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Committee Selection with Multimodal Preferences

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Abstract. We study committee selection with *multimodal preferences*: Assuming a set of candidates A, a set of voters V, and ℓ *layers*, where each voter $v \in V$ has ordinal preferences over the alternatives for each layer separately, the task is to select a committee $S \subseteq A$ of size k. We discuss applications of our model and study the computational complexity of several generalizations of known committee scoring rules (specifically, k-Borda and Chamberlin–Courant) to our setting, as well as discuss domain restrictions for our model. While most problems we encounter are computationally intractable in general, we nevertheless design efficient algorithms for certain cases.

1 Introduction

We consider the following setting (formal definitions are given below): We have a set A of m candidates (i.e., alternatives), a set V of n voters, and ℓ *layers* (i.e., modals). We consider the ordinal model, where voters provide linear orders. The crucial ingredient in our model is that each voter does not correspond to one linear order, but to ℓ linear orders, one for each layer. We are also given a committee size k, and the task is to select a committee of k candidates.

Our model aims to capture situations such as the following: (1) **Uncertain Future**: Consider a scenario of selecting a management committee for a university. In the near future, the university may face different uncertain situations; e.g., it is not clear that the university will receive a big grant from the government or from the industry. Voters (say, faculty) may have different preferences for each of these uncertain future situations: Indeed, perhaps some candidates for the management committee are good to deal with the government while others are good for dealing with industry. Here, we would have one layer corresponding to government and another to industry; importantly, each voter might wish to specify not only one "global" linear order, but two – one for each layer. (2) **Candidate Attributes**: We wish to select a committee based on certain attributes (e.g., proficiency in economics, diplomatic skills, etc.). Using our model, and given that these attributes are set, each voter would provide her preferred ranking over the candidates for each of these attributes separately. Other situations are possible.

Our approach to the problem of selecting committees for our model is to adapt the framework of committee scoring rules (CSRs) [13, 19]. This framework has proven to be very rich and useful in analyzing multiwinner rules. In essence, a CSR selects a committee which maximizes the *total satisfaction* – the sum of *satisfaction* of the voters – where voter satisfaction is defined differently for each CSR based on committee scoring functions – which assigns a value for each position of a committee in the ranking of a vote. We adapt CSRs to our model in several ways; these are defined formally below, but in essence, for a CSR $\mathcal{R}$: Max-$\mathcal{R}$ (Min-$\mathcal{R}$, Sum-$\mathcal{R}$) is a multimodal rule (i.e., a rule where each voter gives ℓ linear orders, one for each layer) which selects a committee which maximizes the *multimodal satisfaction* of a voter, where the multimodal satisfaction of a voter is the maximum (respectively, minimum, sum) over the ℓ layers of her satisfaction from the committee; Pareto-$\mathcal{R}$ is the problem of selecting a Pareto optimal committee, which is a committee S for which no other committee achieves no less total satisfaction than S for all layers and strictly better total satisfaction for at least one layer; and in Vector-$\mathcal{R}$ we are given as input also a vector of required total satisfaction for each layer and shall find a committee that respects this vector.

Our multimodal model generalizes the standard model of multiwinner elections: Indeed, for $\ell = 1$, the two models coincide; in particular, our combinatorial problems (i.e., Max-$\mathcal{R}$, Min-$\mathcal{R}$, Sum-$\mathcal{R}$, Pareto-$\mathcal{R}$, Vector-$\mathcal{R}$) collapse to finding a winning multiwinner committee under $\mathcal{R}$. So, as some CSRs are NP-hard, we inherit their hardness to our models; in particular, we observe this phenomenon when we consider the NP-hard multiwinner CSR Chamberlin–Courant (CC). In addition, we also study the polynomial-time CSR k-Borda and observe that in our more general multimodal model we sometimes also reach computational intractability. This follows from the combinatorial complexity of our more general problems. Indeed, as we discuss throughout the paper, the combinatorial structure of our problems is quite diverse and non trivial.

Relevant Parameters To overcome this computational intractability, we use parameterized complexity and study the effect certain parameters – the number ℓ of layers, the number n of voters, the number m of candidates, and the committee size k – have on the complexity of our problems.

The parameters under study can indeed be small in real-life scenarios. Specifically: (1) in many real life scenarios, we generally require committee members to have certain set of proficiencies. E.g., consider selecting a research paper for a conference; PC members judge a paper on the basis of significance, novelty, correctness, presentation, scope, etc. Therefore, the number ℓ of layers is small in these scenarios; (2) there are many scenarios in which the number n of voters can be small; e.g., the size of a jury in a competition or the size of the board of directors in an organisation, who might be shortlisting projects for their organisation, is small (see, e.g., [11]); (3) there are many scenarios in which the number of candidates is smaller than voters, e.g., in political elections; (4) the committee size k is naturally not very large in real-life scenarios, such as in shortlisting tasks, parliament selection, selecting movies in the plane, etc.

We identify several cases in which efficient parameterized algorithms exist. Furthermore, we adapt the concept of domain restrictions to our multimodal model and study the complexity of solving our problems for restricted multimodal elections.

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Related Work In this paper we generalize several (ordinal) OWA-rules[3] [27] – which is a subclass of the more general class of CSRs [19, 18] – to the setting of multimodal committee elections. Some social choice papers study closely-related models, usually considering attributes of alternatives (similarly in spirit to our second motivating example): Lang and Skowron [24] discuss proportional representation in the context of attribute-based elections. Other scholars [7, 9, 1] consider attribute-based multiwinner elections as well; their goal differs from ours, as they wish to achieve some certain committee diversity. Kagita et al. [22] consider attribute-based multiwinner elections and suggest several appealing axioms in the setting of approval-based elections (we consider ordinal elections).

There is some work on multimodal preferences, e.g., in the context of matching [10] and information retrieval [8]. Our second motivating example has some relation to uncertainty (see, e.g., [2]). Our definition of Pareto-$\mathcal{R}$ somehow relates to other multiobjective optimization views on multiwinner elections; we mention, specifically, the work of Aziz et al. [4], in which the dimensions of optimization are the voters, and the work of Kocot et al. [23], in which the dimensions of optimization are different voting rules.

2 Multiwinner Elections

Next we provide a brief introduction to the standard model of multiwinner elections, multiwinner voting rules, and CSRs. A *multiwinner election* E consists of a set A of m candidates, a set V of n voters, and a committee size k; each voter $v \in V$ provides her preference as a linear order over A (we consider ordinal elections). A *multiwinner voting rule* $\mathcal{R}$ takes as input a multiwinner election E and returns a committee $S \subseteq A$ with $|S| = k$.

Here we adapt two prominent multiwinner voting rules to the multimodal case, namely k-Borda and the rule suggested by Chamberlin and Courant (CC, in short). Both rules are committee scoring rules, in that they select a committee which maximizes the *total satisfaction*, i.e., the sum – over the voters – of the *satisfaction* of the voters, where the satisfaction of a voter from a committee depends on the position of the committee members of the committee at hand in the preference order of the voter. CSRs differ by their definition of such satisfaction: Under k-Borda, the satisfaction of a voter from a committee S is the sum of Borda scores of the committee members in S, where the Borda score of a candidate c ranked in position i in a vote is $m - i$. Under CC, the satisfaction of a voter from a committee S is the Borda score of the committee member from S which is ranked the highest among the candidates in S. E.g., for a vote $a \succ b \succ c \succ d$ and committee $\{b, c\}$, under k-Borda the satisfaction is 3, while under CC the satisfaction is 2.

As argued by certain authors (see, e.g., [16] and [17]), k-Borda and CC represent two extreme multiwinner rules. In essence, k-Borda fits settings in which it is desired to select a committee containing committee members which are ranked highly by the society, while CC fits settings in which it is desired to take into account the diversity of the views apparent in the society. Both rules are thus widely studied. In particular, while finding winning committees under k-Borda can be done in polynomial time (one has to select k candidates with the highest individual Borda scores), CC is NP-hard [26] but FPTfor certain parameters, admit approximation algorithms, and certain heuristics are known to be effective for it [5, 20, 28, 15].

[3] An OWA (ordered weighted average) rule is defined via a scoring vector s of size m and an OWA vector z of size k, and selects a committee that maximizes the sum of voter utilities, where the utility of a voter from a committee she ranks in positions $p_1, \ldots, p_k$ is $\sum_{i \in [k]} z_i \cdot s(p_i)$.

3 Multimodal Multiwinner Elections

In this paper we study adaptations of k-Borda and CC to the setting of multimodal multiwinner elections. A *multimodal multiwinner election* E consists of ℓ multiwinner elections $E_1, \ldots, E_\ell$, all on the same set A of m candidates, the same set $V$of n voters, and the same committee size k; for $z \in [\ell]$, E_z is denoted as the zth *layer*. Note that, in effect, each voter $v \in V$ provides ℓ preference orders, one for each layer. A *multimodal multiwinner voting rule* $\mathcal{R}$ takes a multimodal multiwinner election E and returns a committee $S \subseteq A$ of size k. Next we discuss our different adaptations of CSRs to our multimodal model.

3.1 Max-$\mathcal{R}$, Min-$\mathcal{R}$, and Sum-$\mathcal{R}$

Definition 1 (Max-$\mathcal{R}$, Min-$\mathcal{R}$, Sum-$\mathcal{R}$) *Let $\mathcal{R}$ be a CSR defined via a satisfaction function that returns a value for each vote and committee. Then, we define the following* 3 *multimodal rules: Max-$\mathcal{R}$, Min-$\mathcal{R}$, Sum-$\mathcal{R}$; a winning committee under these rules maximizes the* total multimodal satisfaction*, which is the sum – over the voters – of the* multimodal satisfaction *of the voters, where the multimodal satisfaction is defined differently for each rule: Specifically, under Max-$\mathcal{R}$ (Min-$\mathcal{R}$, Sum-$\mathcal{R}$), the multimodal satisfaction of voter v is the maximum (respectively: minimum, sum) – over the layers – of her satisfaction under $\mathcal{R}$.*

In the decision version of the $\mathcal{T}$-$\mathcal{R}$ problem, for $\mathcal{T} \in \{Max, Min, Sum\}$, we are given a multimodal election $E_1, \ldots, E_\ell$, an integer k, and a committee score R; we shall decide the existence of a size-k committee whose $\mathcal{T}$-$\mathcal{R}$ score is at least R.

Remark 1 *Our adaptations of CSRs to the multimodal setting differ by the way in which we define voter satisfaction w.r.t. the different layers. Intuitively speaking, recalling the "Uncertain Future" example above, Max-$\mathcal{R}$ corresponds to voters which are optimistic; Min-$\mathcal{R}$ corresponds to voters which are pessimistic; and Sum-$\mathcal{R}$ corresponds to voters which believe in a randomly chosen future.*

3.2 Pareto-$\mathcal{R}$ and Vector-$\mathcal{R}$

We consider a further kind of an adaptation of CSRs to our model, based on the concept of Pareto dominance.

Definition 2 (Pareto-$\mathcal{R}$) *Let E' be a multimodal election, let S and S' be two committees for it, and let $\mathcal{R}$ be a CSR. Then, S'* dominates *S if: (1) for each layer $z \in [\ell]$, it holds that the score of S' is not less than the score of S in layer z, according to $\mathcal{R}$; and (2) for at least one layer $z \in [\ell]$ it holds that the score of S' is strictly more than the score of S, according to $\mathcal{R}$. A committee S is not* Pareto dominated *if there is no committee S' which dominates it. The Pareto-$\mathcal{R}$ multimodal rule selects a committee that is not Pareto dominated; i.e., a Pareto-optimal committee.*

We are interested in the computational task of identifying a Pareto-optimal committee, which is not a decision problem. The related problem of deciding whether a Pareto-optimal committee exists is solvable in constant-time, as such a committee always exists.

Observation 1 *A Pareto-optimal committee always exists.*

Proof. Counterpositively, if such a committee does not exist, then there is a cycle of committees, where each committee in the cycle dominates the next. Consider some committee S in this cycle, and consider the vector Z_S containing ℓ entries, where the jth entry of Z_S, i.e., $Z_S[j]$ is the score, under $\mathcal{R}$, for the jth layer, for committee S. Furthermore, denote by z_S the sum of the values of the vector Z_S (i.e., $z_S := \sum_{j\in[\ell]} Z_S[j]$). Observe now that for each pair of consecutive committees in this cycle S, S', we have that $z_S > z_{S'}$. Contradiction now follows as this is a cycle. □

A related decision problem which we do not consider in this paper is the following: Given an instance and a committee, decide whether the given committee is Pareto-optimal. We do, however, consider the following related problem.

Definition 3 (Vector-$\mathcal{R}$) *An instance of Vector-$\mathcal{R}$ gets a multimodal election E and a vector Q of ℓ numbers. The task is to find a committee that gets at least $Q[j]$ score (w.r.t. $\mathcal{R}$) for the election E_j, for each $j \in [\ell]$.*

3.3 Illustrating Example

To illustrate our multimodal multiwinner voting rules we provide the following multimodal election, containing $n := 2$ voters, $m := 4$ candidates, and $\ell := 2$ layers:

	E_1	E_2
$v_1:$	$a \succ b \succ c \succ d$	$c \succ b \succ a \succ d$
$v_2:$	$d \succ a \succ c \succ b$	$a \succ c \succ d \succ b$

For $k := 2$, consider the committee $S = \{b, c\}$. k-Borda satisfaction of v_1 for S in the first layer is 3 and 5 in the second layer. So, Max-k-Borda satisfaction of v_1 is 5. Similarly, the Max-k-Borda satisfaction of v_2 is 2. Hence, the Max-k-Borda score for S is 7. Min-k-Borda satisfaction of voter v_1 and v_2 for the committee S are 3 and 1, respectively; so, Min-k-Borda score for S is 4. Sum-k-Borda satisfaction of v_1 and v_2 for S are 8 and 3, respectively; so, Sum-k-Borda score for S is 11. k-Borda score of S for the first and second layers are 4 and 7, respectively. Consider a committee $S' = \{a, c\}$: k-Borda score of S' for the first and second layers are 7 and 9, respectively; thus, S' dominates S, and hence S is not a Pareto-k-Borda optimal committee. S', however, is a Pareto-k-Borda committee as the k-Borda score of any other committee of size 2 in the second layer is less than 9. S is a winning committee for Vector-k-Borda with the vector $Q = [3, 4]$. CC satisfaction of v_1 for committee S in the first (second) layer is 2 (respectively, 3). Therefore, Max-CC satisfaction of voter v_1 is 3. Similarly, Max-CC satisfaction for v_2 is 2. Hence, the Max-CC score of S is 5. Min-CC satisfaction of v_1 and v_2 for S are 2 and 1, respectively; thus, Min-CC score of S is 3. Sum-CC satisfaction of v_1 and v_2 for S are 5 and 3, respectively; thus, Min-CC score for the committee S is 8. CC score of S for the first and second layers are 3 and 5, respectively. Consider the committee $S' = \{a, c\}$: CC score of S' for the first and second layers are 4 and 6, respectively; thus, S' dominates S, and hence S is not a Pareto-CC committee. Note that S' is a Pareto-CC committee as any other committee of size 2 has less CC score in the second layer. Consider Vector-CC, with the vector $Q = [3, 4]$: Since CC score of S for the first and second layers are 3 and 5, respectively, S is a winning committee according to this rule. S is not a Vector-CC winning committee for, e.g., the vector $Q = [4, 4]$, since CC score of S in the first layer is less than 4.

4 Multimodal $\mathcal{R}$

In this section, we discuss certain properties which hold for any CSR. The first observation apparent from our model is that it generalizes the standard multiwinner model; specifically, for $\ell = 1$. Thus, we inherit computational hardness; in particular, as CC is NP-hard and W[2]-hard [26], we have the following.[4]

Corollary 1 *For each $\mathcal{T} \in \{$Max, Min, Sum, Pareto, Vector$\}$, $\mathcal{T}$-CC is NP-hard and W[2]-hard w.r.t. the committee size k, even if $\ell = 1$.*

As k-Borda is in P [17], we have the following:

Corollary 2 *For each $\mathcal{T} \in \{$Max, Min, Sum, Pareto, Vector$\}$, $\mathcal{T}$-k-Borda is in P when $\ell = 1$.*

Combinatorially speaking, each of Max-$\mathcal{R}$, Min-$\mathcal{R}$, and Sum-$\mathcal{R}$, due to the definition of total satisfaction, incorporate a summation operator over the voters. Sum-$\mathcal{R}$ incorporates a further summation operator – this time over the layers – as the multimodal satisfaction of a voter, for Sum-$\mathcal{R}$, is the summation of her satisfaction over the layers. The following holds as summation is a linear operator.

Proposition 1 *If there is an algorithm running in time $f(n, m, k)$ for some CSR $\mathcal{R}$, then there is an algorithm running in time $f(n\ell, m, k)$ for both Sum-$\mathcal{R}$ and Pareto-$\mathcal{R}$.*

Proof. The algorithm for both works by "flattening" the multimodal election: View the ℓ elections as one big election; i.e., transform the ℓ preference orders of each voter into ℓ voters, each with 1 preference order – one voter for each layer. □

We also have the following related observation.

Observation 2 *A committee that maximizes Sum-$\mathcal{R}$ (i.e., a winner under Sum-$\mathcal{R}$) is Pareto-optimal (i.e., a winner under Pareto-$\mathcal{R}$). Indeed, any algorithm for Sum-$\mathcal{R}$ serves as an algorithm for Pareto-$\mathcal{R}$ (but not necessarily vice versa).*

For a CSR $\mathcal{R}$, Egalitarian-$\mathcal{R}$, considered by Aziz et al. [3], selects a committee which maximizes not the total satisfaction (i.e., the sum of satisfaction over the voters), but the minimum satisfaction over the voters. Betzler et al. [5] also defined a variant of CC, called as Minimax CC Multiwinner. This rule is the same as Egalitarian-CC.

Proposition 2 *For any CSR $\mathcal{R}$, Egalitarian-$\mathcal{R}$ with n voters is equivalent to Min-$\mathcal{R}$ with one voter and n layers.*

Proof. Given an instance of Egalitarian-$\mathcal{R}$ with n voters, we construct a multimodal election with 1 voter v and $\ell := n$ layers, by taking the ith voter of the Egalitarian-$\mathcal{R}$ instance to be the v's preference order in the ith layer. Correctness follows by observing that, in Min-$\mathcal{R}$, we shall have that the score of the winning committee be high in all layers, while in Egalitarian-$\mathcal{R}$, the score of the winning committee shall be high for all (nonmodal) voters.

Similarly, given an instance of Min-$\mathcal{R}$ with a voter v and n layers, we can construct an instance of Egalitarian-$\mathcal{R}$ by taking n voters and preference of ith voter is same as the preference of v in the ith layer. Correctness is similar as discussed above. □

Using similar constructions as in the proof of Proposition 2, we obtain the following result.

[4] As discussed above, Pareto-$\mathcal{R}$ is not a decision problem; we abuse notation and speak of NP-hardness (W-hardness) to mean that no efficient algorithm exists unless P=NP(resp., FPT= W).

Proposition 3 *For any CSR $\mathcal{R}$, winner determination for $\mathcal{R}$ with n voters is equivalent to Sum-$\mathcal{R}$ with one voter and n layers.*

We next give a result connecting Egalitarian-$\mathcal{R}$ and Vector-$\mathcal{R}$.

Proposition 4 *For any CSR $\mathcal{R}$, there is a polynomial time reduction from Egalitarian-$\mathcal{R}$ with n voters to Vector-$\mathcal{R}$ with one voter and n layers.*

Proof. Given an instance (E, R, k) of Egalitarian-$\mathcal{R}$, we construct an instance $(E' = (E_1, \ldots, E_n), Q = (r_1, \ldots, r_n), k)$ of Vector-$\mathcal{R}$: E' is a multimodal election with 1 voter v and $\ell := n$ layers, by taking the preference of ith voter of the instance of Egalitarian-$\mathcal{R}$ to be the v's preference order in the ith layer. For each $i \in [k]$, we set $r_i = R$. In the forward direction, correctness follows as the $\mathcal{R}$-score of a winning committee for an instance (E, R, k) is at least R for each voter. In the reverse direction, since the committee score is at least R in each layer, it is at least R for each voter in V. □

Note that for any CSR $\mathcal{R}$, and for any $\mathcal{T} \in$ {Max, Min, Sum, Pareto, Vector}, $\mathcal{T}$-$\mathcal{R}$ is FPTw.r.t. m as we can go over all possible committees of size k; and output the optimal one.

Remark 2 (Single-winner elections) *We concentrate on multiwinner rules and not on single-winner rules because the setting is quite clear for single-winner rules: Specifically, both k-Borda and CC collapse to Borda for $k = 1$. And there is a polynomial time algorithm which solves {Max, Min, Sum, Pareto, Vector}-Borda by considering all m options. (In fact, this holds for any single-winner scoring rule.)*

5 Multimodal k-Borda

In this section, we study adaptations of k-Borda to the multimodal setting.

Max-k-Borda

We study the parameterized complexity of Max-k-Borda w.r.t. k and prove that the problem is W[1]-hard w.r.t. k even when $\ell = 2$. Towards this, we give a reduction from the W[1]-hard problem Independent Set (IS) [12], in which given a graph G and an integer t; we shall decide the existence of a t-sized set $X \subseteq V(G)$ containing only nonadjacent vertices. Since the reduction can be carried out in polynomial time, NP-hardness follows. Note that this result is tight for ℓ, as the problem is polynomial-time solvable when $\ell = 1$, as it reduces to the nonmodal k-Borda. The core of the reduction is to create a voter v_{xy} for each edge xy, and create two layers such that in layer 1 (layer 2), v_{xy} ranks x first and y last (respectively, y first and x last). Then, we show that selecting candidates corresponding to adjacent vertices in the graph to be committee members results in multimodal satisfaction for v_{xy} which is not sufficient. Next, we prove our result formally.

Theorem 1 *Max-k-Borda is* NP*-hard and* W[1]*-hard w.r.t. k, even for $\ell = 2$.*

Proof. We provide a polynomial time reduction from the W[1]-hard problem Independent Set (IS) [12]. Given an instance (G, t) of IS, we construct an instance $((E_1, \ldots, E_\ell), R, k)$ of Max-k-Borda as follows. Let $|V(G)| = n$ and $|E(G)| = m$. We first construct a set of candidates, A: For each vertex $v \in V(G)$, we add a candidate c_v to the set A; add two sets $C_1 = \{x_1, \ldots, x_{nm(k+1)}\}$ and $C_2 = \{y_1, \ldots, y_{nm(k+1)}\}$ to the set A (we refer to the candidates of sets C_1 and C_2 as *dummy candidates*). We next construct a set of voters, V: For each edge uv in G, we add a voter e_{uv} to the set V. We set the number of layers to be $\ell = 2$; set the bound on the committee score as $R = m(2nmk(k+1) + {}^{k(k+1)}/2)$; and the size of the committee to be $k = t$.

For a voter e_{zw}, let S_{zw} be a set of candidates of the set $A \setminus \{C_1 \cup C_2 \cup \{z, w\}\}$. For a set Z, the symbol $\langle Z \rangle$ denotes that the vertices in the set Z are listed in some arbitrary strict order. Next, for each voter in V, we define the preference list for each layer $i \in [\ell]$: Consider a voter e_{uv} in the voter set V. In the election E_1, the preference list of the voter e_{uv} is $u \succ \langle S_{uv} \rangle \succ \langle C_2 \rangle \succ \langle C_1 \rangle \succ v$. In the election E_2, the preference list of voter e_{uv} is $v \succ \langle S_{uv} \rangle \succ \langle C_2 \rangle \succ \langle C_1 \rangle \succ u$. Next, for all voters e_{pq} in $V \setminus \{e_{uv}\}$, we set the preference list in the election E_1 as $p \succ \langle S_{pq} \rangle \succ \langle C_1 \rangle \succ \langle C_2 \rangle \succ q$ and in the election E_2 as $q \succ \langle S_{pq} \rangle \succ \langle C_1 \rangle \succ \langle C_2 \rangle \succ p$.

The intuition for these preference lists is that, for a voter e_{xy}, both candidates x and y should not belong to the winning committee as, if so, then the committee will not achieve the required committee score. The different ordering of vertices in S_1 and S_2 for the voter e_{uv} are to prevent dummy candidates from entering the committee. In this manner, we ensure that the set of vertices corresponding to candidates in the committee will lead to an independent set in G.

For correctness, we show that (G, t) is a yes-instance of IS iff $((E_1, E_2), R, k)$ is a yes-instance of Max-k-Borda. In the forward direction, let X be a solution to (G, t). We claim that $S = \{c_w \in A \mid w \in X\}$ is a solution to $((E_1, E_2), R, k)$. We first claim that all candidates in S belong to the first $n - 1$ candidates in the preference list of each voter in Vin either E_1 or E_2; dummy candidates do not belong to the committee. Let $w \in X$. If w is an isolated vertex in G, then by the construction of E_1 and E_2, w belongs to the first $n - 1$ candidates in the preference list of every voter in both the elections. Suppose that w is not an isolated vertex. Then, for each $z \in N(w)$, w is the first candidate in the preference list of e_{wz} either in the election E_1 or E_2, by construction. Also, for all the other voters, w belongs to first $n - 1$ candidates in their preference list in both E_1 and E_2. Since total number of candidates is $n + 2nm(k + 1)$, the multimodal satisfaction of any voter for committee S is at least $2nmk(k + 1) + {}^{k(k+1)}/2$. Hence, the total multimodal satisfaction for election E is at least $m(2nmk(k + 1) + {}^{k(k+1)}/2) = R$.

For the backward direction, let S be a solution to $((E_1, E_2), R, k)$. We first note the following properties of S.

Observation 3 *There are no dummy candidates in S.*

Observation 4 *If e_{uv} is a voter in the set of voters V, then both u and v do not belong to S.*

Using Observations 3 and 4, we have that, $X := \{w \in V(G) \mid c_w \in S\}$ is an independent set in G. Theorem 1 now follows. □

Considering the number n of voters as a parameter does not break the computational intractability. The reduction for Theorem 2 is via Multicolored Independent Set.

Theorem 2 *Max-k-Borda is W[1]-hard w.r.t. $n + k$.*

We do identify tractability w.r.t. $n + \ell$, by the following intuitive idea: Recall that, for Max-$\mathcal{R}$, the multimodal satisfaction of a voter for a committee is the maximum, over the layers, of her satisfaction. Therefore, if we guess, for each voter v, the layer $t_v \in [\ell]$ that gives v the maximum satisfaction over the layers, for the solution committee, then it is enough to find a k-Borda committee for the election E', where E' is a multiwinner election on the set of candidates A and voters V, where the preference order for voter v is t_v.

	k-**Borda**	**CC**
Max	NP-h (Thm. 1) W-h w.r.t. k even for $\ell = 2$ (Thm. 1) W-h w.r.t. $n + k$ (Thm. 2) FPTw.r.t. $n + \ell$ (Thm. 3) Poly for constant n (Thm. 4)	NP-h (Cor. 1) W-h w.r.t. k even for $\ell = 1$ (Cor. 1) FPTw.r.t. n (Prop. 7)
Min	NP-h even for $n = 1$ (Prop. 5) W-h w.r.t. $k + \ell$ even for $n = 1$ (Prop. 5)	NP-h even for $n = 1$ (Cor. 4) W-h w.r.t. k even for $\ell = 1$ (Cor. 1) W-h w.r.t. k even for $n = 1$ (Cor. 4) **Open w.r.t.** $n + \ell$
Sum	Poly (Cor. 3)	NP-h even for $n = 1$ (Prop. 8) W-h w.r.t. k even for $n = 1$ (Prop. 8) W-h w.r.t. k even for $\ell = 1$ (Cor. 1) FPTw.r.t. $n + \ell$ (Prop. 9)
Pareto	Poly (Cor. 3)	NP-h (Cor. 1) W-h w.r.t. k even for $\ell = 1$ (Cor. 1) FPTw.r.t. $n + \ell$ (Cor. 5) Poly for constant n (Prop. 10) **Open w.r.t.** n
Vector	NP-h even for $n = 1$ (Prop. 6) W-h w.r.t. $k + \ell$ even for $n = 1$ (Prop. 6)	NP-h even for $n = 1$ (Cor. 6) W-h w.r.t. k even for $\ell = 1$ (Cor. 1) W-h w.r.t. k even for $n = 1$ (Cor. 6) **Open w.r.t.** $n + \ell$

Table 1. Summary of our results. NP,W-h stands for NP,W-hardness.

Algorithm 1 Layer-guessing Algorithm

Input: A multimodal multiwinner election $(E_1, \dots, E_\ell)$ with committee size k and bound on the committee score R.

Output: A committee S of size k with score at least R, if exists, otherwise NO.

▷ Let $\texttt{Borda-Score}(S)$ denote the k-Borda score of committee S

1: $S = \emptyset$, $score_S = 0$
2: **for** each vector $(t_{v_1}, \dots, t_{v_n}) \in [\ell]^n$ **do**
3: Let $E_{(t_{v_1},\dots,t_{v_n})}$ be a multiwinner election on the candidate set A and the voter set V, where the preference list of voter v_i in the election $E_{(t_{v_1},\dots,t_{v_n})}$ is the same as the preference list of v_i in the election $E_{t_{v_i}}$
4: find a k-Borda Committee, S', for $E_{(t_{v_1},\dots,t_{v_n})}$
5: **if** $\texttt{Borda-Score}(S') \geq score_S$ **then**
6: $S = S'$
7: $score_S = \texttt{Borda-Score}(S')$
8: **if** $score_S \geq R$ **then return** S
9: **else return** NO

Theorem 3 *Max-k-Borda is* FPT*w.r.t.* $n + \ell$.

Proof. We run Algorithm 1 for all $R = 1$ to $n(mk - \binom{k}{2})$ (maximum possible Max-k-Borda score for n voters and m candidates) and return a committee with largest possible R. To prove the correctness of the algorithm, we prove that it returns a set S if and only if $((E_1, \dots, E_\ell), R, k)$ is a yes-instance of Max-k-Borda. In the forward direction, suppose that the algorithm returns a set S. We first note that $|S| = k$. Furthermore, there exists a vector $(t_{v_1}, \dots, t_{v_n}) \in [\ell]^n$ such that $\texttt{Borda-Score}(S) \geq R$ for the election $E_{(t_{v_1},\dots,t_{v_n})}$ (the election constructed in Step 3 of Algorithm 1). Now, as in Max-k-Borda the multimodal satisfaction of a voter is the maximum over the layers, it follows that the total multimodal satisfaction for S in the multimodal election $(E_1, \dots, E_\ell)$ cannot be smaller than its k-Borda score in the (nonmodal) election $E_{(t_{v_1},\dots,t_{v_n})}$, which is at least R, by Step 8 of Algorithm 1.

In the reverse direction, let S' be a solution of Max-k-Borda for the multimodal election $((E_1, \dots, E_\ell), R, k)$. For each voter $v_i \in V$, there exists a layer t_{v_i} which contributes to the total satisfaction which is at least R. As Algorithm 1 considers all vectors of such t_{v_i}'s, it will consider also one vector resulting at least R score. The running time of Algorithm 1 is $\mathcal{O}^\star(\ell^n)$[5]. □

Due to running time in Theorem 3, we obtain the following result.

Theorem 4 *Max-k-Borda is in P for constant n.*

Min-k-Borda

We show that Min-k-Borda is intractable even w.r.t. parameter $n + k + \ell$. Aziz et al. [3] proved that Egalitarian-k-Borda is W[1]-hard w.r.t. the number of voters, n, or the committee size, k [Theorem 2]; while the reduction holds for combined parameter $n + k$, as in the reduction, the number of voters is $k + 2\binom{k}{2} + 3$ and the committee size is $k + \binom{k}{2}$. Moreover, their reduction can be carried out in polynomial time. Hence, due to Proposition 2, we obtain the following:

Proposition 5 *Min-k-Borda is* NP*-hard and W*[1]*-hard w.r.t.* $k + \ell$ *even for* $n = 1$.

Sum-k-Borda and Pareto-k-Borda

We obtain tractability in polynomial time; due to Proposition 1 and the fact that k-Borda can be solved in polynomial time [17], we have the following result.

5 $\mathcal{O}^\star$ notation suppresses the polynomial factor. That is, $O(f(k)n^{O(1)}) = \mathcal{O}^\star(f(k))$.

Corollary 3 *Sum-k-Borda and Pareto-k-Borda are in P.*

Vector-k-Borda

As argued for Min-k-Borda, due to Proposition 4, we have:

Proposition 6 *Vector-k-Borda is* NP*-hard and W*[1]*-hard w.r.t.* $k+\ell$ *even for* $n = 1$.

6 Multimodal CC

In this section, we study adaptations of CC to the multimodal setting.

Max-CC

Corollary 1 shows intractability of Max-CC for parameter $k+\ell$. The number of voters, n, helps, though.

Proposition 7 *Max-CC is* FPT*w.r.t.* n.

Proof. We adapt the corresponding algorithm for CC [5]: We guess a clustering of the voters (i.e., a partition of the voter set V into at most k parts, V_z, $z \in [k]$); and then, find a maximum matching in a complete bipartite graph where the partition of voters are to the left and all candidates are to the right, and the weight on an edge between a voter part V_z (on the left) and an candidate c (on the right) equals the sum of multimodal satisfaction given to the voters of V_z by the candidate c. We construct a committee by including all the candidates which are saturated by matching. If the size of maximum matching is less than k, say k', then we include $k - k'$ candidates arbitrarily in the committee. The algorithm outputs a committee that has the largest Max-CC score over all the committees constructed in this algorithm.

For correctness, let S be a solution to Max-CC. Let $P_1, \ldots, P_z$, where $z \leq k$, be the induced partition of the voter set V corresponding to S: Each voter in P_i, for $i \in [z]$, is assigned to the same candidate of S. Let S' be the committee returned by the algorithm when we consider the specific partition $P_1, \ldots, P_z$ (we consider this partition as we consider all possible partitions of V of size at most k). Now, we claim that the total multimodal satisfaction for S' is the same as for S. First, it cannot be larger as S is an optimal solution. Furthermore, let M be a matching obtained by choosing an edge between P_i, $i \in [z]$, and the candidate of S that is assigned to the vertices of P_i. Now, if the committee score for S' is less than S, then the weight of the matching M', which is returned by the algorithm, is less than the weight of the matching M; thus, a contradiction to M' being a maximum matching. Since the algorithm outputs a committee that has the largest Max-CC score over all the committees constructed in this algorithm, the Max-CC score of the committee returned by the algorithm is the same as the Max-CC score of S, as it cannot be larger. The running time is $\mathcal{O}^\star(n^n)$ as the number of parts in a partition cannot be larger than n. □

Min-CC

Corollary 1 shows intractability of Min-CC for parameter $k+\ell$. Due to Proposition 2, and NP-hardness and W[2]-hardness of Egalitarian-CC with respect to the committee size k [5], we obtain:

Corollary 4 *Min-CC is* NP*-hard and W*[2]*-hard w.r.t.* k *even for* $n = 1$.

Sum-CC

Corollary 1 shows intractability of Sum-CC for parameter $k+\ell$. Due to Proposition 3 and the fact that CC is NP-hard and W[2]-hard w.r.t. k [5], we obtain the following result.

Proposition 8 *Sum-CC is* NP*-hard and* W[2]*-hard w.r.t.* k *even for* $n = 1$.

Here, the combined parameter $n+\ell$ does help. Due to Proposition 1 and the fact CC is FPTw.r.t. n [5], we obtain the following result.

Proposition 9 *Sum-CC is* FPT*w.r.t.* $n+\ell$.

Pareto-CC

Here, we identify some tractable cases. For the number of voters, n, we have the following.

Proposition 10 *For constant* n*, Pareto-CC is in P.*

Proof. Given a multimodal election $E = ((E_1, \ldots, E_\ell), k)$, we proceed greedily as follows: Initially, set $k' := k$ and $S = \emptyset$. In each iteration we will add candidates to S and decrease k' accordingly. In iteration i, we consider layer i, $i \in [\ell]$ and perform as follows: If $k' \geq n$, then we add all candidates that are ranked first by at least one voter in layer i, but are not yet in S, and we decrease k' by the number of such candidates. Otherwise, if $k' < n$, then we consider all possibilities of adding k' candidates to S, adding to S such a set which results in no other such set dominating it. Such a set must exist, due to arguments similar to those given for Observation 1. The running time is $\mathcal{O}(\ell + m^{\mathcal{O}(1)})$ as the last iteration can be carried out in polynomial time as k' is constant in the last iteration (since $k' < n$); and all the previous iterations can be performed in $\mathcal{O}(\ell)$ time.

For correctness, assume that the algorithm halts in iteration i, let S be the resulting committee, and let S_i be the set of k' candidates added by the algorithm in the ith iteration (so that $S \setminus S_i$ contains those candidates added in all iterations except the last). Then, any other committee $S' \neq S$ for which $S \setminus S_i \subseteq S'$ cannot dominate S, as S_i was selected to be not dominated. Furthermore, for any other committee $S' \neq S$ for which $S \setminus S_i \nsubseteq S'$, there exists at least one layer $j \in [i-1]$ for which the CC score of S' for election E_j is less than the CC score of S for E_j. □

Due to Observation 2 and Proposition 9, we have the following:

Corollary 5 *Pareto-CC is* FPT*w.r.t.* $n+\ell$.

Vector-CC

Due to Proposition 4 and the fact that Egalitarian-CC is NP-hard and W[2]-hard w.r.t. the committee size k [5], we have following result.

Corollary 6 *Vector-CC is* NP*-hard and* W[2]*-hard w.r.t.* k *even for* $n = 1$.

7 Domain Restrictions

Besides parameterized complexity, here we consider domain restrictions, as another way of coping with the computational intractability of our problems. A domain restriction D is a subset of all possible preference profiles.

A particularly popular domain restriction is the single-peaked domain, originally proposed by Black [6]. For an ordering π over the candidates in a profile, a voter v is *single-peaked w.r.t.* π if, for each pair of candidates a, b it holds that, if either $a \; \pi \; b \; \pi \; top(v)$ or $top(v) \; \pi \; b \; \pi \; a$ holds, where $top(v)$ is the candidate ranked first by v, then v ranks b above a; a (nonmodal) ordinal election $E = (V, A)$ – V is a set of linear orders over A – is *single-peaked* if there is an ordering π such that all voters $v \in V$ are single-peaked w.r.t. π. We refer to the survey of Elkind et al. [14] for a more elaborate exposition on domain restrictions. In particular, a polynomial time algorithm, computing winning committees under CC for single-peaked profiles is known [5].

Next we describe a general way of adapting a domain restriction D (such as, e.g., single-peakedness) to our multimodal model.

Definition 4 (Local-D) *Let D be a (nonmodal) domain restriction. Then, a multimodal profile satisfies* Local-D *if each layer E_z, $z \in [\ell]$, satisfies D.*

While the above definition is general for any nonmodal domain restriction D, here we are interested in D being the single-peaked domain; in particular, Definition 4 implies that a multimodal profile is Local-single-peaked (Local-SP, in short) if each layer is single-peaked – possibly, each w.r.t. a different ordering π. For the single-peaked domain we offer the following, stronger multimodal domain restriction.

Definition 5 (Global-single-peaked) *A multimodal profile is Global-single-peaked (Global-SP, in short) if there is an ordering π over the candidates such that, for each voter $v \in V$ and each layer $z \in [\ell]$, it holds that v's preference order in the zth layer is single-peaked w.r.t. π.*

Global-single-peaked is indeed stronger than Local-single-peaked (the set of Global-single-peaked multimodal profiles is contained in the set of Local-single-peaked multimodal profiles).

Observation 5 *A multimodal election E which is Global-SP is also Local-SP.*

Some intractability results proved in the previous sections carry over also to the Local-SP multimodal domain. Specifically, note that, if a multimodal multiwinner voting rule $\mathcal{R}$ is NP-hard (W-hard) for $n = 1$, then $\mathcal{R}$ is NP-hard (respectively, W-hard) even when he multimodal profile is Local-SP, as a nonmodal election containing only a single voter is always single-peaked (e.g., by setting π to be the preference order of the voter). Thus, we have the following:

Corollary 7 *Min-k-Borda and Vector-k-Borda are* NP*-hard and* W[1]*-hard w.r.t. $k + \ell$ even for Local-SP, and $n = 1$.*

Corollary 8 *Min-CC, Sum-CC, and Vector-CC are NP-hard and* W[2]*-hard w.r.t. k even for Local-SP, and $n = 1$*

We next study the computational complexity of Max-CC for Global-SP profiles, and obtain intractability. Due to Observation 5, this also implies intractability of Max-CC for Local-SP profiles.

Theorem 5 *Max-CC is* NP*-hard even for Global-SP.*

Proof. We describe a polynomial time reduction from the NP-hard problem Vertex Cover [21], in which given a graph G, and an integer k; we shall decide the existence of k-sized subset of vertices, say S, such that for every edge in G at least one of its end-point is in S. Given an instance (G, k) of vertex cover, we construct an instance $(E_1, \ldots, E_\ell, R, k')$ of Max-CC as follows: For every vertex x_i in the graph G, we add a candidate c_{x_i}; and for every edge e in G, we add a voter v_e in the election. We set $\ell = 2$ and $k' = k$.

Next we describe the preference list of the voters for both elections. Let $V(G) = \{x_1, \ldots, x_p\}$. We first fix an ordering of the candidates, say $(c_{x_1}, \ldots, c_{x_p})$. Let $e = x_i x_j$, where $i, j \in [p]$ is an edge in G. For the voter v_e, we have x_i as the first preferred candidate in election E_1 and x_j as the first preferred candidate in election E_2. We rank the other candidates in both elections such that the preference list is single-peaked w.r.t. the axis $(c_{x_1}, \ldots, c_{x_p})$; for example, $x_i \succ x_{i-1} \succ \ldots \succ x_1 \succ x_{i+1} \succ \ldots \succ x_n$. Let the number of edges in G be q. We set $R = (p - 1)q$.

For correctness, in the forward direction, let S be a solution to (G, k). Let S' be the set of candidates corresponding to vertices in S. Since for every voter, its first preferred candidate either in election E_1 or E_2 is in S', Max-CC score of S' is R. Moreover, since $|S| = k$, S' is a solution to $(E_1, \ldots, E_\ell, R, k')$. The reverse direction follows from the fact that every winning committee for $(E_1, \ldots, E_\ell, R, k')$ contains first preferred candidate of every voter in either election E_1 or E_2, otherwise the score of the committee is less than R. $\square$

The complexity of Max-k-Borda, and Pareto-CC for Local-SP profiles remains open.

While Corollaries 7, and 8 show that Local-SP does not break the intractability in certain cases, Global-SP indeed is sometimes more helpful: In particular, we obtain that Sum-CC and Pareto-CC can be solved in polynomial time for Global-SP multimodal profiles by "flattening" the instance of Sum-CC (Pareto-CC), as such a flattening reduces the instance to a CC instance with a nonmodal SP profile (for which polynomial time algorithm exists [Theorem 8, [5]]). The correctness follows similarly as argued in the proof of Proposition 1.

Proposition 11 *Sum-CC and Pareto-CC are in P for Global-SP.*

Remark 3 *It is worth mentioning that the technique, proposed by Peters [25], of formulating winner determination for single-peaked profiles to an integer linear program (ILP) with totally unimodular constraints matrix, does not seem to be useful for our setting: In particular, we could not formulate totally unimodular ILPs for any of ours cases. It will be interesting to either formulate such ILP or show that no such ILP exists. Moreover, we tried to adapt the dynamic programming algorithm for CC under SP profile proposed by Betzler et al. [5] for Max-CC and Min-CC under Global-SP; unfortunately, this route also does not seem to carry over.*

8 Outlook

We have defined a model of multimodal elections, capturing various scenarios in which voters may wish to provide not just one preference order over the set of candidates, but several. We have studied the computational complexity of several adaptations of k-Borda and CC to our model, observing a rich complexity landscape and identifying tractable cases for certain parameters and domain restrictions.

Some future research directions are (1) studying adaptations of further CSRs, such as SNTV, Bloc, and PAV; (2) defining and studying adaptations of non-CSRs, such as STV; and (3) considering other ways of breaking intractability, such as with approximation algorithms.

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