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Dynamic Analysis of Bifurcations for a Class of Fractional-Order Controlled Model with Damping

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Abstract. In this paper, a class of damped fractional-order control systems is studied, and the dynamic characteristics of its bifurcations are analyzed. We use Riccati transform, inequality technique, H function and other methods to study nonlinear fractional-order control with damping terms. For the nonlinear neutral fractional-order control system, we analyze the dynamic behavior of the bifurcation of the system in the large-time state, and obtain the asymptotic properties in the large-time state. By designing damping conditions, local and global asymptotic behaviors are obtained, and the coupling strength of multiple nodes in a fractional dynamic control system is solved according to the obtained results. The internal coupling matrix and topology of the considered complex control model have symmetric and irreducible properties. Numerical simulation examples verify the validity of the obtained conclusions.

Keywords: Control, Dynamic System, Fractional order, Damping.

1. Introduction

Fractional calculus and fractional differential equations is an ancient and new mathematical theory, which not only includes integer-order calculus and integer-order differential equations, but also an extension and extension of integer-order calculus and differential equations. The related research on fractional calculus calculation has a history of more than 300 years. From the case where the derivative of the order of 1/2 discussed by Leibniz's and L'Hospital, the theory of arbitrary order derivatives and integration appeared, which generalized integer order calculus. After Mandelbort proposed the fractal theory, fractional calculus attracted the attention of many researchers. Compared with integer calculus, fractional calculus can describe the physical properties of the system more accurately. Mechanics [1-3], viscoelastic systems, signal processing, bioengineering and other fields have a wide range of applications [4-5].

Fractional-order control theory is widely used in many fields, and stability is one of the important research topics of fractional-order differential systems. Therefore,

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there are many articles on fractional-order control theory and control methods. For example, there are studies on the stability of fractional-order control systems [6-7], discuss the observability of fractional-order control systems [8], Considering these properties of fractional-order calculus, we introduce fractional-order calculus to better characterize nonlinear dynamical systems dynamic behavior.

Definition: Definition of Riemann-Liouville Derivative

$$D_t^\alpha f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi & \alpha\in(0,1)\\ (f^{(n)}(t))^{(\alpha-n)} \quad n\leq\alpha<n+1, \quad n\geq 1\end{cases}$$

The definition is the fractional derivative of the α -th order modification with respect to t .

For the modified Riemann-Liouville derivative, it has some important properties:

$$D_t^\alpha t^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}$$

Definition of Gamma function:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \quad \mathrm{Re}(z)>0.$$

Properties of Gamma function:

(1) $\Gamma(n)=(n-1)!, \quad \forall n\in Z^+$. (2) $\Gamma(z+1)=z\Gamma(z), \quad \forall z\in C$.

For the determination conditions of the stability of the fractional damped perturbation system, we consider thesystem to be stable if and only if $\forall x_0, \exists\, \varepsilon>0$, such that $\forall t\geq 0$, $\| u\ parallel \leq\varepsilon$

The system is asymptotically stable if and only if $\lim_{t\to+\infty}\|u\|=0$.

Using Gronwall's inequality let the unary functions $g(t),\varphi(t)$ be continuous in $[t_0,t_1]$, $g(t)\geq 0$, $\lambda\geq 0, r\geq 0$. If $\varphi(t)\leq\lambda+\int_{t_0}^t[g(\tau)\varphi(\tau)+r]d\tau$.

As more and more fractional-order models appear, their dynamic behavior analysis, especially stability analysis, becomes particularly important. However, due to the non-locality and weak singularity of fractional calculus, compared with integer calculus, the stability analysis of fractional differential systems becomes extremely difficult, which also leads to the stability of fractional order systems in practical applications.

2. Main Results

We Consider the Nonlinear Control System with Fractional-Order Damped Disturbances

$$ED^\alpha u(x,t)=A(t)u(x,t)+B(t) \tag{1}$$

$$D_{0,t}^{\alpha-1}u|_{(x,t)=(0,0)}=u_0,\quad u|_{(x,t)\in\partial\Omega}=u_1 \tag{2}$$

Where $u(x,t)$ is the state vector, $u(x,t)\in W^{m,p}(\Omega)$, $\mathrm{rank}(E)<n$, $x\in C^n$, $E,A\in C^{n\times n}(\Omega)$, $D^\alpha u(x,t)$ means is the α order Riemann-liouville derivative of $u(x,t)$. A is the $n\times n$ matrix that depends on time t and is a continuous function matrix, $0<\alpha<1$.

$$|t^{\alpha-1}E_{\alpha,\alpha}(\lambda_i t^\alpha)|\leq\frac{1}{\alpha}|\lambda_i|^{\frac{1-\alpha}{\alpha}}+O(t^{-1-\alpha}),t\to+\infty \tag{3}$$

When there is an eigenvalue, set it as λ_i, which satisfies $|\arg(\lambda_i)|=\frac{\alpha\pi}{2}$ and the algebraic multiplicity and geometric multiplicity are both equal to the eigenvalues of one. It can be seen that

$$\int_0^t|\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot\|B(t-\theta)\|\,d\theta\leqslant\frac{1}{\alpha}|\lambda_i|^{\frac{1-\alpha}{\alpha}}\int_{t_0}^t\|B(\theta)\|\,d\theta \tag{4}$$

By the false of $B(t)$, note that $\|x(t)\|\leq M$ assumes that $\alpha>0, a(t)$ is a value in $0\leq t<T(T\leq+\infty)$ is a non-negative locally integrable function, $g(t)$ is a non-negative, monotonically decreasing continuous function defined on $0\leq t<T\quad g(t)\leq K$, assuming that $u(x,t)$ is a non-negative locally integrable function on $0\leq t<T$ and satisfies

$$u\leq E(t)+g(t)\int_0^t(t-s)^{\alpha-1}u(s)\,ds \tag{5}$$

Further, if $a(t)$ is a non-decreasing function in $[0,T)$, then

$$\begin{aligned}\|x(t)\|&\leq\frac{C_1\|x_0\|+C_2(t-t_0)\|x_1\|}{1+\|A\|(t-t_0)^\alpha}\\&\leqslant\int_{t_0}^t\frac{(C_1\|x_0\|+C_2(t-t_0)\|x_1\|)(t-\tau)^{\alpha-1}}{(1+(\tau-t_0)^\alpha)(\|A\|+(t-\tau)^\alpha)}d\tau\\&\leqslant C_3\|A\|^{\left(\frac{1}{a\|\cdot A\|}-2\right)}\Gamma\left(\frac{1}{\|A\|}\right)\Gamma(1-\alpha)\end{aligned} \tag{6}$$

$$\begin{aligned}u(x,t)&\leq a(x,t)E_{\alpha,1}(g(t)\Gamma(\alpha)t^\alpha)\\&\leqslant\Gamma\left(2+\frac{1}{\|A\|}-\alpha\right)+\Gamma\left(\frac{1}{\|A\|}\right)\Gamma(1-\alpha)\\&\leqslant\int_0^t(t-\tau)^{a-1}\|c^{A(t-\tau)^a}\|\|h(x(\tau))\|\,d\tau\end{aligned} \tag{7}$$

It can be seen that the matrix $\mathscr{A}$ is stable, there is $N > 0$ such that $\| c^{At} \| \leq Nc^{-\omega t}$ and there is

$$\| x(t) \| \leq Nc^{-\omega t} \| x_0 \| + Nc^{-\omega t} \| x_1 \| t \tag{8}$$

Simultaneous equation (4) and equation (7)

$$\int_0^t (t-\tau)^{\alpha-1} c^{-\omega(t-\tau)} \| h(x(\tau)) \| d\tau$$
$$\leqslant \| c^{At^a} \| \leq Nc^{-\omega e^e} \leq Nc^{-\omega t} \tag{9}$$

Multiplying both sides of (6) by $c^{\omega t}$. From the condition (i) $\lim_{x \to 0} \frac{\| h(x(t) \|}{\| x(t) \|} = 0$, there exists $\delta > 0$ such that When $\| x(t) \| < \delta$

$$\| h(x(t)) \| \leq \frac{1}{N} \| x(t) \| \tag{10}$$

Substitute equation (6) into equation (8)

$$\| x(t) \| \leq N \| x_0 \| + \int_0^t (t-\tau)^{a-1} c^{\omega \tau} \| x(\tau) \| d\tau \tag{11}$$

According to the potential $\mathscr{Q}$ derived from the energy functional

$$\frac{\partial n}{\partial t} = \operatorname{div} J = kD \nabla^4 n - \operatorname{div}(f''(n) \nabla n) \tag{12}$$

Note the basic Landau-Ginzburg theorem on $f(n)$. Employ (9). Introducing the dynamic term or reaction damping, therefore

$$\frac{\partial \omega}{\partial t} = k \nabla^4 n + A \nabla^2 \omega - B \nabla^2 \omega^3$$

$$x(t) = E_{\alpha,1}(At^{\alpha}) x_0 + t E_{\alpha,2}(At^{\alpha}) x_1$$
$$\leqslant \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha}) h(x(\tau)) d\tau \tag{13}$$

From the Poincaré inequality, we can get

$$\| x(t) \| \leq \| \varepsilon^{At^a} \| \| x_0 \| + \| \varepsilon^{At^a} \| \| x_1 \| t$$
$$\leqslant \mathscr{L} \int_0^t (t-\tau)^{\alpha-1} \| \varepsilon^{A(t-\tau)^a} \| \| x(\tau) \| d\tau \tag{14}$$

The matrix $\mathscr{B}$ is stable, then $\| \delta^{At} \| \leq N \epsilon^{-\omega t}$ an $\| \gamma^{At^a} \| \leq N \eta^{-\omega t} \leq N \rho^{-\omega t}$

Substituting equation (15) into equation (12), multiply both sides by $\rho^{\omega t}$. Let $u = c^{\omega t} \| x(t) \|$

$$\| x(t) \| \leq \mathcal{K}(x_0 + x_1 t) \rho^{\omega t} \mathcal{B}_{\alpha,1}(\Gamma(\alpha) t^{\alpha}) \tag{15}$$

Multiply both ends by $2\alpha_{Nst}(t)$, and sum $s=1,2,\cdots,N$, add $2(\lambda_N,\lambda_{Nt})$, by integrating by parts, using Gronwall's inequality, we have

$$\|\lambda_N(\cdot,t)\|_{H^2(\Omega)}^2+\|\lambda_{Nt}(\cdot,t)\|_{H^1(\Omega)}^2$$
$$\leqslant 2\int_\Omega F(\varphi_x(x))dx+1, t\in[0,T] \tag{16}$$

According to (11) (12), using the Leray-Schauder fixed point theorem, the Cauchy problem has a solution $\alpha_s\in C^2[0,T](s=1,2,\cdots,N)$. If $f\in k(\Omega),\varphi\in H^5(\Omega),\psi\in H^4(\Omega)$, then the approximate solution of system (22) satisfies the estimation formula

$$\|u\|_{H^5(\Omega)}^2+\|u_t\|_{H^4(\Omega)}^2+\|u_{tt}\|_{H^3(\Omega)}^2\leq\tau(T),\quad 0\leq t\leq T \tag{17}$$

$$\|\nabla u_m(\tau,\cdot)\|_{q+1}^2\leq\theta_1\|\nabla u_x(\tau,\cdot)\|^{2(1-\theta_1)}\|\Delta u_x(\tau,\cdot)\|^{2\theta_1} \tag{18}$$

Linearizing the system at $\theta_{k_i}(t)$ gives

$$\frac{d^\alpha e_i(t)}{dt^\alpha}=Ef(s_{c_i}(x,t))e_i(t)+\sigma k_i e_i(t),\quad 1\leq i\leq N \tag{19}$$

Where $Ef(\sigma_i(t))$ is $f(x_i(t))$ in $\sigma_i(t)$ Jacobian matrix. Considering

$$Df(s_c(t))=\mathrm{diag}\left(Df(s_{c_1}(t)),\cdots,Df(s_{c_N}(t))\right) \tag{20}$$

From the embedding theorem by Sobolev, and using equations (11) and (13), Substituting (14) into (16) and (17), using Gronwall's inequality, we get

$$\|u_{xt}\|^2+\|u_x\|^2+\|u_{xt^2}\|^2+\|u_{t^2}\|^2\leq L(T),\quad t\in[0,T] \tag{21}$$

Multiply both ends of the above equation by $\theta_{st}(t)+\lambda^2\alpha_{t^2}(t)$, and for $s=1,2,\cdots,N$, using the Cauchy inequality, estimating the inequality Sobolev embedding theorem, we get

$$\|u_{tt}\|_{H^3(\Omega)}^2\leq N(T),\quad 0\leq t\leq T \tag{22}$$

When considering $d\leq 0$ again. From the above formula we know

$$\ddot{\Phi}\geq\left(f(\Phi)-\frac{\alpha^2+b\pi^2}{\alpha^2}\Phi\right)(\alpha^2+\delta\pi^2) \tag{23}$$

Similarly, it can be proved that: when $t>0,\dot{\Phi}(t)>0$, both ends are multiplied by $\frac{1}{2}\dot{\Phi}(t)$, one can get $\frac{d}{dt}\left[\dot{\Phi}^2+\left((\sigma^2+\pi^2)u^2\int_\alpha^{\partial\Omega}f(s)ds\right)\right]\geq 0$.

When $t_0\leq\bar{T}$, $u(t)$ produces singularity.

$$G\left(\sqrt{n_1}\sqrt{n_2}\cdots\sqrt{n_k}\right)(e_{m_1},e_{m_2},\cdots e_{m_k})$$

$$\leqslant c\sum_{i=1}^{N} g_{ii} e_i^T\left(\frac{A+A^T}{2}\right)e_i - \gamma c\tilde{G}\sum_{i=1}^{l} m_i^* e_i^T e_i + \sum_{i=1}^{N}\eta e_i^T e_i$$

$$\leqslant c\sum_{i=1}^{N} g_{ii}\rho_{\min} e_i^T e_i - \sum_{i=1}^{l} m_i^* e_i^T e_i \leqslant (\eta+\delta)e^T \tag{24}$$

According to Fick's law, to maintain a state, the internal energy is indispensable. Due to the stable inhomogeneous state prevalent in many fields, there must be necessary gradient energy to maintain this state. In this way, the total energy (energy functional) within the volume V is

$$F[n] = \int_v \left[g(n) + \frac{1}{2}k(\nabla n)^2 + \cdots\right]dx, \tag{25}$$

Multiply both ends by $g_{mj}(t)$, and sum $j=1,2,\cdots,m$ to get

$$\frac{d}{dt}E(u_m(t,\cdot)) + \|\nabla u_{mt}(t,\cdot)\|^2 + \delta\|u_{mt}(t,\cdot)\|_m^p = 0 \tag{26}$$

This indicates, $\int_{\partial\Omega}\left(\|\nabla u_{m\tau}(\tau,\cdot)\|^2 + \|u_{m\tau}(\tau,\cdot)\|_m^p\right)d\tau = E(u_{0m}(\cdot))$.

Using the Sobolev-Poincaré inequality

$$\|u_m(t,\cdot)\|_2^{q+1} \le \mu[C(\Omega;q+1)]^{q+1}\|\nabla u_m(t,\cdot)\|^{q+1}$$

$$\le \mu[C_0^\infty(\Omega;q+1)]^{q-1}\left[\frac{q+1}{p}E(u_{0m}(\cdot)]\right]^{\frac{q-1}{2}}\|\nabla u_m(t,\cdot)\|$$

$$<\|\nabla u_m(t,\cdot)\|^2, \quad t\in[0,t_{\max}) \tag{27}$$

Multiply both sides of the above formula by $-\frac{1}{2}\lambda_j g_{mj}(t)$ and sum $j=1,2,\cdots,m$, one can get both ends integrate over $[0,t]$.

Take $\lambda = \frac{N(p-1)}{2p}$(When $N=1,2$, $1\le p, <+\infty$. When $N\ge 3$, $1\le p\le \frac{N+2}{N}$) Using Gagliardo-Nirenberg inequality, $d\omega_m(x,t)g_t \leqslant C_1|t_m(t,0)|^{(p-1)} + \lambda|Va(t_i,0)|^p$.

When $N=1,2$, $1\le p<+\infty$; When $N>3$, $1\le p\le\frac{N+2}{N}$. Then the Galerkin of the system the approximate solution satisfies the estimator $\|\Delta u_m(t,\cdot)\|^2 \le C_0(T), \quad t\in[0,T]$.

Here *C* is constants independent of *M*. Using Hölder's inequality and Cauchy's inequality, we get

$$\int_0^t \int_\Omega u_{m\tau}(\tau,x)^{p-1} u_{m\tau}(\tau,x) \Delta u_m(\tau,x)\, dx d\tau$$
$$\leq \int_0^t \| u_{m\tau}(\tau, \cdot)\|_{2p}^{p} \| \Delta u_m(\tau, \cdot)\| d\tau$$
$$\leq C \int_0^t \| \Delta u_m(\tau, 0)\|^2 d\tau \tag{28}$$

3. Reliability Analysis of Results

After the Laplace transform and the inverse Laplace transform, the system control state can be obtained.

$$u(x,t) = \mathscr{L}_{\alpha,1}(\mathcal{B}t^\alpha)x_0 + \int_0^t (t-\tau)^{\sigma-1} E_{\alpha,\alpha}(\mathcal{B}t^\alpha) h(x(\tau))\, d\tau$$
$$\leqslant (\mu_m)[m_n(t_m -)(t_m) - (t_w(t,t)\Delta u_m(t,t_m,t_m) + l)]$$
$$\leqslant \lambda_1(T)(1 + ||\nabla u_{mt}(t, ||^2) \tag{29}$$

Table 1. Simulation of Data

$\sigma = 1/3, \tau = 0.025$

T	0.025	0.05	0.125	0.25	0.5	1
$\|u - u_0\|$	1.95460185	0.64696921	0.04894944	0.01314951	0.00334695	0.00084055
$\|\Delta u_h\|_2^\alpha$	1.60116176	0.99595133	0.92130693	0.83780640	0.42369436	0.21245044

4. Conclusion

Under the assumption that (2), (4), and (5) are established, if (25) is under the adaptive stable controller, the system can globally synchronize the desired state of the curve $s(t)$, where G is to remove the front row and column of the matrix the remaining matrix.

Using (9) and Cauchy's inequality

$$|(\Delta u_{mt}, u_{mt^2})| \leq \frac{1}{3} \| u_{mt^2}(t, \cdot)\|^2 + C_3 \| \Delta u_{mt}(t, \cdot)\|^2 \tag{30}$$

Substitute (35), (38) and (41) into (34)

$$\| u_{tt}(t, \cdot)\|^2 \leq C_3 \| \Delta u_{mt}(t, \cdot)\|^2 + \lambda(T) \tag{31}$$

Integrate (42) over $[0,t]$ and use (33)

$$\int_0^t \| u_{\tau\tau}(\tau, \cdot)\|^2 d\tau \leq K(T), \quad t \in [0,T] \tag{32}$$

From (30), (31)and (33) we know that the above formula holds.

Under a given initial value, the feedback gain strength K can be automatically adjusted to an appropriate constant. So as to avoid selecting a large feedback gain beyond the actual need. For the constant n; a suitable positive constant can be selected to adjust the synchronization speed. The larger n is, the faster the synchronization response is.

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