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A Weak-Strong Competition Model with Robin and Free Boundary Conditions

Shunqin ZHANG, Xuege ZHU, Xiaowei LIU[1]

School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China

Abstract. In this paper, we mainly discuss the Lotka-Volterra competition model with Robin boundary and free boundary conditions, and discuss the long time asymptotic behaviour of solutions in the weak-strong competition case. When $g_\infty < \infty$ the inferior competitor p can not spread successfully as $t \to \infty$. While for the superior competitor q, there are two cases: One is when $g_\infty \le R^*$, q will die out eventually; the other is when $g_\infty > R^*$, q can spread successfully. However, when $g_\infty = \infty$, both p and q have upper and lower bounds.

Keywords. Reaction-diffusion equation, Robin boundary condition, Free boundary condition, Competitive model

1. Introduction

Reaction diffusion equation is a kind of typical semi-linear parabolic partial differential equation, it can be derived from the process of the spread of invasive species, free boundary area is refers to the partial differential equation is unknown, need settlement is given, together with free boundary reaction-diffusion equation of research is one of the important direction of reaction diffusion equation of the research.

At present, Mathematicians have extensively studied competitive models with free boundary conditions. For example, Wang and Zhao [1,2] considered the one-dimensional reaction-diffusion competition model with Dirichlet and Neumann boundary conditions, proved that the alternative nature of invasive species expansion and disappearance was valid under strong-weak and weak-strong conditions, and gave an estimate of the asymptotic expansion rate of the free boundary during species expansion. Guo and Wu [3] studied the strong-weak case with Neumann boundary conditions, and proved that there is a critical value when two species expand, which makes the dominant competitive species always successfully expand when their territory size is higher than this value. Du and Lin [4] proposed a high-dimensional spatial reaction-diffusion competition model with Neumann boundary conditions to describe the spread of invasive species, discussed the strong-weak and weak-strong scenarios, and gave a rough estimate of the expansion rate when expansion occurred.

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1 Corresponding Author, Xiaowei LIU, 3501 Daxue Road, Changqing District, Jinan City, Shandong Province, China; E-mail: liuxw@qlu.edu.cn

The above studies are all aimed at Dirichlet and Neumann boundary conditions, while there are few researches on Robin boundary conditions, because the competition model with Robin boundary is more consistent with the species transmission process in some actual situations, which has theoretical and practical significance [5-7]. Therefore, we mainly research the following Lotka-Volterra model with Robin and free boundary:

$$\begin{cases} p_t = p_{xx} + p(1-p-kq), & t>0, 0<x<g(t), \\ q_t = Dq_{xx} + \gamma q(1-q-hp), & t>0, 0<x<g(t), \\ p(t,0) = bp_x(t,0), q(t,0) = bq_x(t,0), & t>0, \\ p=q=0, \quad g'(t) = -\mu(p_x + \rho q_x), & t>0, \\ g(0) = g_0, & 0 \le x \le g_0 \end{cases} \tag{1}$$

The initial function $p_0(x)$, $q_0(x)$ meets

$$p_0, q_0 \in C^2([0,g_0]), p_0 = bp_0', q_0 = bq_0', p_0, q_0 > 0 \quad in \quad (0,g_0). \tag{2}$$

2. Existence and Uniqueness of Solution

Theorem 2.1 $(p(t,x),q(t,x),g(t))$ is a unique global solution of the problem (1) and for $\alpha \in (0,1)$, $T>0$, we have

$$(p,q,g) \in [C^{\frac{1+\alpha}{2},1+\alpha}(\bar{D}_T)]^2 \times C^{1+\frac{\alpha}{2}}([0,T])$$

and define

$$\bar{D}_T = \{(t,x) \in \mathbb{R}^2 : t \in [0,T], x \in [0,g(t)]\},$$

additionally, we have a $Y>0$, then

$$0 < p(t,x), q(t,x) \le Y, \quad 0 < g'(t) \le Y \quad for \quad 0<t<\infty, \ \lim_{t\to\infty} g'(t) = 0.$$

Theorem 2.2 Suppose that $d,n>0$ and $\Gamma(x)$ satisfies [8, Theorem 2.1]. Suppose that $\Phi \not\equiv 0$ is a nonnegative and bounded, continuous function. Set the following parabolic problem has solution and is unique

$$\begin{cases} q_t - dq_{xx} = q(\Gamma(x) - np), & t>0, 0<x<\infty, \\ q(t,0) = bq_x(t,0), & t>0, \\ q(0,x) = \Phi(x), & 0 \le x < \infty, \end{cases}$$

Then

$$\lim_{t\to\infty} q(t,x) = \hat{q} \quad \text{uniformly in } [0,M) \ \forall M>0 \ . \tag{3}$$

Proposition 2.1 Suppose that $d,n>0$ and $\Gamma(x)$ satisfies [8, Theorem 2.1]. Hence, there exists $\hat{K} > \frac{1}{n}\|\Gamma\|_\infty$ and any $\ell >> 1$, let $\bar{v}_\ell(x) > 0$ be the only solution , then

$$\begin{cases} -dq'' = q(\Gamma(x) - np), & t>0, 0<x<\ell, \\ q(0) = bq'(0), q(\ell) = \hat{K} \end{cases}$$

Then we can get

$$\lim \bar{q}_\ell(x) = \hat{q}(x) \quad \text{consistently in } [0,M) \ \forall M>0 \ .$$

Proposition 2.2 Suppose that $d, n > 0$ and $\Gamma(x)$ satisfies [8, Theorem 2.1]. Let $0 < \omega \le 1$ and $v_\omega^{\pm}(x) > 0$ be the unique solution of

$$\begin{cases} -dq'' = q(\Gamma(x) \pm \omega - nq), & t > 0, 0 < x < \infty, \\ q(0) = bq'(0). \end{cases}$$

Then

$$\lim \overline{q}_\omega^{\pm}(x) = \hat{q}(x) \text{ consistently in } [0, M) \ \forall M > 0$$

3. The Long Time Asymptotic Behaviour of the Solution

According to Theorem 2.1 and Proposition 4.1 of [8], we get this theorem.

Theorem 3.1 The problem (1) has a unique solution (p, q, g). When $g_\infty < \infty$, then

$$\lim_{t\to\infty} \max_{0\le x\le g(t)} p(t,x) = 0. \tag{4}$$

This shows that if the inferior competitor can not spread successfully, it will vanish eventually.

Let $\gamma > 0$ and $\lambda_{\frac{D}{\gamma}}(d,1)$ be the first eigenvalue of the following problem

$$\begin{cases} -\dfrac{D}{\gamma}\varphi'' - \varphi = \lambda\varphi, & 0 < x < d, \\ \varphi(0) = b\varphi'(0). \end{cases} \tag{5}$$

Then $\lambda_{\frac{D}{\gamma}}(d,1)) \ge -r$ for all $c > 0$, $\lambda_{\frac{D}{\gamma}}(d,1) \ge -1$ is strict decreasing in d, γ. Moreover, for the fixed d, there exists a unique $R^* > 0$ so that $\lambda_{\frac{D}{\gamma}}(R^*;1) = 0$.

Theorem 3.2 We assume that $g_\infty < \infty$, let $Q_0(x)$ be the only solution, then

$$\begin{cases} -DQ'' = \gamma Q(1-Q), & 0 < x < g_\infty, \\ Q(0) = bQ'(0), Q(g_\infty)=0. \end{cases} \tag{6}$$

then

$$\text{When } g_\infty \le R^*, \quad \lim_{t\to\infty} \max_{0\le x\le g(t)} q(t,x) = 0; \tag{7}$$

$$\text{When } g_\infty \ge R^*, \quad \lim_{t\to\infty} \max_{0\le x\le g(t)} |q(t,x) - Q_0(x)| = 0. \tag{8}$$

Proof: According to Theorem 3.1, we construct the following function and where $\mathbb{Z}_\tau$ is the only solution,

$$\begin{cases} \mathbb{Z}_t - D\mathbb{Z}_{xx} = \gamma\mathbb{Z}(1 + h\tau - \mathbb{Z}), & t > T, 0 < x < g_\infty, \\ \mathbb{Z}(t,0) = b\mathbb{Z}_x(t,0), \quad \mathbb{Z}(t, g_\infty)=0, & t > T, \\ \mathbb{Z}(T,x) = \varphi_*(x), & 0 < x < g_\infty. \end{cases}$$

There are $0 < \tau \le 1$, $T > 0$ such that $u < \tau$ in $D_T^g = \{(t,x) : t \ge T, 0 \le x \le g(t)\}$. And

$$\varphi_*(x) = \begin{cases} q(T,x), & 0 \le x \le g(T), \\ 0, & g(T) \le x \le g_\infty. \end{cases}$$

So by the comparison principle we have $q \le \mathbb{Z}_\tau$ in D_T^g.

Case 1: For the one hand, when $g_\infty < R^*$. Then we have $\lambda_{\frac{D}{\gamma}}(g_\infty, 1) > 0$. For any given $\tau > 0$ such that $\lambda_{\frac{D}{\gamma}}(g_\infty, 1+h\tau) > 0$. Then the problem

$$\begin{cases} -DQ_\tau'' = \gamma Q_\tau(1+h\tau - Q_\tau), & 0 < x < g_\infty, \\ Q_\tau(0) = bQ_\tau'(0), \quad Q_\tau(g_\infty) = 0, \end{cases} \tag{9}$$

has no non-trivial and non-negative solution, which implies $\lim_{t\to\infty} \mathbb{Z}_\tau(t,x) = 0$ consistently in $[0, g_\infty]$. Therefore

$$\limsup_{t\to\infty} q(t,x) \le 0 \text{ consistently in } [0, M)\ \forall M > 0. \tag{10}$$

Since $q \ge 0$, Therefore, we get

$$\lim_{t\to\infty} q(t,x) = 0 \text{ consistently in } [0, M)\ \forall M > 0. \tag{11}$$

Next, when $g_\infty = R^*$, we can get $\lambda_{\frac{D}{\gamma}}(g_\infty, 1+h\tau) < 0$. Let $Q_\tau(x)$ be a unique solution of problem (9). Recall $q \le \mathbb{Z}_\tau$ in D_T^g, $\lambda_{\frac{D}{\gamma}}(g_\infty, 1) = 0$, it is easy to see that $Q_\tau(x)$ is the continuous function at τ, one has $\lim_{\tau\to 0} Q_\tau(x)$ consistently on $[0, g_\infty]$. As a result, (10) and (11) are still true.

Finally, we assume that (7) is false, there exist $\delta > 0$ and $\{(t_i, x_i)\}_{i=1}^\infty$, with $0 \le x_i < g(t_i)$ and $t_i \to \infty$, such that

$$q(t_i, x_i) \ge 2\delta, \quad i = 1, 2, ... \tag{12}$$

When $0 \le x_i < g_\infty$, there exist a subsequence of $\{x_i\}_{i=1}^\infty$ and $x_0 \in [0, g_\infty]$, such that $x_i \to x_0$. According to (11) and (12) there $x_0 = g_\infty, x_i - g(t_i) \to 0$. By theorem 3.1 and choose $\zeta > 0$ is so small that $\zeta Y < \delta$. We can deduce that $x_i > g_\infty - \zeta$ for all i. Furthermore, by Theorem 2.1 and (12), we conclude

$$|q(t_i, x_i) - q(t_i, g_\infty - \zeta)| = |q_x(t_i, \bar{x}_i)|(x_i - g_\infty + \zeta) \le Y\zeta,$$

$$q(t_i, g_\infty - \zeta) \ge q(t_i, x_i) - |q(t_i, g_\infty - \zeta) - q(t_i, x_i)| \ge q(t_i, x_i) - Y\zeta \ge \delta,$$

where $\bar{x}_i \in (g_\infty - \zeta, x_i)$. It follows from (11) that $\lim_{i\to\infty} q(t_i, g_\infty - \zeta) = 0$. We get a contradiction. So (7) hold.

Case 2: For the other hand, when $g_\infty > R^*$, we have $\lambda_{\frac{D}{\gamma}}(g_\infty, 1) < 0$, then $\lambda_{\frac{D}{\gamma}}(g_\infty, 1+h\tau) < 0$ for all $\tau > 0$. We also have

$$\limsup_{t\to\infty} q(t,x) \le Q_0(x) \text{ consistently on } [0, M)\ \forall M > 0. \tag{13}$$

Now we choose $\eta > 0$ is small such that $g_\infty - \eta > R^*$ and $g_\infty - \eta > g_0$. Therefore, there exists $T > 0$ such that $g(t) > g_\infty - \eta$ for all $t > T$. We also can get $\lambda_{\frac{D}{\gamma}}(g_\infty - \eta, 1 - h\delta) < \lambda_{\frac{D}{\gamma}}(g_\infty - \eta, 1) < \lambda_{\frac{D}{\gamma}}(R^*, 1) = 0$. Moreover, let $Q_\eta > 0$ is the unique solution of

$$\begin{cases} -DQ'' = \gamma(1 - h\tau - Q), & 0 < x < g_\infty - \eta, \\ Q(0) = bQ'(0), \quad Q(g_\infty - \eta) = 0. \end{cases}$$

Then $\lim_{t\to\infty} \mathbb{Z}_\eta(t,x) = Q_\eta(x)$ and $\liminf_{t\to\infty} q(t,x) \ge Q_\eta(x)$ consistently in $[0, g_\infty - \eta]$, it is similar to Case 1 that we have (8) is true.

Theorem 3.3 Supposed that $g(\infty) = \infty$. if $0 < h < 1 \le k$, then the solution $(p(t,x), q(t,x))$ of (1) meets

$$\liminf_{t\to\infty} p(t,x) \ge \underline{p}(x) \quad \limsup_{t\to\infty} p(t,x) \le \overline{p}(x)$$
$$\liminf_{t\to\infty} q(t,x) \ge \underline{q}(x) \quad \limsup_{t\to\infty} q(t,x) \le \overline{q}(x)$$

consistently in $[0, M)$ $\forall M > 0$.

Proof: Step 1: Set $z(t,x)$ be the only solution of

$$\begin{cases} z_t - Dz_{xx} = \gamma z(1 - z), & t > 0, 0 < x < \infty, \\ z(t,0) = bz_x(t,0), & t > 0, \\ z(0,x) = \varphi^*(x). \end{cases}$$

Where we set

$$\varphi^*(x) = \begin{cases} q_0(x) & 0 \le x \le g_0, \\ 0, & x \ge g_0. \end{cases}$$

On the principle of comparison, we get $q(t,x) \le z(t,x)$, where t is a positive constant and $0 \le x \le g(t)$. Replace $Q_0(x)$ with $\overline{q}(x)$ in (6), then $\lim_{t\to\infty} z(t,x) = \overline{q}(x)$ consistently on $[0, M)$ $\forall M > 0$, As $g(\infty) = \infty$, we have

$$\limsup_{t\to\infty} q(t,x) \le \overline{q}(x) \text{ consistently in } [0, M) \ \forall M > 0 .$$

It is similar to step 1 that we can prove $\limsup_{t\to\infty} p(t,x) \le \overline{p}(x)$.

Step 2: As $0 < h < 1 \le k$, we can choose $\varepsilon > 0$ such that $h(1+\varepsilon) < 1$. For any given $\ell >> 1$, exist $T > 0$, then we can get $g(T) > \ell, p(t,x) < \overline{p} + \varepsilon$, $\forall 0 \le x \le \ell, t \ge T$. Moreover, when $\ell >> 1$, set $q_\ell^*(x)$ be a only solution of the following problem

$$\begin{cases} -Dq'' = \gamma q[1 - q - h(1+\varepsilon)], & 0 < x < \ell, \\ q(0) = bq'(0), & q(l) = 0 \end{cases}$$

Because of $q_x(T,0) > 0$ and $q(t,\ell) > 0$, it is a positive constant $\sigma < 1$ such that for all $0 \le x \le \ell$, Since $\overline{p}(x) \le 1$, it is clear that $\sigma q_\ell^*(x)$ is the lower solution of the problem

$$\begin{cases} -Dq'' = \gamma q[1 - q - h(\overline{p} + \varepsilon)], & 0 < x < \ell, \\ q(0) = bq'(0), \quad q(\ell) = 0 \end{cases} \tag{14}$$

Then $q^\ell(t,x)$ be the only solution of

$$\begin{cases} q_t - Dq_{xx} = \gamma q[1 - q - h(\overline{p} + \varepsilon)], & 0 < x < \ell, \\ q(t,0) = qv_x(t,0), \quad q(t,\ell) = 0 \\ q(T,x) = \sigma q_\ell^*(x), & 0 < x < \ell. \end{cases}$$

Then we have $q(t,x) \ge q^{\ell}(t,x)$, $0 \le x \le \ell, t \ge T$, and $q^{\ell}(x)$ increases as t increase. Similarly to above, we get limit $\lim_{t\to\infty} q^{\ell}(t,x) := \underline{q}_{\ell}(x)$. Moreover, there holds uniformly on $[0,\ell]$, Consequently,

$$\liminf_{t\to\infty} q(t,x) \ge \underline{q}_{\ell}(x) \text{ consistently in } [0,\ell]. \tag{15}$$

When $\ell \to \infty$ and $\varepsilon \to 0$, and due to Propositions 2.1 and 2.2. We have

$$\lim_{\ell\to\infty} \underline{q}_{\ell}(x) = \underline{q}(x) \text{ consistently in } [0,M) \ \forall M > 0.$$

Therefore, we have $\liminf_{t\to\infty} q(t,x) \ge \underline{q}(t,x)$ consistently in $[0,M)\ \forall M > 0$.

Step 3: Since $0 < h < 1 \le k$, choose $\varepsilon_0 > 0$ such that $\frac{kh(1+\varepsilon_0)}{k} < 1$. The rest of proof is similar to Step 2, then we can prove $\liminf_{t\to\infty} p(t,x) \ge \underline{p}(t,x)$ consistently in $[0,M)\ \forall M > 0$.

4. Spreading and Vanishing of Criteria

Theorem 4.1 If $0 < h < 1 \le k$, $g_\infty \le \infty$, then $g_\infty \le R_1$ for the problem of (1). then $g_0 \ge R_1$ implies $g_\infty = \infty$.

The proof can refer to [8, Theorem 5.1], where we omit the details.

Now we discuss the case $g_0 < R_1$.

Lemma 4.1 Assume that $0 < h < 1 \le k, g_0 < R_1$. For the problem of (1), if

$$\mu \ge \mu^0 := \max\{1 + \frac{1}{1-h} \| q_0 \|_\infty\} \frac{D}{\rho} (\frac{\pi^2 D}{\gamma})(2\int_0^{s_0} x q_0(x)dx)^{-1},$$

then $g_\infty = \infty$.

Proof: In view of [9, lemma3.5], we can get $g_\infty = \infty$

In view of [8], we have $h < 1$, $\frac{1}{k} < 1$, there exist a unique $R_1 > 0$, $R_2 > 0$ such that $\lambda_1(0, 1-h, R_1) = 0$, $\lambda_1(0, 1-\frac{1}{k}, R_2) = 0$, $1-h$, $1-\frac{1}{k}$ is the principle eigenvalue of

$$\begin{cases} -\frac{D}{\gamma}\varphi'' = \lambda\varphi, & 0 < x < R_1, \\ \varphi(0) = b\varphi'(0), & \varphi(R_1) = 0. \end{cases} \qquad \begin{cases} -\varphi'' = \lambda\varphi, & 0 < x < R_2, \\ \varphi(0) = b\varphi'(0), & \varphi(R_2) = 0. \end{cases} \tag{16}$$

Lemma 4.2 Suppose that $0 < h < 1 \le k$, if $g_0 < \min\{R_1, R_2\}$, then there exist $\mu_0 > 0$, such that $s_\infty \le \infty$ provided $\mu \le \mu_0$.

Proof: This proof method is analogous to [8-10], then we left out details.

Define

$$f(t)=\max\{Cexp\int_0^t \gamma[1-\frac{(1-h)R_1^2}{H^2}]ds,\ \ Cexp\int_0^t[1-\frac{(1-\frac{1}{k})R_2^2}{H^2}]ds\},\ \ t\geq 0,$$

$$\xi(t)=(\xi_0+\mu\sigma\int_0^t f(s)ds)^{\frac{1}{2}},\quad t\geq 0,$$

$$w(t,x)=f(t)q^*(\frac{x}{\xi(t)}),\quad t\geq 0,\quad t>0,0\leq x\leq \xi(t).$$

Therefore, the pair (w,ξ) satisfies

$$\begin{cases} w_t-w_{xx}-w(1-w)>0, & t>0,0<x<\xi(t),\\ w_t-Dw_{xx}-\gamma w(1-w)>0, & t>0,0<x<\xi(t),\\ w(t,0)\geq bw_x(t,0), w(t,\xi)=0, \xi'(t)=-\mu(1+\rho)w_x(t,\xi) & t>0,\\ w(0,x)=C(sin\frac{ax}{\xi_0}+\beta acos\frac{ax}{\xi_0}), & 0\geq x\geq \xi_0, \end{cases}$$

where $\xi(0)=\xi_0>g(0)$ and by comparison principle we have $g(t)\leq\xi(t)$. Taking $t\to\infty$, when $0<\mu\leq\mu_0$, then $s(\infty)<\lim_{t\to\infty}\xi(t)<\infty$.

Theorem 4.2 Suppose that $g_0<\min\{R_1,R_2\}$, there exist $\mu^*\geq\mu_*>0$, it is related to (p_0,q_0,g_0), then $g_\infty=\infty$ when $\mu>\mu^*$, and $g_\infty\leq\infty$ when $\mu\leq\mu^*$.

This proof is analogous to [8, Theorem 5.2].

Reference

[1] Wang M, Zhao J. Free boundary problems for a Lotka–Volterra competition system [J]. Journal of Dynamics and Differential Equations, 2014, 26(3): 655-672.

[2] Wang M, Zhao J. A free boundary problem for the predator–prey model with double free boundaries [J]. Journal of Dynamics and Differential Equations, 2017, 29(3): 957-979.

[3] Guo J S, Wu C H. Dynamics for a two-species competition–diffusion model with two free boundaries [J]. Nonlinearity, 2014, 28(1): 1.

[4] Du Y, Lin Z. The diffusive competition model with a free boundary: invasion of a superior or inferior competitor [J]. arXiv preprint arXiv:1303.0454, 2013.

[5] Wang M. Spreading and vanishing in the diffusive prey–predator model with a free boundary [J]. Communications in Nonlinear Science and Numerical Simulation, 2015, 23(1-3): 311-327.

[6] Bunting G, Du Y, Krakowski K. Spreading speed revisited: analysis of a free boundary model [J]. Networks Heterogeneous Media, 2012, 7(4): 583.

[7] Wang M, Zhang Y. Two kinds of free boundary problems for the diffusive prey–predator model [J]. Nonlinear Analysis: Real World Applications, 2015, 24: 73-82.

[8] Wang M. On some free boundary problems of the prey–predator model [J]. Journal of Differential Equations, 2014, 256(10): 3365-3394.

[9] Liu X, Lou B. Asymptotic behaviour of solutions to diffusion problems with Robin and free boundary conditions[J]. Mathematical Modelling of Natural Phenomena, 2013, 8(3): 18-32.

[10] Zhao J, Wang M. A free boundary problem of a predator–prey model with higher dimension and heterogeneous environment [J]. Nonlinear Analysis: Real World Applications, 2014, 16: 250-263.