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On the Effectiveness of Social Proof Recommendations in Markets with Multiple Products

Vincenzo Auletta[1] and **Diodato Ferraioli**[2] and **Gianluigi Greco**[3]

Abstract. The *social proof marketing strategy* assumes that the marketer provides a novel product for free to some users of a social network and then promptly recommends the product to other users, by informing them that a number of their friends are already using it. In this paper we study this popular marketing strategy in scenarios where the new product enters in markets where two old products are already competing. We show that if customers tend to adopt the product that is the most popular one (over the three alternative products) among their friends, then this marketing strategy allows to maximize the diffusion of the new product only on a narrow class of networks. Moreover, even if we focus on this narrow class of networks, computing the best order of the recommendations is computationally intractable.

Instead, if customers are less prone to change their mind, that is, if they are willing to adopt some product only when an absolute majority of their friends has already agreed on it, then the marketing strategy always works well and, furthermore, an optimal order of recommendations can be computed in polynomial time.

1 INTRODUCTION

In 2011 Google launched Google+, its own social networking service. In order to diffuse Google+, a common marketing strategy has been put in place, consisting of providing a free version of the product to some customers and then *recommending* the product to other potential customers by exhibiting a *social proof* of its quality, i.e., a list of contacts that are influential for them and that are currently using the proposed product. Note that customers may receive their recommendations at different times, by allowing in this way the marketer to include in the social proof not only the first adopters of the novel product, but also those customers that changed their mind by effect of recommendations. In other words, the *order* of recommendations is a very important ingredient of this marketing strategy.

There have been plenty of examples of effective applications of this marketing strategy [11], and many theoretical studies have already provided formal arguments to explain the reasons for its success. For instance, given any set of seeds, we know that there is an order of recommendations (that can be computed efficiently and) that always maximizes the number of final adopters of the sponsored product, regardless of the topology of the social network [15, 17]. Moreover, whereas it can be hard even to approximate within a sublinear factor the selection of the "optimal" seeds when a budget for their selection is given [15, 16], polynomial time algorithms are known to compute a set of seeds leading a minority to become a majority or a bare majority to become a consensus through social influence [25, 5, 7, 8, 10].

However the social proof marketing strategy failed with Google+ and the company announced that the service has been recently shut down [1]. In fact, the failure cannot be due to Google having a scarce knowledge of its customers' influence network. Moreover, it did not depend on a limited choice of initial adopters: Google provided Google+ to a very large set of seeds, for instance to every customer using an Android phone. Hence, it is natural to ask how this failure of the social proof marketing strategy can be explained at the light of the positive theoretical results discussed above.

The crucial observation is that all results described above assume a *binary market*, where customers have to choose among two products only, an innovation and a pre-existing one. However, this is not the setting of the market in which Google+ was entering. Indeed, Google+ has been introduced in a market where several other social networking services were already running, with Facebook and Twitter being the most prominent ones. This suggests that the positive results characterizing the binary markets do not longer hold when moving to markets with multiple competing products.

Our Contribution. The main goal of this paper is to shed light on the effectiveness of the social proof marketing strategy for promoting a novel product in a setting where there are already two competing products[4] and to highlight similarity and differences with the case in which only two products are available. In particular, we look for those network topologies where there is an order of recommendations that maximizes the number of final adopters of the novel product. Moreover, we would like to characterize networks where this optimal order of recommendations not only exists, but it can be computed in polynomial time, regardless of the chosen seeds.

In our model, we assume that the marketer does not lie to customers: indeed, in this setting, lies can be easily verified and they can undermine the credibility of the marketer.

Our analysis highlights that the success of the marketing strategy crucially depends on the *behavior* of the customers, i.e., on how they react to the social proof provided them with the recommendation.

In more details, we first focus on customers who tend to adopt the same product as the one adopted by *most* of their neighbors on the network.[5] This behavior has been used as a reference model in most of the works discussed above and in a number of works related to opinion diffusion in social networks (see, e.g., [26, 15, 16, 25, 5, 10]). Our results for markets with multiple competing products are in

[1] Università degli Studi di Salerno, Italy, email: auletta@unisa.it
[2] Università degli Studi di Salerno, Italy, email: dferraioli@unisa.it
[3] Università della Calabria, Italy, email: gianluigi.greco@unical.it

[4] Since our results are mainly negative, the limitation to only three products, makes our results even stronger. We also remark that the positive results about "absolute majority" customers still hold with more products.

[5] Clearly, as we have three products at hand, the majority can be well formed by less than half of the neighbors.

sharp contrast with those that are known to hold for binary markets. Indeed, we give a complete picture of the problem with respect to the network topology. Our results then prove that *every* social proof marketing strategy leads to maximize the number of adopters of a given alternative only on very trivial topologies (namely, star, path, cycle, and a special six-node graph, that we will refer as bi-triad) (cf. Thm 1). And, even if we simply require that there exists *some* social proof marketing strategy maximizing the number of adopters of the desired product, still the topologies on which this occurs are very limited (cf., Thm 2). Thus, even if it would be expected that the social proof marketing strategy does not work on all topologies, we find highly surprising that this strategy works for so few graph structures. Moreover, through our characterization, we provide a polynomial time algorithm to the marketeer to decide if the marketing strategy can be optimal for the topology at the hand if acyclic, and a proof that no polynomial time algorithm achieves this goal on cyclic networks (cf., Thm 6). We believe that our algorithm may be an useful tool for designing opportune marketing campaign in future.

Then, we consider a more restricted class of customers, who adopt a novel product only when it is already supported by an *absolute majority* of their neighbors in the network. Clearly, over binary markets the two behaviours (majority and absolute majority) coincide. However, over markets where the novel product competes with two other products, it exhibits completely different properties. Indeed, for such kinds of customers, who are less prone to innovations, we show that the social proof marketing strategy is always effective and an optimal order of recommendations can be computed in polynomial time.

Related Works. The framework considered in the paper can be linked to similar models emerged in the literature about the analysis of (discrete) opinion formation in social networks [17, 19, 23]. Indeed, our results can be easily rephrased in terms of opinion diffusion: When more than two opinions are available, a specific kind of manipulation is effective (resp., not effective) for agents with absolute majority (resp., majority) behaviour.

Most of the works appeared on opinion formation focus on binary opinions, and the case of multiple opinions has been considered only rather recently [17, 9]. However, Chierichetti et al. [17] did not address the question of how the dynamics of the opinions can be manipulated and their results are unrelated to ours. Instead, Auletta et al. [9] showed that finding the order of updates, over *all* possible orders, that maximizes the diffusion of an opinion is computationally intractable. In the paper we depart from that work by studying a *specific* class of updates' orders, namely those emerging from the social proof marketing strategy; moreover, we focus on characterizing the power and the effectiveness of this strategy, which is an issue not considered in earlier literature.

Note that some recent literature on opinion diffusion considered more complex behaviors in addition to majority-based ones—see, e.g., [6, 2]. Moreover, recent works considered opinion formation on evolving networks [12, 20, 13]. Understanding to which extent our results can be lifted to these more complex settings is an interesting avenue for further research.

Finally, we remark that our setting is related to the works on competitive diffusion of products on social networks. Differently from our setting, however, most of these works assume that competitors can decide the placements of their seeds on the network. Indeed, some works [14, 29, 18] deal with the problem of approximating the best-response strategies of competitors for the seeds' placement, that is, the strategy to compute the set of seeds that maximizes the adopters of the promoted product given the other promoters' seeds. Other works characterize the cases in which the competition leads to an equilibrium [3, 28]. Finally, performances of these equilibria with respect to the total number of nodes adopting one of the products have also been evaluated [28, 22, 24, 4].

In contrast with these works, in this paper we assume that the seeds placement is given and/or is out of control of the marketer, and we concentrate on the order in which recommendations have to be presented to the customers (with dynamics starting from the initial given configuration). However, we refer the reader to conclusions for a more detailed analysis of our contribution in relation with the seed placement problem.

2 MODEL

Customers and their Behaviour. Let $G = (N, E)$ be a connected and undirected network encoding the social interactions over a set N of customers. For each customer $i \in N$, the set of her neighbors in G will be denoted as $\delta(i) = \{j \mid (i,j) \in E\}$ for short. Each customer $i \in N$ initially owns a product $b_i \in \{black, gray\}$, and a novel product, say *white*, has to be injected in the network.[6]

Let P be the set $\{black, gray, white\}$. A *profile* p is a function $\mathsf{p}\colon N \to P$ mapping customers to products. In particular, we denote with p^0 the profile in which each customer is associated with her own original product, i.e., $\mathsf{p}^0(i) = b_i$ for each $i \in N$. For any subset $A \subseteq N$, we will use p_A to denote the restriction of the profile p to the customers in A only, i.e., p_A is the function $\mathsf{p}_A\colon A \to P$ such that $\mathsf{p}_A(i) = \mathsf{p}(i)$ for every $i \in A$.

The *behaviour* of a customer $i \in N$ is modeled as a function $\mathcal{B}_i$ that, given a profile $\mathsf{p}_{\{i\}\cup\delta(i)}$ for i and her neighbors, returns the product $\mathcal{B}_i(\mathsf{p}_{\{i\}\cup\delta(i)})$ that i prefers. In this work, we will consider two behaviors:

- A customer i has a *majority behaviour* if $\mathcal{B}_i(\mathsf{p}_{\{i\}\cup\delta(i)})$ is the product supported by most of i's neighbors. In case of ties[7] involving the product that is owned by i, we assume that $\mathcal{B}_i(\mathsf{p}_{\{i\}\cup\delta(i)}) = \mathsf{p}(i)$, i.e., the preference of i does not change. For the other ties, we assume an *optimistic* tie-breaking rule: *white* is always preferred, while ties between *black* and *gray* are broken arbitrarily.[8]
- A customer i has an *absolute majority behaviour* if $\mathcal{B}_i(\mathsf{p}_{\{i\}\cup\delta(i)})$ returns the product supported by at least $\left\lfloor \frac{|\delta(i)|}{2} \right\rfloor + 1$ neighbors, if any exists, or the innate product b_i, otherwise. Note that no tie can occur in this case.

A customer $i \in N$ is *stable* in p if $\mathcal{B}_i(\mathsf{p}_{\{i\}\cup\delta(i)}) = \mathsf{p}(i)$, that is, if i has no incentive to adopt a different product. A profile p is stable if all customers are stable in p.

Marketing Strategy. We consider a marketing strategy for the product *white* that is articulated in two phases:

Seeding: At step $t = 1$, a subset $S \subseteq N$ of the customers is seeded with a trial version of the product, i.e., $\mathsf{p}^1(i) = white$ if $i \in S$, and $\mathsf{p}^1(i) = \mathsf{p}^0(i)$, otherwise. As we have pointed out in the Introduction, we assume that the seeding is out of the control of the "marketer". That is, the profile p^1 is actually given as input and our focus is on studying the second of the two phases.

[6] Colors are used here to make graphical representations clearer (*gray* is not an "intermediate" product between *white* and *black*).

[7] For example, a ties occurs when a *gray* node of degree four has two *white* neighbors and two *black* neighbors. Hence, the node, if it has majority behavior, is not stable (*gray* is not a majority in its neighborhood) but we have to break ties among *white* and *black* to decide which product this node have to adopt.

[8] The results we shall provide for such customers are essentially negative. So, our tie-breaking rules make them even stronger.

Recommendation: At each step $t > 1$ and until a stable profile is reached, the "marketer" checks whether there is some unstable customer in p^{t-1} that, according to her behaviour, might be inclined to adopt product *white*. If this is the case, then a customer $i \in N$ of this kind (i.e., with $\mathcal{B}_i(\mathsf{p}^{t-1}_{\{i\}\cup\delta(i)}) = white$) is selected by the marketer and is advertised that a majority of her friends have already adopted the innovation. We assume that this recommendation is effective; hence, it leads to a novel profile p^t such that: $\mathsf{p}^t(i) = white$ and $\mathsf{p}^t(j) = \mathsf{p}^{t-1}(j)$ for every $j \in N \setminus \{i\}$. Note however that if there is no agent $i \in N$ that can adopt the product, then no recommendation is possible and the profile evolves autonomously: p^t is derived from p^{t-1} by changing the product of some arbitrarily chosen unstable agent so that she becomes stable.

Roughly speaking, this marketing strategy is modeling an initial environment in which two products, namely *black* and *gray* are competing. Note that we do not require that the starting profile p^0 is stable, i.e., there may be agents willing to change their product from *black* to *gray* and vice versa.

At the time 1, a new product, namely *white*, enters the network: formally, some nodes (the seeds) are selected to get product *white*. From this time on, the behaviour of the agents cannot be "controlled". In particular, seeds are not forced to stay with the *white* product, and some nodes might be still willing to change from *black* to *gray* and vice versa.

Recommendation are used for "nudge" the nodes to carry out these changes. However, updates can occur even without being "nudged" by the recommendations. These only influence the order of updates: we do not wait till nodes realize which is the majority in their neighborhood, but we exhibit them a social proof (i.e., messages such as "Look! Most of your contacts are using this product") to induce the change. Clearly, this marketing operation is done only towards the promoted product (i.e., *white*). And we assume that recommendations are always and immediately effective: after being advertised by the marketer, customers adopt the innovation before any other update occurs in the network.[9] Whenever no customer can be advertised, instead, the evolution of the profile cannot be controlled by the marketer (as it involves the adoption of products different from *white*).

Evolutions. In this paper, we are not interested in the seed-selection problem (i.e., how we can derive the profile p^1 starting from p^0), whose study is entirely orthogonal to our contribution (we refer the reader to conclusions for a more detailed discussion about the relation between our results and the seed-selection problem). The focus of this work is instead on the effectiveness of the social proof marketing strategy described above with respect to the goal of maximizing the diffusion of the promoted product.

To this aim we would like to compare the number of adopters of the *white* product at the end of the application of the marketing strategy to the maximum number of adopters of the novel product that can be reached at the end of some feasible evolution.[10] Here, we define a *feasible evolution* for the network as a sequence of profiles $\mathsf{p}^1, \mathsf{p}^2, ..., \mathsf{p}^\ell$ such that p^ℓ is a stable profile and, for each $k \in \{1, ..., \ell-1\}$, there is a customer i_k that is unstable in p^k and such that $\mathsf{p}^{k+1}(j) = \mathsf{p}^k(j)$ for every $j \in N \setminus \{i_k\}$ and $\mathsf{p}^{k+1}(i_k) = \mathcal{B}_i\left(\mathsf{p}^k_{\{i_k\}\cup\delta(i_k)}\right)$. Note that in a feasible evolution it is not required that changes occurs contiguously. However, at each time step only one customer is allowed to change her product.

Clearly, feasible evolutions include the ones resulting from the application of the recommendation phase of the marketing strategy, where updates to *white* have priority with respect to other updates. We note, however, that the application of this phase does not univocally determine a feasible evolution for the network. Indeed, at each time step, there might be multiple customers that can be chosen as the target of the marketer and, whenever no manipulable customer exists, the unstable customer that changes her mind is selected non-deterministically. So, the recommendation phase of the Social Proof Marketing Strategy defines in general a number of different feasible evolutions, which we name SPMS evolutions. In a nutshell, these evolutions promote the adoption of the innovation as soon as this is possible, no matter the other products spread over the network.

We remark that it is possible that a customer adopts the *black* or *gray* product in the course of a feasible evolution. In SPMS evolution this can occurs only if there are no customers willing to adopt product *white*, but in general evolutions this can occur at any time.

Problems of Interest. Let us denote by $W(\mathsf{p})$ the number of customers adopting $white$ in a profile p, i.e., $W(\mathsf{p}) = |\{i \mid \mathsf{p}(i) = white\}|$. Then, we say that a feasible evolution $\mathsf{p}^1, \mathsf{p}^2, ..., \mathsf{p}^\ell$ is *optimal* if $W(\mathsf{p}^\ell) \geq W(\hat{\mathsf{p}}^h)$ holds, for each alternative feasible evolution $\hat{\mathsf{p}}^1, \hat{\mathsf{p}}^2, ..., \hat{\mathsf{p}}^h$.

With the above concepts at hand, we can now state the problems that will be addressed in the rest of the paper: Given a starting profile p^1 (resulting from the seeding phase over the original profile p^0), is it the case that all SPMS evolutions are also optimal (i.e., the marketing strategy effectively maximizes the diffusion of the promoted product, regardless of how ties are broken)? Does at least one SPMS evolution always exist that is also optimal (i.e., a way of choosing the next node to update when more than one is available, so that the resulting evolution maximizes the diffusion of *white*)? Can we compute in polynomial time an optimal SPMS evolution (whenever one is guaranteed to exist)? Answers to these questions will be given w.r.t. the behaviour of the customers and the network topologies.

3 MAJORITY BEHAVIOUR

In this section we focus on the case where all customers have majority behavior. We shall mainly present results that are negative. Indeed, we shall show in Section 3.1 that there are few networks for which SPMS evolutions are always optimal. We shall also present in Section 3.2 a few of other networks for which some SPMS evolutions (but not all of them) are optimal. However, not only these networks are still highly constrained, but it is in many cases also NP-hard to efficiently compute on them an optimal SPMS evolution. Eventually, in Section 3.3, we shall also show that these results are tight and rather precisely identify the classes of networks on which the social proof marketing strategy is effective to some extent.

3.1 Every SPMS Evolution is Optimal

We start by considering classes of networks on which *every* SPMS is optimal. Hence, on these networks, any non-deterministic implementation of the recommendation phase is suitable to optimize the propagation of *white* and therefore the marketing strategy can be implemented in polynomial time.

[9] Studying settings where this assumption is relaxed to some extent (e.g., by introducing delays or probabilities of adoptions) is an interesting avenue for further research. However, we highlight that, since we mainly provide negative results, this assumption enforces most of our results.

[10] By a simple potential function argument, it is easy to see that feasible evolutions always converge to a stable state both for customers with majority and absolute majority behavior. Indeed, each sequence of profiles converging to a stable one and where, at each step, one agent that is not stable changes her current opinion, is a feasible evolution—regardless of the order of updates.

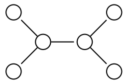

Figure 1. The bi-triad.

Actually, it turns out that the class of graphs enjoing this properties is very limited, since it consists only of paths, rings, stars, and the six-node graphs showed in Figure 1, that we will name *bi-triad*.

Theorem 1. *For customers with majority behaviour, if G is a star, a path, a cycle, or a bi-triad, then every* SPMS *evolution is optimal.*

Proof Sketch. Assume that G is a star and let c be its center node. Let p^1 be the starting profile. We observe that if either $\mathsf{p}^1(c) = white$ or *white* is the most popular product among the neighbors of c, then all the SPSM evolutions always lead to a consensus on *white*. On the other hand, for the remaining profiles, all the feasible evolutions reach a stable profile where no customer has product *white*.

Consider the case where G is a path or a cycle. In these networks all nodes have degree either 1 or 2. This implies that if in a profile p^t there are two neighbors, i and j, adopting the same product, then both customers are stable and they do not change their products in all feasible evolutions starting from p^t. It can be checked that, for every customer i that has no neighbors with the same product, either i adopts *white* in every SPSM evolution or there is no feasible evolution ending in a stable profile where i adopts *white*.

Finally, for the case that G is a bi-triad, the result follows by a simple case analysis. □

We will show next that the characterization above is tight for acyclic networks. Indeed, for all networks different from paths, stars, and bi-triad, it happens that the optimality of SPMS evolution depends on the order in which customers change to non-*white* products.

3.2 There is an Optimal SPMS Evolution

We now move to isolate a class of networks where *some* (but not *all*) SPMS evolutions works well. Moreover, we show that identifying such optimal SPMS evolutions turns out to be an intractable problem. In particular, we shall evidence that the source of the intractability lies in the uncertainty about how the two competing products will spread over the network—which is clearly outside of the control of the manipulator. From the practical viewpoint, the result tells us that there is no a-priori guarantee that the social proof marketing strategy is effective on these networks.

Let us formally define the class of networks on which we focus in this section.

Definition 1. The class of *augmented quasi-paths* (*augmented cycles*, resp.) contains all acyclic (cyclic, resp.) networks such that each node of degree 3 or at least 5 has all neighbors of degree 1 except at most one.

The name of the class above derives by the observation that networks in the class that do not have any node of degree 4, must be either a cycle, or a quasi-path, defined as follows.

Definition 2. The class of *quasi-paths* consists of all paths whose extremes are allowed to have degree 3 or more (but with remaining neighbors of degree 1).

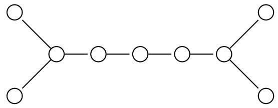

Figure 2. A quasi-path network. Note that stars, paths and the bi-triad are degenerate quasi-paths.

An example of quasi-path is depicted in Figure 2. As we will see below, customers of degree 4 play a special role in the hardness of computing an optimal SPMS evolution.

Theorem 2. *For customers with majority behaviour, if G is augmented quasi-path or an augmented cycle, then an* SPMS *evolution always exists that is also optimal.*

Proof Sketch. First, observe that a kind of *white*-monotonicity holds: given a customer u, if in a profile p u is willing to adopt the *white* product, then it will also adopt this product in the profile p' achieved from p by setting to *white* the product of a non-*white* neighbor of u.

Let now consider the feasible evolution $\mathsf{p}^1, ..., \mathsf{p}^\ell$ that maximizes the numbers of *white* in the stable profile p^ℓ. We shall show that either this sequence is SPMS or it can be transformed in a SPMS evolution $\mathsf{p}^1 = \tilde{\mathsf{p}}^1, ..., \tilde{\mathsf{p}}^k$ ending in a profile with at least the same number of final adopters of the innovation.

To this aim, let us define, for each profile p, $w(\mathsf{p})$ as the set of agents that want to adopt product *white* in p. Then for $i = 2$ until $w(\tilde{\mathsf{p}}^{i-1}) \neq \emptyset$, we build $\tilde{\mathsf{p}}^i$ from $\tilde{\mathsf{p}}^{i-1}$, by updating to *white* the product of some $u \in w(\tilde{\mathsf{p}}^{i-1})$.

Let $\tilde{\mathsf{p}}^k$ be the profile at the end of this phase. Until a new node willing to become *white* appears, we apply all non-*white* changes of the original sequence that are still allowed in $\tilde{\mathsf{p}}^k$ exactly in the same order as in the original sequence.

Then, we can repeat the first phase of the procedure (i.e., change all nodes willing to become *white*) until for every node u such that $\mathsf{p}^\ell(u) = white$, it occurs that $\tilde{\mathsf{p}}^{k'}(u) = white$ too.

Finally, we complete the sequence of profiles $\tilde{\mathsf{p}}^1, ..., \tilde{\mathsf{p}}^{k''}$, by stabilizing those nodes that are not stable in $\tilde{\mathsf{p}}^{k''}$, exactly in the same order as done in the original sequence. □

However, for these networks, selecting the SPMS evolution that maximizes the diffusion of *white* is computationally infeasible.

Theorem 3. *For customers with majority behaviour, computing an* SPMS *evolution that is also optimal is* NP*-hard even on classes of networks that are augmented quasi-paths or augmented cycles.*

Proof Sketch. We exhibit a reduction from the well-known 3-SAT problem. Given a Boolean formula Φ in conjunctive normal form and where each clause contains three distinct literals, we build the network G_Φ (and the profile p^1 given by the colors associated to the customers in the illustration) as follows. For each variable x, G_Φ contains (as a subgraph) the gadget shown in Figure 3; for each clause with precisely one negated variable, G_Φ contains a gadget as in Figure 4; for each clause where all variables are negated, G_Φ contains a gadget as in Figure 5; finally, for clauses with precisely two negated variables (resp., no negated variable), G_Φ contains a gadget as in Figure 4 (resp., Figure 5), but where colors black and gray are swapped. We set n to be the length of the long tail of clause gadgets. Note that G_Φ is either an augmented quasi-path or an augmented quasi-cycle.

We shall show that the formula Φ is satisfiable if, and only if, there exists a SPMS evolution converging to a stable profile where

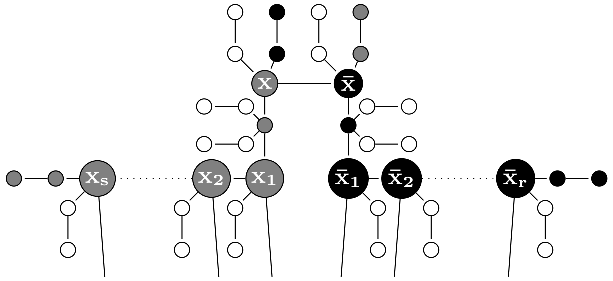

Figure 3. The gadget for variable x. We assume that x appears positively (resp., negatively) in s (resp., r) clauses. The edges without an endpoint are between the given node with its corresponding node in the clause gadget.

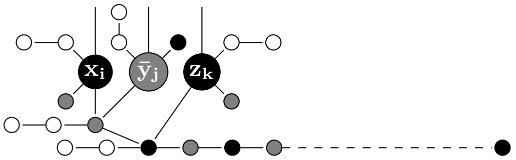

Figure 4. The gadget for a clause with one negated variable

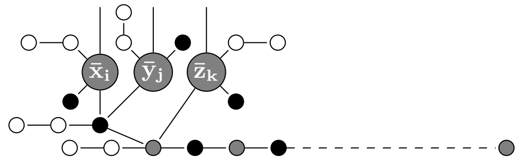

Figure 5. The gadgets for a clause with three negated variables

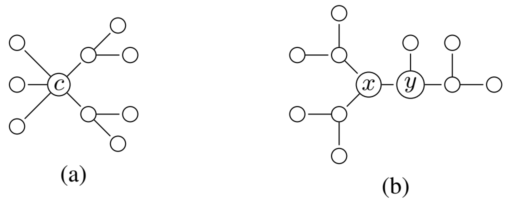

Figure 6. Examples of quasi-stars.

all customers in the clause gadgets adopt product *white*. Hence, by taking n large enough, any algorithm computing an optimal SPMS evolution is able to distinguish whether Φ is satisfiable or not.

In order to have a clear intuition on the salient features of the reduction, observe that in p^1 no customer is willing to adopt product *white*. Moreover, for every variable x, the two customers x and $\bar{x}$ would like to exchange the product they currently adopt. Then, if one of these two customers, say x, adopts the product currently held by $\bar{x}$, then all customers on its side will eventually adopt *white*, and $\bar{x}$ and all customers on the side of $\bar{x}$ become stable. □

In the proof of Theorem 3 a fundamental role is played by nodes with degree 4: this is necessary. Indeed, if we exclude these nodes we have that G is either a cycle (for which we know that each SPMS evolution is optimal) or a quasi-path, for which Proposition 1 shows that an optimal SPMS is easy to compute.

Proposition 1. *For customers with majority behaviour, if G is a quasi-path, then an optimal* SPMS *evolution can be computed in polynomial time.*

Besides augmented quasi-paths, even for the following quasi-star networks an optimal SPMS evolution always exists.

Definition 3. A *quasi-star* network satisfies one of the following conditions:

(a) It consists of a center customer c of degree 3 or at least 5, with all neighbors x such that either x has degree 1 or it has degree 2 or 3 and all the neighbors of x different from c have degree 1; or
(b) It consists of two neighbors x and y of degree 3 whose remaining neighbors either have degree 1 or have degree 3 and all neighbors of degree 1.

An example of these networks is depicted in Figure 6.

Next Proposition states that on these graphs an optimal SPMS evolution always exists. Moreover, given the finiteness of these networks, this evolution can be clearly computed efficiently.

Proposition 2. *For customers with majority behaviour, if G is a quasi-star, then an optimal* SPMS *evolution always exists.*

3.3 Tightness of the Characterization

To complete the picture of our analysis of customers with majority behaviour, we observe that the classes we have identified on which the social proof marketing strategy works are essentially the largest possible ones. We formalize this claim by focusing on networks that are acyclic. First, we prove the tightness of Theorem 1.

Theorem 4. *For customers with majority behaviour, if G is an acyclic graph that is not a star, a path, or a bi-triad, then there is an* SPMS *evolution that is not optimal.*

Proof Sketch. If a graph is neither a star nor a path or a cycle, then it must be the case that either there is a node u of degree at least 3 with a neighbor v with degree either 2 or at least 4, or there is a line of three nodes, u, v, and w, of degree 3.

In the first case, consider the initial profile represented in Figure 7. Consider then the following SPMS evolution: u adopts product *black*, then v adopts product *white*, finally x_1 and x_2 adopt product *gray* if they are not stable (indeed, if the degree of x_1 and x_2 is 1 or 2, then they are stable), and possibly also u changes back to *gray*. Note that this evolution ends with $W + 1$ *white*. Consider instead the following evolution: x_1 and x_2 adopt the *gray* product (they always have an incentive to do so, even if their degree is 1). It is immediate to see that after that no node has incentive to change (except possibly y if its degree is 1). In particular, since no node want to assume the *white* product, the sequence is SPMS. However, it ends up in a stable profile with at most W *white* nodes.

In the second case, consider the initial profile given in Figure 8. Consider then the following SPMS evolution: w adopts product *gray*, and then the non-stable neighbors of w' and w'' stabilize themselves (note that it they change their color, they do not assume *white*). Note that this evolution ends in a stable state, and let W be the number of *white* nodes in this state. Consider instead the following evolution: w' and w'' adopts *black*, v changes to *black*, and then the non-stable neighbors of v, w' and w'' stabilize themselves (note that it they

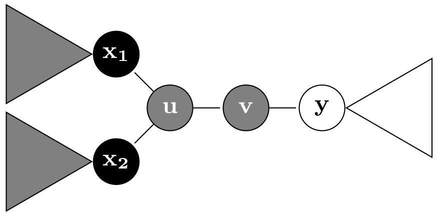

Figure 7. A starting profile from which there is a SPMS evolution that is not optimal.

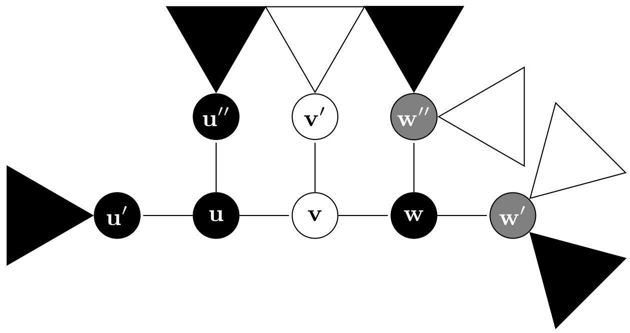

Figure 8. A starting profile from which there is a SPMS evolution that is not optimal.

change their color, they do not assume *white*). Note that *white* neighbors of w' or w'' change their product in the first sequence if and only if they will do also in the second one. Hence the second evolution ends in a stable state in which there are at most $W-1$ *white* nodes. Observe that, since no node want to assume the *white* product, this sequence is SPMS. □

Next, we prove that Theorem 2 and Proposition 2 enumerate all acyclic network topologies on which there is an optimal SPMS.

Theorem 5. *Let G be any acyclic network defined on customers with majority behaviour. If G is neither a quasi-star not an augmented quasi-path, then an initial profile p^1 always exists for which no* SPMS *evolution is optimal.*

Proof Sketch. Consider the following classes of networks:

H1: It consists of a customer x of degree either 3 or at least 5, with a neighbor y of degree either 2 or at least 4, and a neighbor z of degree at least 3;

H2: It consists of a customer x of degree at least 5, with at least three neighbors of degree at least 2 such that their neighbors all have degree 1;

H3: It consists of a customer x of degree either 3 or at least 5, with two neighbors of degree 2, and at least one of them has a neighbor of degree at least 2, whereas the other one has a neighbor that either has degree 1 or at most $\left\lfloor \frac{\delta(y)-1}{2} \right\rfloor$ neighbors of degree 1;

H4: It consists of a line (x, y, z, w) of four customers of degree 3 with x having a neighbor of degree at least 2.

By simple inspection one can verify that if G is neither a quasi-star nor an augmented quasi-path, then G contains a subgraph that belongs to one of these classes.

We can then prove that if an acyclic network G contains a subgraph belonging to one of these classes, then the maximum number of adopters of the *white* product cannot be achieved by any SPMS evolutions. Below we detail the case of the class **H1**, with the proof for the other classes being similar.

Consider the following profile p^1: $\mathsf{p}^1(x) = black$, $\mathsf{p}^1(y) = black$, and $\mathsf{p}^1(z) = gray$. Then, let $x_1, \ldots, x_{|\delta(y)|-2}$ be the neighbors of x different from y and z. We partition them in three sets: $\mathcal{B}_x = \{x_1, \ldots, x_\kappa\}$, $\mathcal{G}_x = \{x_{\kappa+1}, \ldots, x_{\kappa+\lambda}\}$, and $\mathcal{W}_x = \{x_{\kappa+\lambda+1}, \ldots, x_{|\delta(x)|-2}\}$, where $\kappa = \lfloor|\delta(x) - 1|/2\rfloor - 1$ and $\lambda = \lceil|\delta(x)|/2\rceil - 1$ (observe that $\mathcal{B}_x$ is empty if $|\delta(x)| = 3$, $\mathcal{W}_x$ is empty if $|\delta(x)|$ is odd). We set to $black$ the product of all customers in the subtree whose root is in $\mathcal{B}_x$, to $white$ the product of all customers in the subtree whose root is in $\mathcal{W}_x$, and to $gray$ the product of all customers in the subtree whose root is in $\mathcal{G}_x$.

Similarly, let $y_1, \ldots, y_{|\delta(y)|-1}$ be the neighbors of y different from x. We partition them in three sets: $\mathcal{B}_y = \{y_1, \ldots, y_k\}$, $\mathcal{W}_y = \{y_{k+1}, \ldots, y_{k+\ell}\}$, and $\mathcal{G}_y = \{y_{k+\ell+1}, \ldots, y_{|\delta(y)|-1}\}$, where $k = \lfloor|\delta(y)|/2\rfloor - 1$ and $\ell = \lfloor|\delta(y)|/2\rfloor$ (observe that $\mathcal{B}_y$ is empty if $|\delta(y)| = 1$, $\mathcal{G}_y$ is empty if $|\delta(y)|$ is even, and $|\mathcal{G}_y| + 1 \leq |\mathcal{W}_y| = |\mathcal{B}_y| + 1$). In the profile p^1, we set to $black$ the product of all customers in the subtree whose root is in $\mathcal{B}_y$, to $white$ the product of all customers in the subtree whose root is in $\mathcal{W}_y$, and to $gray$ the product of all customers in the subtree whose root is in $\mathcal{G}_y$. We finally set to $white$ the product of remaining customers (this include the (at least two) neighbors of z).

Now, let us denote by W^1 the number of customers with product *white* in p^1, except those having degree 1 in the neighborhood of x and y. Observe that in p^1 the customers that are not stable are: x (for which $\mathcal{B}_x(\mathsf{p}^1_{\{x\}\cup\delta(x)}) = gray$), z, (for which $\mathcal{B}_z(\mathsf{p}^1_{\{z\}\cup\delta(z)}) = white$, and the neighbors of degree 1 of x, y, and z (that would like to adopt the same product as their neighbors and thus not *white*).

Then it is not hard to check that any SPMS evolution leads to a profile where at most $W^1 + 1$ customers adopted *white*. However, if x adopts the $gray$ product before any adoption of *white*, then we can find an evolution that leads to a stable configuration with at least $W^1 + 2$ $white$ customers. □

The arguments in the proof of Theorem 5 can be extended to prove that SPMS evolutions are not optimal on large classes of *cyclic* networks that are neither cycles nor augmented cycles. However, a succinct and sharp characterization cannot be achieved in this case, in the light of the following complexity-theoretical argument.

Theorem 6. *For customers with majority behaviour, it is* coNP*-hard to decide whether, for a given network G that is neither a cycle nor an augmented cycle, an optimal* SPMS *evolution exists for each initial profile.*

Proof Sketch. Consider the NP-hard ONE-IN-THREE POSITIVE 3-SAT problem of deciding whether, given an instance Φ of 3-SAT where each of its m clauses consists of three positive variables, there is a truth assignment such that each clause is satisfied by precisely one variable [21]. W.l.o.g., we assume that each variable occurs precisely in three clauses. We build a network G_Φ as follows.

For each variable X in the set $\mathcal{V}$ of the variables occurring in Φ, the set N of the customers contains a customer $\hat{X}$ and G_Φ contains as a subgraph the network G^X depicted in Figure 9. Moreover, for each clause c_j, if X, Y and Z are the variables occurring in it, then G_Φ contains the subgraphs G_j, $G_j^{X,Y}$, $G_j^{X,Z}$, and $G_j^{Y,Z}$ as depicted in Figure 9. Finally, for each variable X, the customer $\hat{X}$ has three further neighbors as depicted in the bottom part of Figure 9, one of them being the distinguished customer R. Note that R is connected to all the customers associated to the variables in $\mathcal{V}$.

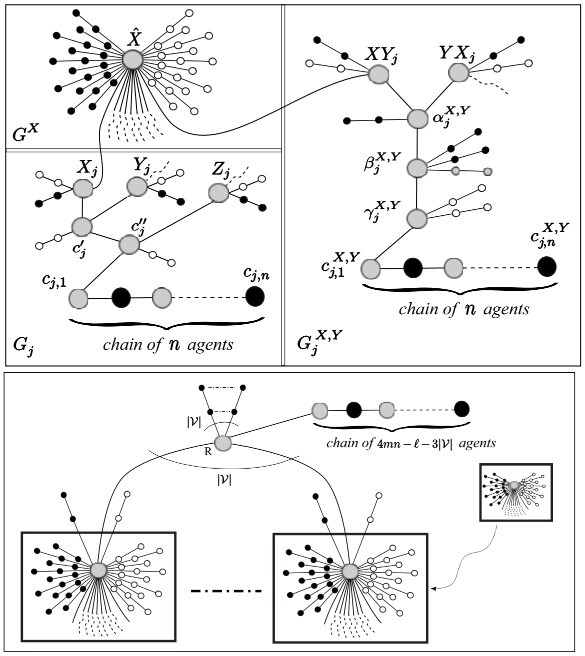

Figure 9. Gadgets exploited in the proof of Theorem 3.

Now, any SPMS evolution is such that all customers associated to the variables will adopt $white$, and in turn R and the chain attached to it, and all chains in the subgraphs G^X. Let ℓ (in the chain attached to R) be the maximum number of customers outside the various chains that will eventually hold opinion $white$ in this evolution. By construction and by taking the value of n large enough, it can be checked that this evolution is optimal if, and only if, there is no assignment that is a "yes" instance to ONE-IN-THREE POSITIVE 3-SAT. In particular, if an assignment σ of this kind exists, then we can build a feasible evolution (which is not SPMS) such that $\hat{X}$ takes product $white$ (resp., $black$) if X evaluates to true (resp., false) in σ. This leads to a profile where all chains, but the one attached to R, adopt product $white$. □

4 ABSOLUTE MAJORITY BEHAVIOUR

We now move to analyze customers having an absolute majority behaviour. We shall show that, for these customers, the social proof marketing strategy still works, no matter the underlying network topology. A crucial role in our analysis is played by the following "monotonicity" property.

Definition 4. The behavior $\mathcal{B}_i$ of a customer $i \in N$ is *locally monotone* if, for every pair of profiles p and p' that differ only in the product assigned to some neighbor $j \in \delta(i)$ and such that $\{\mathsf{p}(j), \mathsf{p}'(j)\} \cap \{white\} = \emptyset$, it holds that $\mathcal{B}_i(\mathsf{p}_{\{i\}\cup\delta(i)}) \neq white$ if, and only if, $\mathcal{B}_i(\mathsf{p}'_{\{i\}\cup\delta(i)}) \neq white$. □

In words, the behavior of a customer $i \in N$ satisfies the above property if, whenever i does not want to adopt $white$, then she will not change her mind if one of her neighbors adopts a product different from $white$. Observe that, armed with this property, we are able to prove the following theorem.

Theorem 7. *For customers with locally monotone behavior, every* SPMS *evolution is also optimal.*

Proof Sketch. We first prove that for customers with locally monotone behavior, an optimal evolution exists that first updates the product of each customer that is willing to become *white* until one exists, and then it stabilizes remaining customers. Note that during the first phase the order of *white* changes does not matter. Moreover, by local monotonicity, the stabilization phase cannot enable any new node to become *white*, regardless of the order in which these changes are done. Hence, we can conclude that every SPMS evolution is optimal. □

It is immediate to check that absolute majority customers are locally monotone. We then have the following corollary.

Corollary 1. *For customers with absolute majority behavior, every* SPMS *evolution is also optimal.*

Absolute majority behavior allows also to map our setting with three products to the setting with just two products.

Lemma 1. *If there is a* SPMS *evolution* $\tilde{\mathsf{p}}^1, \dots, \tilde{\mathsf{p}}^t$ *for customers with majority behavior, then there is a* SPMS *evolution* $\mathsf{p}^1, \dots, \mathsf{p}^t$ *for customers with absolute majority behavior, such that* $\tilde{\mathsf{p}}^1(i) = white$ *if* $\mathsf{p}^1(i) = white$*, and* $\tilde{\mathsf{p}}^1(i) = black$ *if* $\mathsf{p}^1(i) \in \{black, gray\}$ *and* $W(\mathsf{p}^t) = W(\tilde{\mathsf{p}}^t)$.

Proof Sketch. We can make the latter mimic the former evolution. Indeed, a majority of *white* in the binary setting corresponds to an absolute majority when multiple products are available. Similarly, the adoption of a non-*white* product in the former setting implies that either that product is an absolute majority in the latter setting, or that no product is supported by an absolute majority, which causes the given customer i to assume her innate product b_i. □

Roughly speaking, Lemma 1 states that the diffusion of innovation in a market with two pre-existing competing products is equivalent to the diffusion of the same new product in a market with a single pre-existing product, whenever the behavior of customers is locally monotone with respect to the new product. Hence, for the former setting, all the properties known for the latter setting still hold. In particular, Lemma 1 allows us to borrow other recent results from binary markets. For instance, the follow result follows from [5] and [10].

Theorem 8. *For every non-clique G with an odd number of customers, there is a set of seeds S, with $|S| < |N|/2$ from which any* SPMS *evolution leads to a stable profile* p^* *such that* $W(o^*) > |N|/2$.

Moreover, for every G, there is a set of seed S, with $|S| \leq \lceil |N|/2 \rceil$ from which any SPMS *evolution leads to a stable profile* p^* *such that* $W(o^*) = |N|$. *Both sets are computable in polynomial time.*

5 CONCLUSION

In this paper we analyzed the effectiveness of social proof marketing strategies with respect to maximizing the adopters of a product in a market with multiple competing products.

We remark that our perspective is different from the seed-selection perspective, in which one is interested in finding a set of seeds maximizing the diffusion of the promoted product. In our setting, instead, we focus on the design of the marketing strategy and we assume that the seeding is given before the marketing strategy takes place and that it is not under the control of the marketer. Our results describe the effectiveness of the social proof marketing strategy, which emerged as being not able—when customers' behavior is to follow the majority of their neighbors—of ensuring the maximum possible spread of the innovation (except few cases). In particular, our intractability results are not about finding the optimal seeds, but they are about finding the optimal way of propagating *white* given some initial configuration of the seeds (so that they are entirely different in nature if compared with earlier results in the literature).

Nevertheless, some of our results can be still seen in a seed-selection perspective: in particular, some of our results highlight that, when customers' behavior is to follow the majority of their neighbors, for every network topology (except simple cases), there are seeds from which SPMS evolutions lead to sub-optimal stable configurations. Hence, it would be interesting to understand what happens when we merge the two perspectives, i.e., when we allow the manipulator to both choose the seeds, and the nodes receiving the social proof recommendation at each step. For instance, can we always select seeds so that SPMS evolutions are optimal starting from the corresponding profile?

We also highlight that we hereby do not consider the marketer as operating in a game-theoretic setting. Whereas this would clearly be an interesting direction to follow in future, in this work we only consider the optimization task of marketers. Our results can be seen as stating that the best-response of each marketer cannot be to simply run an SPMS evolution, as instead it is the case with only two product in the market.

In this work we focused only on two customers' behavior, namely majority and absolute majority. This is motivated by our goal to compare the effectiveness of the marketing strategy in a setting with multiple products to the known results for binary products. Since the latter ones mainly consider customers with majority behavior, we focused only on the most natural extensions of this behavior to more than two products. However, our analysis shows sharp differences between majority to absolute majority behaviours. Clearly, among these two extremal behaviors, there is a wide range of different customers' behaviors. Hence, another direction for further work is to extend our analysis to such intermediate cases.

Finally, note that we assumed that seeds are selected only once. However, in many cases it is possible to inject new seeds in the network from time to time. Can such an adaptive seeding [27] strategy help in making more effective the social proof marketing strategy?

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