Election Score Can Be Harder than Winner

Zack Fitzsimmons[1] and **Edith Hemaspaandra**[2]

Abstract.   Voting rules based on scores generally determine the winner by computing the score of each candidate and the winner is the candidate with the best score. It would be natural to expect that computing the winner of an election is at least as hard as computing the score of a candidate. We show that this is not always the case. In particular, we show that for Young elections for dichotomous preferences the winner problem is easy, while determining the score of a candidate is hard. This complexity behavior has not been seen before and is unusual. The easiness of the winner problem for dichotomous Young crucially uses the fact that dichotomous preferences guarantee the transitivity of the majority relation. In addition to dichotomous preferences we also look at single-peaked preferences, the most well-studied domain restriction that guarantees the transitivity of the majority relation. We show that for the three major hard voting rules and their natural variants, dichotomous Young is the only case where winner is easy and score is hard. This also solves an open question from Lackner and Peters (AAAI 2017), by providing a polynomial-time algorithm for Dodgson score for single-peaked electorates.

1 Introduction

Voting rules based on scores generally find a winner by computing the score of each candidate and a candidate is a winner if they have the best score. Most voting rules have easy score problems and so finding a winner in this way is also easy. The three most-commonly studied voting rules where this is not the case are Young [38], Dodgson [11], and Kemeny [23] (and its qualitative variant Slater [33]). The score problem for each of these rules is NP-complete and each corresponding winner problem is Θ_2^p-complete [2, 30, 20, 21].

In general it would seem that determining if a candidate is a winner of an election is at least as hard as determining the score of a candidate. We show that this is not always the case. We show that for Young elections for dichotomous preferences the score problem is hard and the winner problem is easy. This is a behavior that has not been seen before in the computational study of voting. Dichotomous preferences are a very natural way for voters to state their preferences, where they approve of a subset of the candidates and disapprove the others. In many situations voters may not be able to state strict preferences over all of the candidates, but can state their approval/disapproval for each candidate. Approval voting is the most well-known voting rule that uses dichotomous preferences as input. Additionally, most voting rules have definitions that apply or can be naturally extended to dichotomous preferences, and can keep many of the desirable social-choice properties they have in the total-order case (e.g., this is the case for Kemeny [6]).

The behavior of Young for dichotomous preferences is quite unusual and does not occur for any of the obvious variations of the setting. When we instead consider strongYoung elections, in which the score is based on how far removed the candidate is from becoming a Condorcet winner (rather than a weak Condorcet winner as in Young), the score and winner problems for dichotomous preferences are hard. This is also the case when the electorate is trichotomous (each voter ranks their most preferred, middle preferred, and least preferred candidates). Additionally, natural variants of Dodgson and Kemeny for dichotomous preferences do not exhibit Young's anomalous behavior: Their score and winner problems are in P.

The computational easiness of the winner problem for Young for dichotomous preferences follows from the fact that when the electorate is dichotomous, the majority relation is transitive. To further explore this effect, we consider additional preference domains. We examine how having single-peaked preferences and how having single-crossing preferences affects the complexity of the winner and score problems for Young, Kemeny, and Dodgson elections. Single-peaked preferences [5] and single-crossing preferences [27] are the two most-commonly studied domain restrictions that guarantee transitivity of the majority relation and they each model the preferences of the voters with respect to a single polarizing issue. In the case of single-peaked preferences, the axis is a total ordering of the candidates where candidates on the leftmost/rightmost ends of the axis represent the extremes of the issue. In the case of single-crossing preferences, the voters can be ordered along an axis (in a political setting we can think of of the leftmost (rightmost) voters as the most-liberal (most-conservative), and for each choice between two alternatives, the preferences of the voters swaps at most once when moving left-to-right along the axis. We follow the model from Walsh [37] where the single-peaked or single-crossing axis is given as part of the input. However, each of these can be computed in polynomial time [4, 12]. When electorates are single-peaked or single-crossing many important computational problems that are computationally difficult in general become easy (see, e.g., [14, 7, 32, 26]).

We consider the complexity of the score and winner problems for Young, strongYoung, Kemeny, Dodgson, and weakDodgson elections for single-peaked and for single-crossing preferences. Except for Dodgson and weakDodgson score for single-crossing preferences, which remain open, all score and winner problems are in P. In proving our results, we solve an open problem from Lackner and Peters [24] by showing that computing the Dodgson score of a candidate is in P for single-peaked preferences.

Our study of the relationship between score and winner is significant in two ways. The first one is that it shows that we should not always use the score problem to compute the winner (as in dichotomous Young). The second is that by carefully looking at the score problem, we may come up with easier winner problem algorithms (as in Dodgson for single-peaked preferences).

[1] College of the Holy Cross, Worcester, MA USA
[2] Rochester Institute of Technology, Rochester, NY USA

2   Preliminaries

An election consists of a set of candidates C, and a set of voters V. Each voter $v \in V$ has a corresponding vote (preference order) over the set of candidates. The most commonly studied model is for voters to state their preferences as a total order, i.e., strictly ranking all of the candidates from most to least preferred. Another natural way for voters to state their preferences is as a dichotomous order, where each voter approves of a subset of the candidates and disapproves of the remaining candidates. More formally, given a set of candidates C, a dichotomous vote v partitions C into two sets, A and B (which may be empty), with the vote written as $(A > B)$, such that for all $a \in A$ and for all $b \in B$, v states $a > b$ (where $>$ denotes strict preference) and v does not state strict preference between candidates in A nor between candidates in B.

A voting rule is a mapping from an election (a set of candidates and a set of voters) to a subset of the candidate set referred to as the winner(s). We are interested in the relationship between the complexity of the score and winner problems for voting rules so we focus on voting rules with computationally difficult score problems: Young [38], Dodgson [11], and Kemeny [23], and some of their related variants.

For our results and to define the voting rules mentioned above, it will be useful to refer to Condorcet winners and weak Condorcet winners. A Condorcet winner is a candidate in an election that beats each other candidate by pairwise majority. Similarly, a weak Condorcet winner is a candidate that beats-or-ties each other candidate by pairwise majority.

The Young score of a candidate is the size of a largest subset of voters for which the candidate is a weak Condorcet winner. A candidate is a Young winner if the candidate has maximum Young score. Notice that the definition of Young applies to dichotomous preferences. We also consider the common slight variant of Young called strongYoung, in which the goal is to make a candidate a Condorcet winner (instead of a weak Condorcet winner).

For total order votes, the Dodgson score of a candidate is the fewest number of swaps between adjacent candidates in the voters' rankings such that the candidate can become a Condorcet winner. A candidate is a Dodgson winner if the candidate has minimum Dodgson score. The definition for weakDodgson is the same except we consider if a candidate can become a weak Condorcet winner. For dichotomous preferences, a natural analogue for Dodgson is to move candidates between the two groups in a dichotomous vote, which keeps the votes dichotomous, e.g., given the vote $(\{a, b\} > \{c, d\})$ we can move c up to get $(\{a, b, c\} > \{d\})$ with one move.

A total order $>$ is a Kemeny consensus if the sum of Kendall tau distance to the voters is minimal, i.e., $\sum_{a>b} N(b, a)$ is minimal, where $N(b, a)$ denotes the number of voters that state $b > a$. The Kemeny score of a candidate p is the minimal sum of Kendall tau distance to the voters for a total order that has p as the most preferred. For dichotomous preferences, we consider two variants of Kemeny introduced by Zwicker [39] called $(2, 2)$-Kemeny and $(2, m)$-Kemeny. For both rules, the votes are dichotomous.[3] In $(2, 2)$-Kemeny, the consensus is also dichotomous. In $(2, m)$-Kemeny, the consensus is a total order. Having a total order consensus allows expressing more information about the electorate, which sometimes may be more appropriate [1]. In $(2, 2)$-Kemeny and $(2, m)$-Kemeny, the score of ranking $>$ is $\sum_{a>b}(N(a, b) - N(b, a))$ and we are look-ing to *maximize* the score.

The Slater rule is the qualitative version of the Kemeny rule. A total order $>$ is a Slater order if it is closest to the pairwise majority relation $>_m$ induced by the voters, in the sense that we maximize the score, the number of ordered pairs of candidates (a, b) such that ($a > b$ if and only if $a >_m b$). Slater winners are those candidates that are ranked first in a Slater order. The score of a candidate p is the maximal score of a Slater order that ranks p first.

The definition of Slater order also naturally extends to weak orders. A k-chotomous order $>$ is a k-chotomous Slater order if it is a k-chotomous order closest to $>_m$ in the sense described above. In $(2, k)$-Slater, the voters are dichotomous and the Slater order is k-chotomous. In $(2, m)$-Slater, the voters are dichotomous and the Slater order is a total order.

We now define the score and winner decision problems with Young as an example. Note that in Young we are looking to *maximize* the score (as is the case for $(2, 2)$-Kemeny and $(2, m)$-Kemeny), but in Dodgson and Kemeny we are looking to *minimize* the score and so the decision problems for these rules must be adjusted accordingly.

Name: YoungScore
Given: An election (C, V), a candidate $p \in C$, and a number k.
Question: Is the Young score of p at least k?

Name: YoungWinner
Given: An election (C, V) and a candidate $p \in C$.
Question: Does p have highest Young score?

Our computational results involve the classes P, NP, and Θ_2^p. The class Θ_2^p was first studied by Papadimitriou and Zachos [28], named by Wagner [36], and shown by Hemachandra [19] to be equivalent to $P_{||}^{NP}$, the class of problems solvable by a polynomial-time oracle machine that asks all of its queries to an NP oracle in parallel. Note that $NP \cup coNP \subseteq \Theta_2^p \subseteq P^{NP}$.

3   Dichotomous Preferences

For total order votes, the winner problems for Young, strongYoung, Dodgson, weakDodgson, and Kemeny are Θ_2^p-complete [7, 30, 20, 7, 21]. The Θ_2^p upper bounds for these problems are shown as follows: Use the associated NP-complete score problem, and compute the scores of all candidates in parallel in polynomial time.

A voting rule is weakCondorcet-consistent if on every input that has at least one weak Condorcet winner, the winners of the voting rule are exactly the weak Condorcet winners. If an election has dichotomous votes then it has at least one weak Condorcet winner [22], which implies the following theorem.

Theorem 1 *For each voting rule that is weakCondorcet-consistent for dichotomous preferences, the winner problem for dichotomous preferences is in* P.

3.1   Young Elections

Theorem 2 *For dichotomous preferences, YoungWinner is in* P *and YoungScore is* NP-*complete.*

Proof.   Since Young is weakCondorcet-consistent [15], Young-Winner in P follows from Theorem 1.

We will now show that YoungScore remains NP-complete for dichotomous preferences. And so the "natural" way of deciding Young-Winner by using YoungScore as an oracle is not optimal.

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[3] The definition of dichotomous votes used by Zwicker [39] requires that each of the two groups are nonempty. However, this does not make a difference in the results in our paper.

We reduce from Independent-Set. Given a graph $G = (V, E)$, let the candidate set be $E \cup \{p\}$ and let the voter set consist of the following voters.

- For each vertex $v \in V$, one voter corresponding to v voting ($\{e \in E \mid v \in e\} > \cdots$).
- One voter voting ($\{p\} > \cdots$).

Note that the Young score of p is $\alpha(G) + 1$, where α is the independence number of G, i.e., the size of a maximum independent set of G. This score is realized by the voter that ranks p first and a set of voters corresponding to a maximum-size independent set of G. ❑

Note that the proof of the theorem above does not contradict the previous statement that YoungWinner is in P, since p is clearly in general not a Young winner. Also note that this construction does not give NP-completeness for YoungScore for total orders. Not surprisingly, it turns out that that problem is also NP-complete (even to approximate [8]). Somewhat surprisingly, a direct proof of NP-completeness is not given in the literature, but it is implicit in the proof of Rothe, Spakowski, and Vogel [30].[4]

It may be surprising that the construction for total orders is harder than the one for dichotomous preferences, because it may seem that a problem on total orders would be harder than the analogous problem for dichotomous preferences. However, for the plurality rule, control by adding voters can be NP-complete for votes with ties [16] while it is in P for total orders [3]. And it is easy to see that dichotomous votes suffice to get NP-completeness.

Add an additional vote ($\{p\} > \cdots$) to get the following.

Theorem 3 *strongYoungScore for dichotomous preferences is* NP-*complete.*

We now consider the ranking problem, which is the problem that asks given two candidates p and r whether the score of p is at least the score of r. This will be used as an intermediate problem to show the hardness of strongYoungWinner and for Young, to further contrast the complexity of the score, ranking, and winner problems. The construction also shows that the YoungLoser problem, which asks if a candidate has lowest Young score, is Θ_2^p-complete, in stark contrast to the YoungWinner problem.

Theorem 4 *YoungRanking for dichotomous preferences is* Θ_2^p-*complete.*

Proof. Reduce from Min-Card-Independent-Set-Compare, in which we are given two graphs G and H with the same number of vertices and we ask if $\alpha(G) \geq \alpha(H)$. This problem is Θ_2^p-complete [35] (for an explicit proof, see [34]).

Without loss of generality, assume that G and H both contain at least one edge (so that $\alpha(G) < \|V(H)\|$ and $\alpha(H) < \|V(G)\|$) and that the sets of vertices are disjoint. Our reduction generalizes the YoungScore reduction from the proof of Theorem 2. We will ensure that the Young score of p is $\alpha(G) + 1 + \|V(H)\| + 1$ and that the Young score of r is $\alpha(H) + 1 + \|V(G)\| + 1$. This proves the theorem, since $\|V(G)\| = \|V(H)\|$.

Let the candidate set be $E(G) \cup E(H) \cup \{p, r\}$ and let the voter set consist of the following voters.

Type I For each $v \in V(G)$, one voter corresponding to v voting ($\{r\} \cup E(H) \cup \{e \in E(G) \mid v \in e\} > \cdots$).
Type II One voter voting ($E(H) \cup \{p, r\} > \cdots$).
Type III For each $v \in V(H)$, one voter corresponding to v voting ($\{p\} \cup E(G) \cup \{e \in E(H) \mid v \in e\} > \cdots$).
Type IV One voter voting ($E(G) \cup \{p, r\} > \cdots$).

Note that to realize the Young score of p, we should always include all the Type III and Type IV voters. The rest of the argument is as in the proof of Theorem 2. Note that p ties-or-beats each candidate in $E(H)$, since we are including $\alpha(G) < \|V(H)\|$ Type I votes. ❑

Add a vote ($E(H) \cup \{p, r\} > \cdots$) and a vote ($E(G) \cup \{p, r\} > \cdots$) to get the following.

Theorem 5 *strongYoungRanking for dichotomous preferences is* Θ_2^p-*complete.*

As stated in Theorem 2, YoungWinner is in P for dichotomous votes. In contrast, the winner problem for strongYoung for dichotomous votes is Θ_2^p-complete.

Theorem 6 *strongYoungWinner for dichotomous preferences is* Θ_2^p-*complete.*

Proof. The main insight here is that we can always make sure that a candidate c's strongYoung score is 0, by adding a candidate c' and making sure that c and c' are tied in every vote. We adapt the construction used to show that strongYoungRanking for dichotomous preferences is Θ_2^p-complete from Theorem 5. For each $e \in E(G) \cup E(H)$, add a candidate e' and make sure that e and e' are tied in every vote. ❑

It is interesting to see that our Θ_2^p-completeness proofs for dichotomous preferences are also significantly easier than those for total orders [30].

Note that the approach from Theorem 6 does not work for YoungWinner. Rothe, Spakowski, and Vogel [30] reduce the strongYoungRanking to strongYoungWinner problem by replacing each candidate g other than c and d by $\|V\|$ new candidates $g_0, \ldots, g_{\|V\|-1}$, and by replacing the occurrence of g in the ith voter by $g_i > g_{i+1} > \cdots > g_{i+\|V\|-1}$ (modulo $\|V\|$). This does not change the scores of c and d, but ensures that the strongYoung score of every other candidate is at most 1, and so we can ensure that these candidates are never winners in the image of the reduction. Note that this construction does not work for dichotomous preferences (which is consistent with the Theorem 2 result that YoungWinner for dichotomous preferences is in P). The construction also does not work for trichotomous preferences, or indeed for any k-chotomous preferences.

However, we can adapt the construction used to show that YoungRanking for dichotomous preferences is Θ_2^p-complete from Theorem 4.

Theorem 7 *YoungWinner for trichotomous preferences is* Θ_2^p-*complete.*

Proof. The main insight is that we can make sure that candidate c's Young score is relatively low, by adding candidates c' and c'', tied with c in each original vote, and adding, for some big integer B, the following $6B$ voters.

- B voters voting ($c > c' > \cdots$).
- B voters voting ($\cdots > c > c'$).
- B voters voting ($c' > c'' > \cdots$).

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[4] It is interesting to note that the NP-completeness of YoungScore does not directly follow from the Θ_2^p-completeness of YoungWinner and strongYoungWinner: Under the assumption that NP does not have p-measure 0, there exists a set A that is NP-complete under truth-table reductions, but not NP-complete (under many-one reductions) [25]. Note that YoungWinner and strongYoungWinner are in P_{tt}^A, though A is not NP-complete.

- B voters voting $(\cdots > c' > c'')$.
- B voters voting $(c'' > c > \cdots)$.
- B voters voting $(\cdots > c'' > c)$.

Note that the Young scores of c, c', and c'' are at most $n - 2B$, where n is the number of voters.

We adapt the construction used to show that YoungRanking is Θ_2^p-hard from Theorem 4. For each $e \in E(G) \cup E(H)$, we add e' and e'', we replace each occurrence of e in the original votes by e, e', e'', and we add $6\|V(G)\|$ voters as specified above, taking $B = \|V(G)\|$. For readability, we write $\widehat{E}$ for $\{e, e', e'' \mid e \in E\}$. This gives the following set of voters.

Type I   For each vertex $v \in V(G)$,

- One voter corresponding to v voting
  $(\{r\} \cup \widehat{E(H)} \cup \{e \in \widehat{E(G)} \mid v \in e\} > \cdots)$.

Type II

- One voter voting $(\widehat{E(H)} \cup \{p, r\} > \cdots)$.

Type III   For each vertex $v \in V(H)$,

- One voter corresponding to v voting
  $(\{p\} \cup \widehat{E(G)} \cup \{e \in \widehat{E(H)} \mid v \in e\} > \cdots)$.

Type IV

- One voter voting $(\widehat{E(G)} \cup \{p, r\} > \cdots)$.

Type V   For each $e \in V(G) \cup V(H)$,

- $\|V(G)\|$ voters voting $(e > e' > \cdots)$.
- $\|V(G)\|$ voters voting $(\cdots > e > e')$.
- $\|V(G)\|$ voters voting $(e' > e'' > \cdots)$.
- $\|V(G)\|$ voters voting $(\cdots > e' > e'')$.
- $\|V(G)\|$ voters voting $(e'' > e > \cdots)$.
- $\|V(G)\|$ voters voting $(\cdots > e'' > e)$.

It is immediate that for all $c \neq p, r$, $\text{YoungScore}(c) \leq n - 2\|V(G)\|$. We can show that $\text{YoungScore}(p) = n - \|V(G)\| + \alpha(G)$ and $\text{YoungScore}(r) = n - \|V(H)\| + \alpha(H)$. Since $\|V(G)\| = \|V(H)\|$, it follows that $\alpha(G) \geq \alpha(H)$ if and only if p is a Young winner, which completes the proof. Details of the proof can be found in the technical report [17].   ❑

3.2   Dodgson Elections

We now show that the complexity behavior of Young for dichotomous preferences does not occur for Dodgson and Kemeny for dichotomous preferences.

Recall that for Dodgson for dichotomous preferences a candidate can only be swapped up from disapproved to approved or swapped down from approved to disapproved. So unlike in the case for total orders, to determine the score of a candidate we cannot just consider swaps that move that candidate up. For example, given the vote $(\{a, b\} > \{c, d\})$, for c to beat a pairwise, we first need to move c up to get $(\{a, b, c\} > \{d\})$ and then move a down to get $(\{b, c\} > \{a, d\})$. (We could of course also do these moves in the reverse order.) With this in mind, we can show that DodgsonScore and weakDodgsonScore are each in P (and so are the corresponding winner problems).

Theorem 8   *DodgsonScore and weakDodgsonScore, for dichotomous preferences, are each in* P.

Proof.   First notice that moving p up from the set of disapproved candidates in a vote to the set of approved candidates decreases $N(a, p) - N(p, a)$ by 1 for every candidate $a \in C - \{p\}$. And it is not possible to decrease $N(a, p) - N(p, a)$ by more than 1 with one move. So, the weakDodgson score of p is the max over all candidates $a \in C - \{p\}$ of $N(a, p) - N(p, a)$: We can make p a weak Condorcet winner by moving p up from disapproved to approved in that many votes and it is easy to see that there are enough votes to do this. So, weakDodgsonScore is in P.

For DodgsonScore, if p starts out as a Condorcet winner, the score is 0. Otherwise, we need to move p up in one more vote compared to what was needed to make p a weak Condorcet winner. If no such vote exists then for some candidate a, $a > p$ in each original vote. In this case, move p up in every vote, and then for every candidate a for which a and p are tied, move that candidate down. So, DodgsonScore is also in P.   ❑

3.3   Kemeny Elections

Zwicker [39] shows that the winner problems for $(2, 2)$-Kemeny and $(2, m)$-Kemeny are in P. An easy way to see this for $(2, 2)$-Kemeny is because this is the same as the mean rule [39]. Computing the score of a candidate p is a little harder, since we need to rank p (tied for) first in a dichotomous ranking, and so we are not ranking according to approval score (as in the mean rule). However, a similar argument to what is used in the proof of Lemma 3 from Zwicker [39], which computes a k-chotomous consensus, can be used to show that the score problem for $(2, 2)$-Kemeny is also in P.

For Kemeny versions that require a total order consensus (such as $(2, m)$-Kemeny) we will show that the score problem polynomial-time Turing-reduces to the winner problem, which implies that the score problem is in P if the winner problem is in P. And it follows that $(2, m)$-KemenyScore is in P.

Theorem 9   *For each preference domain $\mathcal{D}$ that is closed under deletion of candidates, KemenyScore for $\mathcal{D}$-preferences polynomial-time Turing-reduces to KemenyWinner for $\mathcal{D}$-preferences with a total order consensus.*

Proof.   The reduction in the theorem above works in the following way. To compute the Kemeny score of candidate p, we need to compute the score of a total order that ranks p first. So, put p first in the total order. The contribution of p to the score is independent of how the remaining candidates are ordered (for example, p contributes $\sum_{c \neq p}(N(p, c) - N(c, p))$ for $(2, m)$-Kemeny). For an optimal order, we need to order $C - \{p\}$ such that the order restricted to those candidates is optimal, i.e., we need to compute a Kemeny consensus of the electorate restricted to $C - \{p\}$.

So, delete p and repeatedly query whether a candidate a is a winner. If so, put a next in the order and delete a. This builds an optimal total order with p first. It is clear this argument holds for preference domains that are closed under deletion of candidates and this includes total order, dichotomous, single-peaked, and single-crossing preferences.   ❑

3.4   Slater Elections

For total-order votes, the score and winner problems for Slater are each NP-hard [2].

We first mention that the analogous result to Theorem 9 holds for Slater, with the proof using essentially the same argument.

Theorem 10 *For each preference domain $\mathcal{D}$ that is closed under deletion of candidates, SlaterScore for $\mathcal{D}$-preferences polynomial-time Turing-reduces to SlaterWinner for $\mathcal{D}$-preferences with a total order consensus.*

We now consider the complexity of the score and winner problems for Slater for dichotomous votes.

Theorem 11 *For all fixed k, the winner and score problems for $(2, k)$-Slater and $(2, m)$-Slater are in P.*

Proof. Since Slater is weakCondorcet-consistent, it follows from Theorem 1 that the winner problem for $(2, m)$-Slater is in P, and thus Theorem 10 implies that $(2, m)$-SlaterScore is in P.

It remains to show that $(2, k)$-SlaterScore is in P. We know from Inada [22] that the induced majority relation $>_m$ is a weak order. Below, we will prove the following lemma.

Lemma 12 *Let $>_m$ be a weak order majority relation and let $>$ be a k-chotomous weak order. If there exist two candidates a, b such that $a >_m b$ and $b > a$, then switching a and b in $>$ will give a k-chotomous weak order with higher Slater score.*

From this lemma, it follows that the Slater score of candidate p is witnessed by an order $>$ such that p is a top-ranked candidate and such that for all $a, b \neq p$, if $a >_m b$, then not $b > a$. What this means is that, apart from candidate p, the order witnessing the Slater score is not inconsistent with the majority relation.

Since k is constant, there are polynomially many sequences of positive integers $\ell_1, \ldots, \ell_k$ that add to m. View such a sequence as a k-chotomous order not inconsistent with $>_m$ that ranks p first. Note that this order is not unique, but the Slater score of such an order is unique. This gives a polynomial-time algorithm to compute the Slater score of p. ❑

It remains to prove Lemma 12.
Proof. It is immediate that switching a and b gives a k-chotomous order. It remains to show that the score of this order, which we will call $>'$, is higher than that of the original order $>$.

First note that $\{a, b\}$ contributes 0 to the Slater score of $>$ and 2 to the Slater score of $>'$, since $a >_m b$ if and only if $a >' b$ and $b >_m a$ if and only if $b >' a$. For $c, d \notin \{a, b\}$, the contribution of $\{c, d\}$ to the Slater score of $>$ and $>'$ are the same. We will now show that for every $c \notin \{a, b\}$, the contribution of $\{a, c\}$ plus the contribution of $\{b, c\}$ to the Slater score of $>'$ is never less than the contribution to the Slater score of $>$.

This proceeds by a simple case distinction shown in Table 1. We have five possibilities for $>_m$ and five possibilities for $>$. We handle each of the 25 cases separately for clarity. ❑

4 Single-Peaked Preferences

Single-peaked preferences model the preferences of the electorate with respect to a one-dimensional axis L, a total ordering of the candidates, where each voter has a single most preferred candidate (their peak) and candidates farther to the leftmost (rightmost) ends of the axis are strictly less preferred. More formally, for every triple of candidates $aLbLc$ or $cLbLa$, for every voter $v \in V$ if v states $a > b$ then v states $b > c$.

A real-life scenario for single-peaked preferences is voting for the temperature for a room. When presented with several options (candidates) for the temperature, the preferences of the voters can be explained by a left-to-right ordering of the options from the lowest to the highest, where each voter has a most-preferred temperature, and has strictly decreasing preferences for lower temperatures and has strictly decreasing preferences for higher temperatures.

The pairwise majority relation for single-peaked preferences is transitive. Brandt et al. [7] show that it follows from their Theorem 3.2 (the analogue of Theorem 1) that the winner problems for Kemeny, Young, and weakDodgson are in P. They also show that the winner problems for strongYoung and Dodgson are in P.

So, what happens to the single-peaked score problems of the rules mentioned above?

4.1 Dodgson Elections

Peters [29] states that "... while Brandt et al. (2015) give an efficient algorithm for finding a Dodgson *winner* in the case of single-peaked preferences, the problem of efficiently calculating *scores* appears to be open and non-trivial." The reason for this nontriviality is that after swapping, the electorate is not required to be single-peaked (see [7, Footnote 5]). This makes single-peaked Dodgson very different from single-peaked Kemeny and Young, where we will never get non-single-peaked electorates in the computation. We will show below that for Young and Kemeny, the single-peaked score problem is in P.

More surprisingly, we also show the following theorem.

Theorem 13 *DodgsonScore and weakDodgsonScore for single-peaked preferences are in P.*

Proof. We start with an example that shows the core argument of our algorithm. Suppose the societal axis is $a_1 L a_2 L a_3 L a_4 L p$ and suppose our election consists of the following types of votes. (In contrast to the results in the previous section, this result is about total-order votes. Here we are using sets to denote those candidates strictly ranked in that location of a total-order vote, e.g., the total-order votes $(x > y > z)$ and $(y > x > z)$ are each votes of the form $(\{x, y\} > z)$.)

- 10 votes of the form $(\{a_1, a_2, a_3, a_4\} > p)$.
- 50 votes of the form $(\{a_2, a_3, a_4\} > p > \cdots)$.
- 10 votes of the form $(\{a_3, a_4\} > p > \cdots)$.
- 20 votes of the form $(a_4 > p > \cdots)$.
- 11 votes of the form $(p > \cdots)$.

In order for p to become a Condorcet winner, every candidate other than p can be preferred to p by at most 50 voters. p does not need any votes over a_1, and we leave the 10 votes of the form $(\{a_1, a_2, a_3, a_4\} > p)$ as is. p needs 10 votes over a_2. In 10 of the votes of the form $(\{a_2, a_3, a_4\} > p > \cdots)$, we swap p to the top of the order (using 30 swaps total) and we leave the remaining 40 votes of this form as is. Note that in the resulting set of votes:

- 10 votes of the form $(\{a_1, a_2, a_3, a_4\} > p)$.
- 40 votes of the form $(\{a_2, a_3, a_4\} > p > \cdots)$.
- 10 votes of the form $(p > \{a_2, a_3, a_4\} > \cdots)$.

a_2, a_3, and a_4 are preferred to p by 50 voters. This implies that we need to swap p to the top of the preference order in all remaining votes, i.e., in all votes of the form $(\{a_3, a_4\} > p > \cdots)$ and in all

	$c > b > a$ $c >' a >' b$	$\{b,c\} > a$ $\{a,c\} >' b$	$b > c > a$ $a >' c >' b$	$b > \{a,c\}$ $a >' \{b,c\}$	$b > a > c$ $a >' b >' c$
$c >_m a >_m b$	2+2; 2+2	2+1; 1+2	2+0; 0+2	1+0; 0+1	0+0; 0+0
$\{a,c\} >_m b$	1+2; 1+2	1+1; 2+2	1+0; 1+2	2+0; 1+1	1+0; 1+0
$a >_m c >_m b$	0+2; 0+2	0+1; 1+2	0+0; 2+2	1+0; 2+1	2+0; 2+0
$a >_m \{b,c\}$	0+1; 0+1	0+2; 1+1	0+1; 2+1	1+1; 2+2	2+1; 2+1
$a >_m b >_m c$	0+1; 0+0	0+1; 1+0	0+2; 2+0	1+2; 2+1	2+2; 2+2

Table 1. Case distinctions for the proof of Lemma 12. The content of each cell is the $\{a,c\}$ contribution for $>$ "+" the $\{b,c\}$ contribution for $>$, and the $\{a,c\}$ contribution for $>'$ "+" the $\{b,c\}$ contribution for $>'$.

votes of the form $(a_4 > p > \cdots)$. Note that no swap is wasted, and so p has a Dodgson score of 70.

We now show that this approach can be generalized to all single-peaked electorates. Recall that we follow the model from Walsh [37] where the axis L is given as part of the input. In the model where L is not given it suffices to compute any L (which can be done in polynomial time [4]) since the Dodgson score of a candidate depends on only the electorate. Let the societal axis be $a_1 L a_2 L \ldots L a_{m_a} L p L b_{m_b} L \ldots L b_2 L b_1$. Our goal is to compute the (Dodgson or weakDodgson) score of p. Let H (for "half") be the maximum number of voters that can prefer c to p while making p a winner. To be precise, in the case of Dodgson winner, H is $\lfloor (n-1)/2 \rfloor$, and in the case of weakDodgson winner, H is $\lfloor n/2 \rfloor$, where n is the number of voters. In the example above for Dodgson winner, H is 50. We will show that we can make p a winner without wasting swaps, i.e., in the election that makes p a winner, c is preferred to p by exactly H voters for every candidate c such that $N(c,p) > H$ (where $N(c,p)$ denotes the number of voters that prefer c to p in the original election), and by $N(c,p)$ voters for every candidate c such that $N(c,p) \leq H$. Since we are not wasting swaps, this witnesses the Dodgson score of p (with value $\sum_{c \neq p, N(c,p) > H}(N(c,p) - H)$).

Note that in each single-peaked vote, p is preferred to all a candidates or to all b candidates. So, when we swap p up in a vote, this will help p against a candidates or against b candidates, but not against both. This implies that we can treat the a candidates and b candidates separately. We will show that we can ensure that each a candidate is preferred to p by at most H voters without wasting swaps. The same holds for the b candidates, which proves the theorem.

As in the example, we classify the voters in terms of the set of a candidates that are preferred to p. For every $i, 1 \leq i \leq m_a$, let k_i be the number of voters with a vote of the form $(\{a_i, a_{i+1}, \ldots, a_{m_a}\} > p > \cdots)$ and let k_0 be the number of voters that prefer p to all a candidates. So, $N(a_i, p) = \sum_{j=1}^{i} k_j$, the total number of voters is $n = \sum_{j=0}^{m_a} k_j$, and H is defined as above.

If for all $1 \leq i \leq m_a$, $N(a_i, p) \leq H$, then p does not need to gain any points over a candidates. Otherwise, let i_0 be the smallest index such that $N(a_{i_0}, p) > H$. Note that $N(a_{i_0}, p) - H \leq k_{i_0}$. (This is immediate if $i_0 = 1$. If $i_0 > 1$, $N(a_{i_0-1}, p) \leq H$, and thus $N(a_{i_0}, p) \leq H + k_{i_0}$.) We now swap p to top of the preference order of $N(a_{i_0}, p) - H$ voters with a vote of the form $(\{a_{i_0}, a_{i_0+1}, \ldots, a_{m_a}\} > p > \cdots)$. Also, for all $i, i_0 < i \leq m_a$, we swap p to the top of the preference order of all the voters with a vote of the form $(\{a_i, a_{i+1}, \ldots, a_{m_a}\} > p > \cdots)$.

We now show that after swapping, each a candidate is preferred to p by at most H voters without wasting swaps. That is, a_i is preferred to p by H voters for all $i \geq i_0$ and a_i is preferred to p by $N(a_i, p)$ voters for all $i < i_0$.

For $i < i_0$, since we never swap p over a_i, it is immediate that a_i

is preferred to p by the same voters as in the original election. For $i \geq i_0$, a_i is preferred to p by all unchanged voters that prefer a_i to p. And there are exactly $\sum_{i=1}^{i_0-1} k_i + k_{i_0} - (N(a_{i_0}, p) - H) = N(a_{i_0}, p) - (N(a_{i_0}, p) - H) = H$ such voters.   ❏

The proof of the above theorem shows that for single-peaked electorates, the Dodgson score of p is equal to $\sum_{c \neq p, N(c,p) > H}(N(c,p) - H)$. This also gives a simple algorithm for DodgsonWinner. Though this problem was known to be in P [7], that algorithm was more complicated, since it did not look at the form of the entire single-peaked electorate in the way we do in our proof.

4.2 Young Elections

It is easy to see that we can adapt the construction for Dodgson from the previous section to Young. For every voter that we swap p to the top of the preference order of in the Dodgson construction, we now delete that voter. It is easy to see that this gives the minimum number of voters to delete in order for p to become a Condorcet (weak Condorcet) winner, and so the number of remaining voters is exactly the strongYoung (Young) score of p. This result also follows from the result that constructive control by deleting voters for Condorcet and weak Condorcet elections are each in P for single-peaked preferences [7].

4.3 Kemeny Elections

Brandt et al. [7] show that KemenyWinner for single-peaked preferences is in P. Since we are computing a total order consensus, it follows from Theorem 9 that KemenyScore for single-peaked preferences Turing-reduces to KemenyWinner for single-peaked preferences, and is thus also in P.

4.4 Slater Elections

For single-peaked preferences, computing a Slater winner is in P (see Conitzer [9]). As with Kemeny, since we are computing a total order consensus, it follows from Theorem 10 that SlaterScore for single-peaked preferences Turing-reduces to SlaterWinner for single-peaked preferences, and is thus also in P.

5 Single-Crossing Preferences

Another important domain restriction that ensures the majority relation is transitive is the single-crossing restriction [27], where the voters can be ordered along a one-dimensional axis $L = v_1 L v_2 L \ldots L v_n$ such that for each pair of candidates $a, b \in C$ all of the voters that state $a > b$ precede the voters that state $b > a$, i.e., there is a single crossing point for each pair.

A real-life scenario for single-crossing preferences is the traditional liberal-conservative political spectrum, for the case where voters' preferences depend only on where they fall on the political spectrum. The voters can be ordered along an axis where the leftmost (rightmost) voters are the most-liberal (most-conservative), and for each choice between two alternatives, the preferences of the voters swaps at most once when moving left-to-right along the axis. If the alternatives can also be placed on this spectrum (this happens for example when the alternatives correspond to political candidates), the preferences will also be single-peaked.[5]

As in the case for dichotomous preferences and for single-peaked preferences, we immediately obtain that YoungWinner and weakDodgsonWinner are each in P. It follows from Magiera and Faliszewski [26] that strongYoungScore and strongYoungWinner are each in P. As mentioned by Cornaz et al. [10], KemenyWinner for single-crossing elections is in P. And due to the weakCondorcet-consistency of Slater, SlaterWinner for single-crossing elections is also in P. This immediately implies using Theorem 9 that KemenyScore is in P, and using Theorem 10 that SlaterScore is in P.

An important property for the complexity is the fact that single-crossing elections have a variant of the median voter theorem, in that the median voter(s) represent the majority relation [31, 18]. The following corollary extracts the properties that we need.

Corollary 14 *Let $v_1 L v_2 L \ldots L v_n$ be a single-crossing order of voters. A candidate p is a weak Condorcet winner if and only if*

- *n is odd and p is the most preferred candidate of the median voter.*
- *n is even and for every $a \neq p$, a is preferred to p by at most one of the two median voters (and $a > p$ by exactly one of the median voters if and only if p and a are tied).*

Theorem 15 *YoungScore for single-crossing preferences is in* P.

Proof. Let L be the single-crossing order. (Recall that we follow the model from Walsh [37] where the axis is given as part of the input. In the model where L is not given it suffices to compute any L, which can be done in polynomial time [12], since the Young score of a candidate depends on only the electorate.) For every voter v that has p at the top of its preference order, keep a maximum odd number of voters such that v is the median voter. If the number of remaining voters is greater than the current best score, this becomes the current best score.

For every pair of voters v and w, if there is a candidate $a \neq p$ that is preferred to p by v and w, go to the next loop iteration. Otherwise, keep a maximum even number of voters such that v and w are the two median voters. If the number of remaining voters is greater than the current best score, this becomes the current best score. ❑

Theorem 16 *DodgsonWinner for single-crossing preferences is in* P.

Proof. This is trivial if there are an odd number of voters. So, assume that the number of voters is even. We will show that every Dodgson winner is a weak Condorcet winner, and that the Dodgson scores of weak Condorcet winners are easy to compute. This immediately implies the theorem, since the Dodgson winners are the weak Condorcet winners with lowest Dodgson score.

Suppose p is a weak Condorcet winner. It follows from Corollary 14 that for every candidate $a \neq p$, p is preferred to a by at least one of the median voters and a is preferred to p by one of the median voters if and only if a is tied with p pairwise. To make p a Condorcet winner with a minimal number of swaps, it suffices to swap p to the top of the two median voters. This gives a Dodgson score of $\|\{a \in C - \{p\} \mid p \text{ ties } a\}\|$.

If q is not a weak Condorcet winner, then there is a weak Condorcet winner p such that p beats q pairwise. It is easy to see that the Dodgson score of q is greater than the Dodgson score of p, since for every candidate $a \in C - \{p, q\}$, if p needs a vote over a, then so does q. In addition, q needs two votes over p. It follows that the Dodgson score of q is greater than $\|\{a \in C - \{p\} \mid p \text{ ties } a\}\|$ which is the Dodgson score of p. ❑

It is interesting to see that the algorithm for DodgsonWinner for single-crossing preferences from the previous theorem is similar to the algorithm for single-peaked preferences from Brandt et al. [7], in that there it also was shown that only weak Condorcet winners can be winners and how to compute the Dodgson score of a weak Condorcet winner. In Theorem 13, we finally solved the open problem of computing the Dodgson and weakDodgson scores for single-peaked preferences. We have not managed to solve the complexity of these problems for single-crossing electorates. Recall that it was crucial that in the single-peaked case we could always realize the score without wasting swaps. That is not the case in the single-crossing case, as shown by the following simple example. This gives some indication that the single-crossing case may be harder to handle.

Example 17 Consider the following four voters, single-crossing with respect to the ordering $v_1 L v_2 L v_3 L v_4$.

- v_1 voting $(a > b > p > c)$.
- v_2 voting $(a > b > p > c)$.
- v_3 voting $(a > c > p > b)$.
- v_4 voting $(a > c > p > b)$.

To become a Condorcet winner p needs one vote over b, one vote over c, and three votes over a. However, the three votes over a can only be obtained by wasting an extra swap over either b or c. So the Dodgson score of p is six.

6 Future Work

A concrete open question is the complexity of Dodgson score for single-crossing elections. Using the phrasing of Peters [29]: While we give an efficient algorithm for finding a Dodgson *winner* in the case of single-*crossing* preferences, the problem of efficiently calculating *scores* appears to be open and nontrivial.

We also point out that there are other options for "dichotomous Kemeny" depending on the amount a tie contributes to the distance. Kemeny [23] adds 0.5 and Fagin et al. [13] considers all penalty values between 0 and 1. It should be noted that $(2, 2)$-Kemeny does not correspond to any of these penalty values, since this rule depends solely on the induced weighted majority graph (making it a C2 rule in the sense of Fishburn [15]), whereas the others do not. Computing a dichotomous consensus for $p > 0$ is a very interesting challenge. (For the $p = 0$ case, this can be solved in polynomial time via network flow. Computing a total order consensus is equivalent to $(2, m)$-Kemeny and thus also in P.)

Finally, we would like to find out what makes the Young rule behave so counterintuitively. That is, we would like to find out what

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[5] Not all single-crossing preferences are single-peaked (see Example 17). Single-peaked preferences are also not necessarily single-crossing. For example, in the voting for the temperature in a room example, the (single-peaked) votes $(16 > 18 > 21 > 25)$, $(18 > 21 > 25 > 16)$, and $(21 > 18 > 16 > 25)$ are not single-crossing.

properties of the Young rule cause the winner problem to be easy for dichotomous preferences while the score, ranking, and loser problems are hard.

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