

Election. An *election* is a tuple $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$, where $\mathcal{C}$ is a set of *candidates* and $\Pi_\mathcal{V}$ a multiset of *votes*, defined as permutations (linear orders) over $\mathcal{C}$. For two candidates $c, c' \in \mathcal{C}$ and a vote $\pi \in \Pi_\mathcal{V}$, we say c is *ranked above* c' or π *prefers* c to c' if $\pi(c) < \pi(c')$. Here, $\pi(c)$ is the *position* of c in π, i.e., $\pi(c) = |\{c' \in \mathcal{C} : \pi(c') < \pi(c)\}| + 1$. For two subsets $X, Y \subseteq \mathcal{C}$ of candidates, a vote with preference $X \succ Y$ means that this vote prefers every $x \in X$ to every $y \in Y$. For brevity, we use $x \succ y$ for $\{x\} \succ \{y\}$. For $C \subseteq \mathcal{C}$ and a vote $\pi \in \Pi_\mathcal{V}$, let $\pi(C) = \{\pi(c) : c \in C\}$. In addition, let $\pi^C : C \to [|C|]$ be π restricted to C so that for $c, c' \in C$, $\pi(c) < \pi(c')$ implies $\pi^C(c) < \pi^C(c')$. Let $\Pi_\mathcal{V}^C = \{\pi^C : \pi \in \Pi_\mathcal{V}\}$. Hence, $(C, \Pi_\mathcal{V}^C)$ is the election $(\mathcal{C}, \Pi_\mathcal{V})$ restricted to C. We use $N_\mathcal{E}(c, c')$ to denote the number of votes preferring c to c' in $\mathcal{E}$. We drop $\mathcal{E}$ from the notation when it is clear from the context which election is considered. For two candidates c and c' in $\mathcal{C}$, we say c *beats* c' if $N(c, c') > N(c', c)$, and c *ties* c' if $N(c, c') = N(c', c)$.

Voting correspondence. A *voting correspondence* φ is a function that maps an election $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$ to a non-empty subset $\varphi(\mathcal{E})$ of $\mathcal{C}$. We call the elements in $\varphi(\mathcal{E})$ the *winners* of $\mathcal{E}$ with respect to φ. The following voting correspondences are related to our study.

r-Approval Each vote approves exactly its top-r candidates. Winners are those with the most total approvals. Throughout this paper, r is assumed to be a constant, unless stated otherwise.

Borda Every vote π gives $m - \pi(c)$ points to every candidate c and the winners are the ones with the highest total score. Here, m is the number of candidates.

Copeland$^\alpha$ ($0 \leq \alpha \leq 1$) For a candidate c, let $B(c)$ (resp. $T(c)$) be the set of candidates beaten by c (resp. tie with c). The Copeland$^\alpha$ score of c is $|B(c)| + \alpha \cdot |T(c)|$. A Copeland$^\alpha$ winner is a candidate with the highest score.

Maximin The Maximin score of a candidate $c \in \mathcal{C}$ is $\min_{c' \in \mathcal{C} \setminus \{c\}} N(c, c')$. Maximin winners are those with the highest Maximin score.

Condorcet The *Condorcet winner* of an election is the candidate that beats all other candidates. It is well-known that each election has either zero or exactly one Condorcet winner. In addition, both Copeland$^\alpha$ and Maximin select only the Condorcet winner if it exists. In this paper, the Condorcet correspondence refers to as the following one: if the Condorcet winner exists, it is the unique winner; otherwise, all candidates win.

Nearly single-peakedness. A vote π over $\mathcal{C}$ is single-peaked with respect to a linear order $\lhd$ of $\mathcal{C}$ if for every three candidates $a, b, c \in \mathcal{C}$ such that $a \lhd b \lhd c$ or $c \lhd b \lhd a$, it holds that $\pi(c) < \pi(b)$ implies $\pi(b) < \pi(a)$. An election $(\mathcal{C}, \Pi_\mathcal{V})$ is single-peaked with respect to a linear order $\lhd$ if all votes in $\Pi_\mathcal{V}$ are single-peaked with respect to $\lhd$. Such a linear order $\lhd$ is called an axis of $(\mathcal{C}, \Pi_\mathcal{V})$. Single-peaked elections were first studied by Black [7]. An election $(\mathcal{C}, \Pi_\mathcal{V})$ is k-axes single-peaked if there are k axes $\lhd_1, \ldots, \lhd_k$ such that every $\pi \in \Pi_\mathcal{V}$ is single-peaked with respect to at least one of $\{\lhd_1, \ldots, \lhd_k\}$. We call such a set $\{\lhd_1, \ldots, \lhd_k\}$ a k-axes witness of the election. In addition, $(\mathcal{C}, \Pi_\mathcal{V})$ is a k-CP election if there is a k-partition $(C_1, \ldots, C_k)$ of $\mathcal{C}$ such that for all $i \in [k]$, $(C_i, \Pi_\mathcal{V}^{C_i})$ is single-peaked.

Problem formulation. For a voting correspondence φ, we study the following two problems.

CONSTRUCTIVE CONTROL BY ADDING VOTES (CCAV)

Input: An election $(\mathcal{C}, \Pi_\mathcal{V})$, a distinguished candidate $p \in \mathcal{C}$, a multiset $\Pi_\mathcal{W}$ of votes over $\mathcal{C}$, and a positive integer ℓ.

Question: Is there $\Pi_W \subseteq \Pi_\mathcal{W}$ such that $|\Pi_W| \leq \ell$ and p uniquely wins $(\mathcal{C}, \Pi_\mathcal{V} \cup \Pi_W)$ with respect to φ?

In the above definition, votes in $\Pi_\mathcal{V}$ and $\Pi_\mathcal{W}$ are referred to as registered votes and unregistered votes, respectively.

CONSTRUCTIVE CONTROL BY DELETING VOTES (CCDV)

Input: An election $(\mathcal{C}, \Pi_\mathcal{V})$, a distinguished candidate $p \in \mathcal{C}$, and a positive integer ℓ.

Question: Is there $\Pi_V \subseteq \Pi_\mathcal{V}$ such that $|\Pi_V| \leq \ell$ and p uniquely wins the election $(\mathcal{C}, \Pi_\mathcal{V} \setminus \Pi_V)$ with respect to φ?

In this paper, we study CCAV and CCDV in k-CP (k-axes) elections. For CCAV, we mean that $(\mathcal{C}, \Pi_\mathcal{V} \cup \Pi_\mathcal{W})$ is a k-CP (k-axes) election. For NP-hardness results, we are only interested in the minimum values of k for which CCAV and CCDV in k-CP (k-axes) elections are NP-hard. It is easy to see that for voting correspondences considered in this paper, if CCAV and CCDV in k-CP (resp. k-axes) elections are NP-hard, so are they in $(k+1)$-CP (resp. $(k+1)$-axes) elections. Our hardness results are based on the following problem.

RESTRICTED EXACT COVER BY 3-SETS (RX3C)

Input: A universe $U = \{c_1, c_2, \ldots, c_{3\kappa}\}$ and a collection $S = \{s_1, s_2, \ldots, s_{3\kappa}\}$ of 3-subsets of U such that each $c_x \in U$ occurs in exactly three elements of S.

Question: Is there an $S' \subseteq S$ such that $|S'| = \kappa$ and each $c_x \in U$ appears in exactly one element of S'?

The RX3C problem is NP-hard [23].

We assume the reader is familiar with parameterized complexity [13, 14].

Table 1. A summary of the complexity of CCAV and CCDV. Here, "P" stands for "polynomial-time solvable" and "SP" stands for "single-peaked". Our results are in boldface. The numbers next to our results point to the corresponding theorems or corollary. The FPT results for Condorcet are with respect to k.

	CCAV					CCDV				
	SP	$(k \geq 2)$-axes	2-CP	$(k \geq 3)$-CP	general	SP	$(k \geq 2)$-axes	2-CP	$(k \geq 3)$-CP	general
r-approval	P [22]	**FPT** (1) w.r.t. $k+r$	P [42]	$r \leq 3$: P [27] $r \geq 4$: **NP-h** (2)	$r \leq 3$: P [22] $r \geq 4$: NP-h [27]	P [22]	$r \leq 2$: P [27] $r \geq 3$: **NP-h** (3)			$r \leq 2$: P [27] $r \geq 3$: NP-h [27]
Borda	NP-h [39]				NP-h [34]	NP-h [39]				NP-h [30]
Condorcet	P [8]	**FPT** (4)	open	**NP-h** (5)	NP-h [4]	P [8]	**FPT** (4)	open	**NP-h** (5)	NP-h [4]
Copeland$^{\alpha \in [0,1)}$	open	**NP-h** (6)	NP-h [41]		NP-h [21]	open	**NP-h** (6)	NP-h [41]		NP-h [21]
Copeland^1	P [8]	**NP-h** (6)	**NP-h** (7)		NP-h [21]	P [8]	**NP-h** (6)	**NP-h** (7)		NP-h [21]
Maximin	P [8]	**NP-h** (6)	**NP-h** (7)		NP-h [19]	P [8]	**NP-h** (6)	**NP-h** (7)		NP-h [19]

3 r-Approval

In general elections, CCAV and CCDV for r-approval are NP-hard even when r is a constant ($r \geq 4$ for CCAV and $r \geq 3$ for CCDV) [27, 28]. However, when restricted to single-peaked elections, both problems become polynomial-time solvable even when r is not a constant [22]. We complement these results by first showing that CCAV for r-approval in k-axes elections is FPT with respect to the combined parameter $k + r$. Our FPT-algorithm is based on the following two observations

Observation 1 *If a vote is single-peaked with respect to an axis $\lhd$, then all approved candidates in the vote lie consecutively in $\lhd$.*

Observation 2 *For every Yes-instance of the CCAV problem, every vote in an optimal solution approves the distinguished candidate.*

The above observations suggest that to solve an instance, we need only to focus on a limited number of candidates—the candidates at most "r far away" from the distinguished candidate in a k-axes witness of the given election.

Theorem 1 *CCAV for r-approval in k-axes elections is FPT with respect to the combined parameter $k + r$.*

PROOF. Let $((\mathcal{C}, \Pi_{\mathcal{V}}), p \in \mathcal{C}, \Pi_{\mathcal{W}}, \ell)$ be a CCAV instance, where $(\mathcal{C}, \Pi_{\mathcal{V}} \cup \Pi_{\mathcal{W}})$ is a k-axes election. For each $c \in \mathcal{C}$, let $\mathsf{sc}(c)$ be the score of c with respect to $\Pi_{\mathcal{V}}$, i.e., $\mathsf{sc}(c)$ is the number of votes in $\Pi_{\mathcal{V}}$ approving c. Let Π_p be the multiset of all votes $\pi \in \Pi_{\mathcal{W}}$ such that $\pi(p) \leq r$. For each vote $\pi \in \Pi_p$, let $C(\pi)$ be the set of candidates ranked in the top-r positions, i.e., $C(\pi) = \{c \in \mathcal{C} : \pi(c) \leq r\}$. Moreover, let $B = \bigcup_{\pi \in \Pi_p} C(\pi) \setminus \{p\}$. Due to Observation 2, every optimal solution is a subset of Π_p. Moreover, adding a vote in Π_p never prevents p from winning. Hence, if the given instance is a Yes-instance, there must be a feasible solution consisting of exactly $\min\{|\Pi_p|, \ell\}$ votes. We reset $\ell := \min\{|\Pi_p|, \ell\}$, and seek a feasible solution with ℓ votes in Π_p. Obviously, the final score of p is $\mathsf{sc}(p) + \ell$. If there is a candidate $c \in \mathcal{C}$ such that $\mathsf{sc}(c) \geq \mathsf{sc}(p) + \ell$, the given instance must be a NO-instance. Assume that this is not the case. The question is then whether there are ℓ votes in Π_p such that for every $c \in B$ at most $\mathsf{sc}(p) + \ell - \mathsf{sc}(c) - 1$ of these votes approve c. This can be solved in FPT time with respect to $|B|$. To this end, we give an integer linear programming (ILP) formulation with the number of variables being bounded by a function of $|B|$. For ease of exposition, we call every vote $\pi \in \Pi_p$ a β-vote where $\beta = C(\pi) \setminus \{p\}$. We create for each subset $\beta \subseteq B$ an integer variable x_β which indicates the number of β-votes that are included in the solution. The restrictions are as follows. Let n_β be the number of β-votes in Π_p. First, for each variable x_β, we require that $0 \leq x_\beta \leq n_\beta$. Second, the sum of all variables should be ℓ, i.e., $\sum_{\beta \subseteq B} x_\beta = \ell$. Third, for each $c \in B$, it must be that $\mathsf{sc}(c) + \sum_{c \in \beta} x_\beta \leq \mathsf{sc}(p) + \ell - 1$. By a result of Lenstra [26], this ILP can be solved in FPT time with respect to $|B|$. Due to Observation 1, B contains at most $k \cdot 2(r-1)$ candidates. The theorem follows. □

Note that the FPT-algorithm in the proof of Theorem 1 does not need any k-axes witness of the given election. What is important is that when the given election is k-axes single-peaked, the cardinality of the set B is bounded from above by $k \cdot 2(r-1)$. The framework in the proof does not apply to CCDV for r-approval in k-axes elections. The reason is that any optimal solution of CCDV contains only votes disapproving the distinguished candidate. Hence, we cannot only confine ourselves to a limited number of candidates.

Now we consider k-CP elections. Yang and Guo [42] developed a polynomial-time algorithm for CCAV for r-approval in 2-peaked elections. As 2-CP elections are a special case of 2-peaked elections, their polynomial-time algorithm directly applies to CCAV for r-approval in 2-CP elections. However, if k increases just by one, we show that the problem becomes NP-hard even for $r = 4$.

Theorem 2 *CCAV for r-approval in 3-CP elections is NP-hard for every $r \geq 4$.*

Now we turn our attention to CCDV. Yang and Guo [41] proved that CCDV for r-approval in 2-peaked elections is NP-hard even for $r = 3$. We strengthen their result by showing that the problem remains NP-hard even when restricted to elections that are both 2-axes single-peaked and 2-CP single-peaked. Our reduction is completely different from the one in [41]. In fact, to establish our result, we resort to a property of 3-regular bipartite graphs which has not been used in the proof of Yang and Guo [41]. The 3-regular bipartite graph in our reduction comes from the graph-representation of the RX3C problem. In general, this property says that for every 3-regular bipartite graph there are two linear orders over the vertices so that every edge of the graph is between two consecutive vertices in at least one of the two orders. We believe that this property is of independent interest. Recall that 3-regular graphs are those whose vertices are all of degree 3.

Lemma 1 *Let G be a 3-regular bipartite graph with vertex set $V(G)$ and edge set $E(G)$. Then, there are two linear orders $\lhd_1$ and $\lhd_2$ over $V(G)$ and a partition (A_1, A_2) of $E(G)$ such that for every $i \in \{1, 2\}$ and for every edge $(u, v) \in A_i$, it holds that u and v are consecutive in $\lhd_i$.*

The following lemma is also used in our reduction.

Lemma 2 *Let $\lhd$ be a linear order over $\mathcal{C}$ and let $C \subseteq \mathcal{C}$ be a subset of candidates that are consecutive in $\lhd$. Then we can construct a linear order π over $\mathcal{C}$ such that all candidates in C are ranked above all candidates not in C, and π is single-peaked with respect to $\lhd$.*

Now we are ready to unfold the NP-hardness of CCDV for r-approval in 2-axes and 2-CP single-peaked elections.

Theorem 3 *For every $r \geq 3$, CCDV for r-approval restricted to elections that are both 2-axes and 2-CP single-peaked is NP-hard.*

PROOF. Let $(U = \{c_1, \ldots, c_{3\kappa}\}, S = \{s_1, \ldots, s_{3\kappa}\})$ be an instance of RX3C. We create a CCDV instance with the following components. We only give the proof for the case where $r = 3$.

Candidates $\mathcal{C}$. We create in total $15\kappa + 5$ candidates. In particular, for each $c_x \in U$, we create two candidates c_x^1 and c_x^2. In addition, we create five candidates p, q_1, q_2, q_3, and q_4, where p is the distinguished candidate. Let $C_1 = \{c_x^i : i \in \{1,2\}, c_x \in U\} \cup \{p, q_1, q_2, q_3, q_4\}$. Finally, for each $s = \{c_x, c_y, c_z\} \in S$, we create three candidates $c_x(s)$, $c_y(s)$, and $c_z(s)$. Let C_2 be the set of all candidates corresponding to elements in S. Let $\mathcal{C} = C_1 \cup C_2$.

Votes $\Pi_\mathcal{V}$. We only specify here the approved candidates in each created vote, then after the correctness proof we utilize Lemma 2 to specify the linear preferences of all votes so that they are both 2-axes single-peaked and 2-CP single-peaked. First, we create one vote approving p, q_1, q_2 and one vote approving p, q_3, q_4. In addition, for each $s = \{c_x, c_y, c_z\} \in S$, we create four votes as follows:

- π_s approving $c_x(s)$, $c_y(s)$, $c_z(s)$;
- π_s^x approving $c_x(s)$, c_x^1, c_x^2;
- π_s^y approving $c_y(s)$, c_y^1, c_y^2; and
- π_s^z approving $c_z(s)$, c_z^1, c_z^2.

It is easy to verify that the winning set is $C_1 \setminus \{p, q_1, q_2, q_3, q_4\}$. Precisely, every winning candidate has score 3, p has score 2, every q_i where $i \in [4]$ has score 1, and every candidate in C_2 has score 2. Finally, we set $\ell = 7\kappa$, i.e., we delete at most 7κ votes.

The above construction clearly takes polynomial time. In the following, we prove the correctness of the reduction.

($\Rightarrow$) Assume that $S' \subseteq S$ is an exact set cover of U. Consider the election after deleting the following 7κ votes:

- all κ votes π_s such that $s \in S'$; and
- for each $s = \{c_x, c_y, c_z\} \notin S'$, all three votes π_s^x, π_s^y, and π_s^z.

Due to the construction and the fact that S' is an exact set cover, p has score 2 and every other candidate has score 1 after deleting these votes, implying that p becomes the unique winner.

($\Leftarrow$) Assume that Π_V is a subset of $\Pi_\mathcal{V}$ with minimal cardinality such that $|\Pi_V| \leq \ell = 7\kappa$ and p becomes the unique winner after deleting all votes in Π_V. Due to the minimality of Π_V, no vote in Π_V approves p. Hence, p has score 2 after deleting all votes in Π_V. Let $\Pi_S = \{\pi_s : s \in S\}$ and $\Pi^U = \{\pi_s^x : s \in S, c_x \in s\}$. For every $\pi_s \notin \Pi_V \cap \Pi_S$, all three votes π_s^x, π_s^y, π_s^z, where $s = \{c_x, c_y, c_z\}$, must be included in Π_V, since otherwise one of $c_x(s)$, $c_y(s)$, and $c_z(s)$ would have score 2 after deleting all votes in Π_V. Let $t = |\Pi_V \cap \Pi_S|$. It follows from the above analysis that $|\Pi_V| \geq t + 3(3\kappa - t) = 9\kappa - 2t$, implying that $t \geq \kappa$. On the other hand, as there are 6κ candidates in $C_1 \setminus \{p, q_1, q_2, q_3, q_4\}$ and every vote in Π^U approves two of these candidates, to decrease their scores to at most 1, we need to delete at least $9\kappa - 3\kappa = 6\kappa$ votes, i.e., $|\Pi_V \cap \Pi^U| \geq 6\kappa$. This directly implies that $t = \kappa$ and $|\Pi_V| = 7\kappa$. Let $S' = \{s \in S : \pi_s \in \Pi_V\}$. Clearly, $|S'| = t = \kappa$. Due to the above analysis, for every $\pi_s \in \Pi_V$, none of π_s^x, π_s^y, and π_s^z is in Π_V, since otherwise there would be more than 7κ votes in Π_V. As a result, if there are two $s, s' \in S'$ which contain a common element $c_x \in U$, then c_x^1 (and c_x^2) would have score at least 2 after the deletion of all votes in Π_V, contradicting that p is the unique winner. So, the 3-subsets in S' must be pairwise disjoint, implying that S' is an exact set cover of U.

Finally, we show that the election constructed above is both a 2-axes election and a 2-CP election. We first show that it is 2-axes single-peaked. To this end, we show that there exist two axes $\lhd_1$ and $\lhd_2$ over $\mathcal{C}$ such that for every vote constructed above, the approved candidates in the vote are consecutive in at least one of $\lhd_1$ and $\lhd_2$. Note that the RX3C instance (U, S) can be represented by a 3-regular bipartite graph with vertex-partition (U, S). In addition, there is an edge between some $c \in U$ and $s \in S$ if and only if $c \in s$. Due to Lemma 1, there are two linear orders $\lhd_1'$ and $\lhd_2'$ over $U \cup S$ such that for every edge (c, s) in the graph where $c \in s \in S$ the two vertices c and s are consecutive in one of these two orders. We first construct a linear order $\lhd_1^*$ (resp. $\lhd_2^*$) over $\mathcal{C}$ based on $\lhd_1'$ (resp. $\lhd_2'$). First, we let $\lhd_1^*$ (resp. $\lhd_2^*$) be a copy of $\lhd_1'$ (resp. $\lhd_2'$) and then we do the following modification.

- For each $c_x \in U$ where $x \in [3\kappa]$, we replace c_x with the two candidates c_x^1 and c_x^2 corresponding to c_x in $\lhd_1^*$ (resp. $\lhd_2^*$). The relative order between c_x^1 and c_x^2 in $\lhd_1^*$ (resp. $\lhd_2^*$) does not matter.
- For each $s = \{c_x, c_y, c_z\} \in S$ where $\{x, y, z\} \subseteq [3\kappa]$, we replace s in $\lhd_1^*$ (resp. $\lhd_2^*$) with the three candidates $c_x(s)$, $c_y(s)$, and $c_z(s)$ created for the element s. The relative order among these three candidates is determined as follows. If s is not the first element in $\lhd_1'$ (resp. $\lhd_2'$), let c_i, $i \in [3\kappa]$, be the element ordered immediately before s in $\lhd_1'$ (resp. $\lhd_2'$), i.e., c_i and s are consecutive in $\lhd_1'$ (resp. $\lhd_2'$) and $c_i \lhd_1' s$ (resp. $c_i \lhd_2' s$). If $i \in \{x, y, z\}$, we require that $c_i(s)$ is ordered before everyone in $\{c_x(s), c_y(s), c_z(s)\} \setminus \{c_i(s)\}$ so that the three candidates c_i^1, c_i^2, and $c_i(s)$ are consecutive. Symmetrically, if s is not the last element in $\lhd_1'$ (resp. $\lhd_2'$), and c_j denotes the element ordered immediately after s in $\lhd_1'$ (resp. $\lhd_2'$) we have the following requirement: if $j \in \{x, y, z\}$, we require that $c_j(s)$ is the last one among $c_x(s)$, $c_y(s)$, and $c_z(s)$, so that the three candidates c_j^1, c_j^2, and $c_j(s)$ are consecutive. We order $c_x(s)$, $c_y(s)$, and $c_z(s)$ so that the above requirements are fulfilled.

Given the final $\lhd_1^*$ and $\lhd_2^*$, let $\lhd_1 = (q_1, q_2, p, q_3, q_4, \lhd_1^*)$ and $\lhd_2 = (q_1, q_2, p, q_3, q_4, \lhd_2^*)$. Clearly, the three candidates p, q_1, and q_2 are consecutive in both $\lhd_1$ and $\lhd_2$, and the three candidates p, q_3, and q_4 are consecutive in both $\lhd_1$ and $\lhd_2$ too. Due to Lemma 2, we can complete the linear order of the vote approving exactly p, q_1, and q_2 (resp. p, q_3, and q_4) so that it is single-peaked with respect to $\lhd_1$ and, moreover, p, q_1, and q_2 (resp. p, q_3, and q_4) are the top-3 candidates. Let $s \in S$ be a 3-subset in S. In $\lhd_1$ and $\lhd_2$, all the three candidates created for s are consecutive. Let $c_x \in s$ be an element in s and let us consider the vote π_s^x which approves $c_x(s)$, c_x^1, and c_x^2. Clearly, (c_x, s) is an edge in the above mentioned 3-regular graph. Then, due to Lemma 1, c_x and s are consecutive in at least one of the original orders $\lhd_1'$ and $\lhd_2'$, say, without loss of generality, $\lhd_1'$. Then due to the definition of $\lhd_1^*$, the three candidates c_x^1, c_x^2, and $c_x(s)$ are consecutive in $\lhd_1^*$. Therefore, all the three votes created for s can be completed into linear-order votes which are single-peaked with respect at least one of $\lhd_1$ and $\lhd_2$, and whose top-3 candidates are exactly those that are approved in these votes. This completes the proof that the constructed election is a 2-axes election with $\{\lhd_1, \lhd_2\}$ being the witness.

Now, we show that the above election is also a 2-CP election. To this end, it suffices to show that $(C_i, \Pi_{\mathcal{V}}^{C_i})$ is single-peaked for each $i \in [2]$. Let $\lhd_1$ be an order of C_1 such that for every $c_x, x \in [3\kappa-1]$, the two candidates corresponding to c_x are ordered before the two candidates corresponding to c_{x+1}. Moreover, for each $c_x \in U$, c_x^1 is ordered before c_x^2. Furthermore, the candidates p, q_1, q_2, q_3, and q_4 are ordered after all the other candidates in C_1 and they are ordered as (q_1, q_2, p, q_3, q_4). Let $\lhd_2$ be an order of C_2 such that for every $s \in S$, the three candidates corresponding to s are ordered consecutively (the relative order among them does not matter). Clearly, for each $i \in [2]$ the approved candidates restricted to C_i in every vote lie consecutively in $\lhd_i$. In this case, we can specify the preferences so that each vote restricted to C_i is single-peaked with respect to $\lhd_i$ as follows. Let π be a vote. Let A_1 and A_2 be the sets of approved candidates of π included in C_1 and in C_2, respectively. Due to Lemma 2, we can specify the preference $\succ_1$ (reap. $\succ_2$) of π restricted to C_1 (resp. C_2) so that A_1 (resp. A_2) are ranked consecutively above all the other candidates and the preference is single peaked with respect to $\lhd_1$ (resp. $\lhd_2$). Then, we define the preference of π over the whole set of candidates as $A_1 \succ A_2 \succ C_1 \setminus A_1 \succ C_2 \setminus A_2$, where the preferences among candidates in A_1, and among candidates in $C_1 \setminus A_1$ are specified by $\succ_1$, and the preferences among candidates in A_2, and among candidates in $C_2 \setminus A_2$ are specified by $\succ_2$. This preference is 2-CP single-peaked with respect to the partition (C_1, C_2). □

4 Condorcet Consistent Voting

In this section, we study CCAV and CCDV for several Condorcet consistent voting correspondences, i.e., voting correspondences which select exactly the Condorcet winner whenever it exists. We first show some FPT results for Condorcet. Our results rely on an FPT-algorithm for the MIXED INTEGER PROGRAMMING WITH SIMPLE PIECEWISE LINEAR TRANSFORMATIONS problem (MIPWSPLT). This problem is a generalization of integer linear programming (ILP) with the entries of the input matrix being replaced with piecewise linear convex or concave functions. Bredereck et al. [9] recently proved that MIPWSPLT is FPT with respect to the number of variables. To establish our FPT result, we need only a special case of the MIPWSPLT problem which is defined as follows.

INTEGER PROGRAMMING WITH SIMPLE PIECEWISE LINEAR TRANSFORMATIONS (IPWSPLT)

Input: A collection of $s \cdot t$ piecewise linear concave functions $\{f_{i,j} : i \in [s], j \in [t]\}$, and a vector $b \in \mathbb{Z}^s$.

Question: Is there a vector $x = \langle x_1, x_2, \ldots, x_t \rangle$ of t integers such that for every $i \in [s]$, it holds that

$$\sum_{j=1}^{t} f_{i,j}(x_j) \leq b_i? \qquad (1)$$

Lemma 3 ([9]) *IPWSPLT is solvable in $O^*(t^{2.5t+o(t)})$ time.*

Note that the result in [9] holds for the variant of IPWSPLT where the less than sign is replaced with the greater than sign or the equal sign in (1).

For a vote π over a set $\mathcal{C}$ of candidates and a candidate $c \in \mathcal{C}$, let $\mathsf{Ab}(\pi, c)$ (resp. $\mathsf{Be}(\pi, c)$) be the set of all candidates ranked above (resp. below) c in π, i.e., $\mathsf{Ab}(\pi, c) = \{c' \in \mathcal{C} : \pi(c') < \pi(c)\}$ (resp. $\mathsf{Be}(\pi, c) = \{c' \in \mathcal{C} : \pi(c') > \pi(c)\}$).

Theorem 4 *CCAV and CCDV for Condorcet in k-axes elections are FPT with respect to k when a k-axes witness of the given election is provided.*

PROOF. [CCDV] We prove the theorem by giving an IPWSPLT formulation with the number of variables being bounded by a function of k. Let $((\mathcal{C}, \Pi_{\mathcal{V}}), p \in \mathcal{C}, \ell)$ be a given CCDV instance where $(\mathcal{C}, \Pi_{\mathcal{V}})$ is a k-axes election with respect to k-axes $\lhd_1, \ldots, \lhd_k$. We solve the instance as follows. Let $\Pi_{\mathcal{V}_1}, \ldots, \Pi_{\mathcal{V}_k}$ be a partition of $\Pi_{\mathcal{V}}$ such that for every $i \in [k]$, all votes in $\Pi_{\mathcal{V}_i}$ are single-peaked with respect to $\lhd_i$. Observe that for each $\pi \in \Pi_{\mathcal{V}_i}$, $i \in [k]$, all candidates ranked above the distinguished candidate p are consecutively in $\lhd_i$. Moreover, either all of them lie on the left-side of p or all of them lie on the right-side of p in $\lhd_i$. For each $\Pi_{\mathcal{V}_i}$, $i \in [k]$, let $\Pi_{\mathcal{V}_i}^L$ (resp. $\Pi_{\mathcal{V}_i}^R$) be the multiset of all votes in $\Pi_{\mathcal{V}_i}$ where all candidates ranked above p lie on the left-side (resp. right-side) of p in $\lhd_i$. Precisely,

$$\Pi_{\mathcal{V}_i}^L = \{\pi \in \Pi_{\mathcal{V}_i} : \forall (c \in \mathcal{C} \setminus \{p\}, \pi(c) < \pi(p))[c \lhd_i p]\}$$

and

$$\Pi_{\mathcal{V}_i}^R = \{\pi \in \Pi_{\mathcal{V}_i} : \forall (c \in \mathcal{C} \setminus \{p\}, \pi(c) < \pi(p))[p \lhd_i c]\} .$$

For each $i \in [k]$ and $X \in \{L, R\}$, let $t_i^X = |\Pi_{\mathcal{V}_i}^X|$, and if $t_i^X > 0$ let $(\pi_{(i,1)}^X, \pi_{(i,2)}^X, \ldots, \pi_{(i,t_i^X)}^X)$ be an order over $\Pi_{\mathcal{V}_i}^X$ such that $\pi_{(i,x)}^X(p) \geq \pi_{(i,x+1)}^X(p)$ for all $x \in [t_i^X - 1]$. An observation is that for every $x \in [t_i^X - 1]$, $\mathsf{Ab}(\pi_{(i,x+1)}^X, p) \subseteq \mathsf{Ab}(\pi_{(i,x)}^X, p)$, i.e., the candidates ranked above p in $\pi_{(i,x+1)}^X$ are also ranked above p in $\pi_{(i,x)}^X$. This implies that there is an optimal solution such that for each $i \in [k]$ and $X \in \{L, R\}$ such that $t_i^X > 0$, this solution includes either none of $\Pi_{\mathcal{V}_i}^X$, or it includes all votes $\pi_{(i,x)}^X$ such that $x \in [y]$ for some positive integer $y \leq t_i^X$ and excludes all the other votes in $\Pi_{\mathcal{V}_i}^X$. Based on the observation, we create an instance of IPWSPLT as follows. We create in total $2k$ variables. In particular, for each axis $\lhd_i$, $i \in [k]$, we create two integer variables denoted by x_i^L and x_i^R, where x_i^L (resp. x_i^R) indicates how many votes in $\Pi_{\mathcal{V}_i}^L$ (resp. $\Pi_{\mathcal{V}_i}^R$) are included in the solution. Let $(c_1, c_2, \ldots, c_m)$ be any arbitrary but fixed order of $\mathcal{C} \setminus \{p\}$, where m is the number of candidates minus one. For each $i \in [k]$, each $X \in \{L, R\}$, and every candidate $c \in \mathcal{C} \setminus \{p\}$, we define a piecewise linear concave function $f_{i,X,c} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows. First, $f_{i,X,c}(0) = 0$. Second, for each non-negative integer $x \leq t_i^X$, $f_{i,X,c}(x)$ is the number of votes in $\{\pi_{(i,1)}^X, \ldots, \pi_{(i,x)}^X\}$ that rank c above p. For each integer $x > t_i^X$ we have that $f_{i,X,c}(x) = f_{i,X,c}(t_i^X)$. Finally, for a real number x between integers $y \geq 0$ and $y + 1$, we have that

$$f_{i,X,c}(x) = f_{i,X,c}(y) + (x - y) \cdot (f_{i,X,c}(y+1) - f_{i,X,c}(y)) .$$

The restrictions are as follows.

- For every $i \in [k]$ and every $X \in \{L, R\}$, we have that $x_i^X \in \mathbb{N}$ and $0 \leq x_i^X \leq t_i^X$.
- Since we seek a feasible solution of size at most ℓ, we have that

$$\sum_{i \in [k]} (x_i^L + x_i^R) \leq \ell.$$

- To ensure that p is the Condorcet winner in the final election, for each candidate $c \in \mathcal{C} \setminus \{p\}$, we have that

$$2\left(N(c,p) - \sum_{\substack{i\in[k]\\ X\in\{L,R\}}} f_{i,X,c}\left(x_i^X\right)\right) < |\Pi_{\mathcal{V}}| - \sum_{\substack{i\in[k]\\ X\in\{L,R\}}} x_i^X.$$

In the above inequality, $N(c,p)$ is calculated with respect to $\Pi_{\mathcal{V}}$. The right side is the number of votes in the final election, and the left side is double the number of votes ranking c above p in the final election. This inequality ensures that p beats c in the final election. The above programming can be solved in FPT time with respect to k by the algorithm studied in [9]. □

Now we consider Condorcet in k-CP elections. Yang and Guo [42] proved that CCAV for Condorcet in 3-peaked elections and CCDV for Condorcet in 4-peaked elections are NP-hard. We strengthen their results by showing that CCAV and CCDV for Condorcet are NP-hard in 3-CP elections, a subclass of 3-peaked elections.

Theorem 5 *CCAV and CCDV for Condorcet in 3-CP elections are NP-hard.*

PROOF. [CCAV] We reduce the RX3C problem to CCAV. Let $(U = \{c_1, \ldots, c_{3\kappa}\}, S = \{s_1, \ldots, s_{3\kappa}\})$ be an RX3C instance. The components of the CCAV instance are as follows.

Candidates $\mathcal{C}$. We create in total 12κ+1 candidates. Precisely, for each $c_x \in U$, we create one candidate c'_x. Let $C_1 = \{c'_x : c_x \in U\}$. In addition, for each $s = \{c_x, c_y, c_z\} \in S$, we create three candidates $c_x(s)$, $c_y(s)$, $c_z(s)$ corresponding to c_x, c_y, c_z, respectively. Let C_2 be the set of candidates corresponding to the 3-subsets in S. Finally, we create a distinguished candidate p. Let $C_3 = \{p\}$.

Let $\lhd_1 = (c'_1, \ldots, c'_{3\kappa})$ be the order of C_1 according to the indices of the candidates. Moreover, let $\lhd_2$ be any arbitrary order of C_2 such that for each $s \in S$ the three candidates corresponding to s are ordered consecutively.

Registered Votes $\Pi_{\mathcal{V}}$. We create in total $5\kappa - 3$ votes, each of which ranks p in the last position. The positions of other candidates in a vote are set in a way so that the vote restricted to C_1 and C_2 is single-peaked with respect to $\lhd_1$ and $\lhd_2$, respectively.

Unregistered Votes $\Pi_{\mathcal{W}}$. The unregistered votes are created according to S. In particular, for each $s = \{c_x, c_y, c_z\} \in S$, we create four votes π_s, π_s^x, π_s^y, and π_s^z such that:

- $\pi_s(\{c_x(s), c_y(s), c_z(s)\}) = \{1, 2, 3\}$, $\pi_s(p) = 4$;
- $\pi_s^x(c_x(s)) = 1$, $\pi_s^x(c'_x) = 2$, $\pi_s^x(p) = 3$;
- $\pi_s^y(c_y(s)) = 1$, $\pi_s^y(c'_y) = 2$, $\pi_s^y(p) = 3$; and
- $\pi_s^z(c_z(s)) = 1$, $\pi_s^z(c'_z) = 2$, $\pi_s^z(p) = 3$.

The exact positions of $c_x(s)$, $c_y(s)$, $c_z(s)$ in π_s, and the positions of the remaining candidates in each of the above four votes are set in a way so that the votes restricted to $\lhd_1$ and $\lhd_2$ are single-peaked. Let $\Pi_S = \{\pi_s : s \in S\}$ and $\Pi^U = \{\pi_s^x : s \in S, c_x \in s\}$.

Finally, we set $\ell = 5\kappa$. The above construction clearly takes polynomial time. It remains to prove the correctness.

($\Rightarrow$) Assume that $S' \subseteq S$ is an exact set cover of U. Consider the election after adding the following 5κ votes:

1. All 2κ votes π_s such that $s \notin S'$;
2. For each $s = \{c_x, c_y, c_z\} \in S'$, all the three votes π_s^x, π_s^y, and π_s^z.

Clearly, the final election has in total $10\kappa - 3$ votes. Moreover, for each $c \in C_2$ at most one of the above added 5κ votes ranks c above p. As a result, there are at most $(5\kappa - 3) + 1 = 5\kappa - 2$ votes ranking c above p, implying that p beats every candidate in C_2 in the final election. As S' is an exact set cover, for each c'_x, among the 5κ added votes only the vote π_s, corresponding to $s \in S'$ such that $c_x \in s$, ranks c'_x above p. Analogous to the above analysis, we know that p beats every candidate in C_1 in the final election. In summary, p becomes the Condorcet winner after adding the above 5κ votes.

($\Leftarrow$) Assume that $\Pi_W \subseteq \Pi_{\mathcal{W}}$ such that $|\Pi_W| \leq \ell = 5\kappa$ and p becomes the Condorcet winner after adding all votes in Π_W. As p is not the Condorcet winner with respect to the registered vote constructed above, it holds that $|\Pi_W| \geq 1$. Then, we can observe that $|\Pi_W| = 5\kappa$ must hold, since otherwise there is at least one candidate which ranked above p in at least $(5\kappa - 3) + 1 = 5\kappa - 2$ votes and hence is not beaten by p in the final election. Moreover, Π_W contains exactly 2κ votes in Π_S. The reason is as follows. If Π_W contains less than 2κ votes in Π_S, then Π_W contains more than 3κ votes from Π^U. This implies that there are two votes in $\Pi_W \cap \Pi^U$ both of which rank a common candidate $c \in C_2$ above p, leading to c not being beaten by p in the final election. On the other hand, if Π_W contains some vote $\pi_s \in \Pi_S$ where $s = \{c_x, c_y, c_z\}$, then none of π_s^x, π_s^y, and π_s^z can be included in Π_W, since otherwise due to the construction of the votes, one of $c_x(s)$, $c_y(s)$, $c_z(s)$ is not beaten by p in the final election. Hence, if Π_W contains t votes in Π_S, then $|\Pi_W| \leq t + 3(3\kappa - t) = 9\kappa - 2t$. If $t > 2\kappa$, then $|\Pi_W| < 5\kappa$, a contradiction. Therefore, it must be that $t = 2\kappa$, i.e., $|\Pi_S \cap \Pi_W| = 2\kappa$. Let $S' = \{s \in S : \pi_s \notin \Pi_W\}$. Due to the above analysis, it holds that $|S'| = 3\kappa - |\Pi_W \cap \Pi_S| = 3\kappa - 2\kappa = \kappa$. Moreover, for each π_s where $s \in S'$ and $s = \{c_x, c_y, c_z\}$, all three votes π_s^x, π_s^y, π_s^z are in Π_W (otherwise Π_W contains less than 5κ votes). As for each candidate $c \in C_1$ there can be at most one vote in Π_W ranking c above p, it follows that S' is an exact set cover. □

Now we discuss CCAV and CCDV for Copeland$^\alpha$, where $0 \leq \alpha \leq 1$, and Maximin in k-axes elections for small values of k. In a sharp contrast to the fixed-parameter tractability of CCAV and CCDV for Condorcet in k-axes elections, the same problems for both Copeland$^\alpha$ and Maximin are NP-hard even for $k = 2$. A general explanation of the complexity difference is that to make the distinguished candidate p the Condorcet winner, we need only to focus on the comparisons between p and every other candidate. In other words, if two votes rank the same set of candidates above p, they have the same impact on the solution. However, in Copeland$^\alpha$ and Maximin this does not hold.

Theorem 6 *CCAV and CCDV for Copeland$^\alpha$, $0 \leq \alpha \leq 1$, and Maximin in 2-axes elections are NP-hard.*

Note that the NP-hardness of CCAV and CCDV for Copeland$^\alpha$, $0 \leq \alpha < 1$ in elections with single-peaked width 2, established by Yang and Guo [41], implies the NP-hardness of the same problems in 2-CP elections because any election with single-peaked width k is a k'-CP election for some $k' \leq k$ [17].

For Copeland^1 and Maximin in elections with single-peaked width 2, Yang and Guo [41] proved that CCAV and CCDV are polynomial-time solvable. We show that both problems become NP-hard when extended to 2-CP elections.

Theorem 7 *CCAV and CCDV for Copeland^1 and Maximin in 2-CP elections are NP-hard.*

5 Recognition of Nearly Single-Peakedness

It is known that determining whether an election is single-peaked (1-axis) is polynomial-time solvable [5, 12, 18]. For every $k \geq 3$, Erdélyi, Lackner, and Pfandler [17] proved that determining whether an election is a k-axes election is NP-hard. We complement these results by showing that determining whether an election is a 2-axes election is polynomial-time solvable, completely filling the complexity gap of the problem with respect to k. To this end, we reduce the problem to the 2-SATISFIABILITY problem (2SAT).

2-SATISFIABILITY	
Input:	A set of Boolean variables and a collection of clauses each of which consists of two literals.
Question:	Is there a truth-assignment which satisfies all the given clauses?

It is well-known that the 2SAT problem can be solved in linear time in the number of clauses [1, 2, 24].

Single-peaked elections have a nice characterization [3] which is useful for our study.

Definition 1 (Worst-diverse structure (WD-structure)) *A WD-structure in an election $(\mathcal{C}, \Pi_{\mathcal{V}})$ is a 6-tuple $(\pi_x, \pi_y, \pi_z, a, b, c)$ such that*

- $\pi_x, \pi_y, \pi_z \in \Pi_{\mathcal{V}}$, $a, b, c \in \mathcal{C}$;
- $\pi_x(a) > \max\{\pi_x(b), \pi_x(c)\}$;
- $\pi_y(b) > \max\{\pi_y(a), \pi_y(c)\}$; *and*
- $\pi_z(c) > \max\{\pi_z(a), \pi_z(b)\}$.

Three votes WD-conflict if there are three candidates forming a WD-structure with them.

Definition 2 (α-structure) *An α-structure in an election $(\mathcal{C}, \Pi_{\mathcal{V}})$ is a 6-tuple $(\pi_x, \pi_y, a, b, c, d)$ such that*

- $\pi_x, \pi_y \in \Pi_{\mathcal{V}}$, $a, b, c, d \in \mathcal{C}$;
- $\pi_x(a) < \pi_x(b) < \pi_x(c)$, $\pi_x(d) < \pi_x(b)$; *and*
- $\pi_y(c) < \pi_y(b) < \pi_y(a)$, $\pi_y(d) < \pi_y(b)$.

Two votes α-conflict if there are four candidates forming an α-structure with them. The following lemma was studied by Ballester and Haeringer [3].

Lemma 4 ([3]) *An election $(\mathcal{C}, \Pi_{\mathcal{V}})$ is single-peaked if and only if there are no WD-structures and α-structures in $(\mathcal{C}, \Pi_{\mathcal{V}})$.*

Armed with the above lemma, we are able to develop the polynomial-time algorithm for recognizing 2-axes elections.

Theorem 8 *Determining whether an election is a 2-axes election is polynomial-time solvable.*

PROOF. Let $(\mathcal{C}, \Pi_{\mathcal{V}})$ be an election. The problem is equivalent to seeking a partition (Π_T, Π_F) of $\Pi_{\mathcal{V}}$ so that both $(\mathcal{C}, \Pi_T)$ and $(\mathcal{C}, \Pi_F)$ are single-peaked. We reduce the problem to the 2SAT problem as follows. For each vote $\pi \in \Pi_{\mathcal{V}}$, we create a variable $x(\pi)$. Hence, a partition (Π_T, Π_F) of $\Pi_{\mathcal{V}}$ corresponds to a truth-assignment of the variables, and vice versa: variables corresponding to votes in Π_T are assigned true and variables corresponding to votes in Π_F are assigned false.

If there are no WD-structures in $(\mathcal{C}, \Pi_{\mathcal{V}})$, we create the clauses as follows. For every two votes π and π' which α-conflict, we create two clauses $(x(\pi), x(\pi'))$ and $\left(\overline{x(\pi)}, \overline{x(\pi')}\right)$. In order to satisfy both clauses, $x(\pi)$ and $x(\pi')$ must be assigned differently, and thus π and π' are partitioned into different sets. From Lemma 4, there is a truth-assignment satisfying all clauses if and only if $(\mathcal{C}, \Pi_{\mathcal{V}})$ is a 2-axes election. Assume now that there are WD-structures in $(\mathcal{C}, \Pi_{\mathcal{V}})$. Let a, b, and c be three candidates in a WD-structure. Let (Π_a, Π_b, Π_c) be a partition of $\Pi_{\mathcal{V}}$ such that, among a, b, and c, Π_a consists of all votes ranking a last, Π_b all votes ranking b last, and Π_c all votes ranking c last. Observe that Π_a, Π_b, and Π_c are all nonempty. Moreover, if $(\mathcal{C}, \Pi_{\mathcal{V}})$ is 2-axes single-peaked, then none of Π_T and Π_F contains three votes from Π_a, Π_b, and Π_c, respectively, where Π_T and Π_F are as discussed above. Due to symmetry of Π_T and Π_F, we have three cases to consider: (1) $\Pi_a \subseteq \Pi_T$, $\Pi_b \subseteq \Pi_F$, (2) $\Pi_a \subseteq \Pi_T$, $\Pi_c \subseteq \Pi_F$, and (3) $\Pi_b \subseteq \Pi_T$, $\Pi_c \subseteq \Pi_F$. We analyze only Case 1. The analysis for the other cases are similar. First, if $(\mathcal{C}, \Pi_a)$ or $(\mathcal{C}, \Pi_b)$ are not single-peaked, we discard this case. Assume now that both $(\mathcal{C}, \Pi_a)$ and $(\mathcal{C}, \Pi_b)$ are single-peaked. Then, we create the clauses as follows. We shall ensure that there is a partition (Π_T, Π_F) of $\Pi_{\mathcal{V}}$ such that $(\mathcal{C}, \Pi_T)$, $(\mathcal{C}, \Pi_F)$ are single-peaked, $\Pi_a \subseteq \Pi_T$ and $\Pi_b \subseteq \Pi_F$, if and only if the 2SAT instance has a truth-assignment satisfying all the following clauses.

- If there is a vote $\pi \in \Pi_c$ which α-conflicts with one vote in Π_a (resp. Π_b), or WD-conflicts with two votes in Π_a (resp. Π_b), then π must be included in Π_F (resp. Π_T). In this case, we create a clause $(\overline{x(\pi)})$ (resp. $(x(\pi))$).
- If there are two votes $\pi, \pi' \in \Pi_c$ which WD-conflict with one vote in Π_a (resp. Π_b), we create one clause $(\overline{x(\pi)}, \overline{x(\pi')})$ (resp. $(x(\pi), x(\pi'))$), to ensure that at least one of $\{\pi, \pi'\}$ is in Π_F (resp. Π_T).

One may wonder that there might be three votes $\pi, \pi', \pi'' \in \Pi_c$ that WD-conflict. We don't need to consider this case since it has been implicitly dealt with in the second type of clauses. Assume that $(\pi, \pi', \pi'', d, d', d'')$ is a WD-structure, where $d, d', d'' \in \mathcal{C}$ and $\pi, \pi', \pi'' \in \Pi_c$. Let π_1 and π_2 be two arbitrary votes from Π_a and Π_b, respectively. If the last ranked candidates among d, d', d'' in π_1 and π_2 are the same, say, d, then, π' and π'' WD-conflict with both π_1 and π_2. Hence, two clauses $(x(\pi'), x(\pi''))$ and $(\overline{x(\pi')}, \overline{x(\pi'')})$ have been created due to the above discussion. In order to satisfy these two clauses, $x(\pi')$ and $x(\pi'')$ must be assigned different values and the votes π' and π'' are partitioned into different sets accordingly. On the other hand, assume that the last ranked candidates among d, d', d'' in π_1 and π_2 are different. Without loss of generality, assume that π_1 ranks d in the last and π_2 ranks d' in the last. Then, π' and π'' WD-conflict with π_1, and π and π'' WD-conflict with π_2. Due to the above discussion, we have two clauses $(\overline{x(\pi')}, \overline{x(\pi'')})$ and $(x(\pi), x(\pi''))$. Again, to satisfy these two clauses, $x(\pi)$, $x(\pi')$, and $x(\pi'')$ cannot be assigned the same value, leading to π, π', π'' not being partitioned into the same set. □

6 Conclusion

Aiming at pinpointing the complexity border of CCAV and CCDV between single-peaked elections and general elections, we have studied these problems in k-axes elections and k-CP elections and obtained many tractability and intractability results. We particularly studied the voting correspondences r-approval, Condorcet, Maximin, and Copeland$^\alpha$, $0 \leq \alpha \leq 1$. Our study closed many gaps left in the literature. We refer to Table 1 for a summary of our results. Though that our focus in this paper is the unique-winner model of CCAV and CCDV, it should be pointed out that all our results, including polynomial-time solvability results, FPT results, and NP-hardness results hold for the nonunique-winner model of CCAV and CCDV as well. Recall that in the nonunique-winner model, the goal is to make the distinguished candidate a winner, but not necessarily the unique winner.

In addition, we proved that determining whether an election is a k-axes election is polynomial-time solvable for $k = 2$. Given that the problem is polynomial-time solvable for $k = 1$ [3] and NP-hard for every $k \geq 3$ [17], our result closes the final complexity gap of the problem with respect to k.

There remain several open questions (see Table 1) for future research. In particular, the complexity of CCAV and CCDV for Copeland$^\alpha$, $0 \leq \alpha < 1$, in single-peaked elections remained open. In addition, investigating the complexity of other voting problems, e.g., DESTRUCTIVE CONTROL BY ADDING/DELETING VOTES/CANDIDATES, BRIBERY, in nearly single-peaked elections is another promising topic for future research.

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