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Topological Structure of $QW_G(f)$ Set

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Abstract: According to the definition of quasiweak almost periodic point, we introduce the definition of $QW_G(f)$ in the paper. We studied topological structure of *G*-quasiweak almost periodic point set. We have the following two results: (1) Let (Y, d') be a compact $G-$metric space and $f: X \to X$ be equicontinuous. Let $\{f_n\}_{n=1}^{\infty}$ be *G*-strongly uniform converge to the map *f* and $x_k \in QW_G(f_n)$. If $\lim_{k\to\infty} x_k = x$, then $x \in QW_G(f)$; (2) Let (Y, d') be a compact *G*-metric space and $f: X \to X$ be equicontinuous. If $\{f_n\}_{n=1}^{\infty}$ is *G*-strongly uniform converge to *f*, then we have $\limsup QW_G(f_n) \subset QW_G(f)$.The conclusions results generalize the corresponding results given in [Journal of Southwest China Normal University: Natural Science Edition 44(2019), 40-44].

Keywords. Topological structure, metric *G*-space, *G*-strongly uniform convergence

1. Introduction

Strongly uniform convergence is an important tool to deal with function sequence map, series with function terms and generalized integral with parametric variables and is also an important concept in topological dynamical systems. It has become one of the main topics of nonlinear science and is widely used in many fields such as engineering technology, computer, biology, informatics and economics. Therefore, strongly uniform convergence dynamical system has attracted great attention of scholars at home and abroad. The research results can be found in [1-17]. For example, Ji [1] proved that if sequence maps $\{f_n\}_{n=1}^{\infty}$ are strongly uniform converge to the map f where f is equicontinuous, then we can get that $\limsup QW(f_n) \subset QW(f)$; Wang [2] proved that $\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} W(f_n)$ is a subset of $W(f)$ under strongly uniform convergence.

According to the definition, we know that *G*-strongly uniform convergence can deduce strongly uniform convergence, but the opposite is not necessarily true. Therefore, the concept of *G*-strongly uniform convergence is different from strongly uniform convergence.

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In this paper, it is studied that topological structure of G-quasiweak almost periodic point set under strongly uniform convergence of group action. We have the following two results.

Theorem2.3 Let (Y,d') be a compact $G-$ metric space and $f:X\to X$ be equicontinuous. Let $\{f_n\}_{n=1}^{\infty}$ be G-strongly uniform converge to the map f and $x_k\in QW_G(f_n)$. If $\lim\limits_{k\to\infty} x_k = x$, then $x\in QW_G(f)$.

Theorem2.4 Let (Y,d') be a compact $G-$ metric space and $f:X\to X$ be equicontinuous. If $\{f_n\}_{n=1}^{\infty}$ is G-strongly uniform converge to f, then we have $\limsup QW_G(f_n)\subset QW_G(f)$.

2. G-Quasi Weak Qlmost Periodic Point

Throughout this section, the definition of equicontinuity can be found in [1] and the definitions of $G-$ strongly uniform convergence, metric space, metric G-space and $QW_G(f)$ can be found in [17-20], respectively.

Lemma2.1(see[21]) Let (Y,d') be a metric G-space and G be compact. For any $\varepsilon>0$ there exists $0<\delta_1<\varepsilon$ such that $d(x,y)<\delta_1$ implies

$$d'(gx,gy)<\varepsilon \text{ for all } g\in G.$$

Lemma2.2 Let (Y,d') be a metric G-space and $\{f_n\}_{n=1}^{\infty}$ be G-strongly uniform converge to f. If $x\in QW_G(f_n)$, then $x\in QW_G(f)$.

Proof: Since sequence map $\{f_n\}_{n=1}^{\infty}$ are $G-$strongly uniform converge to the map f, for any $\varepsilon>0$ there exist a positive integer $N_1\in N_+$ such that $n\ge N_1$ implies

$$d'(pf_n^{\,l}(y),sf^{\,l}(y))<\frac{\varepsilon}{2} \text{ for any } x\in X,\ p,s\in G \text{ and } l\in N_+ \tag{1}$$

Let $m\in N_+$ satisfying $m\ge N_1$. Then $x\in QW_G(f_m)$. Hence for the above $\frac{\varepsilon}{2}>0$ there exists $N_2>0$ and $\{n_i\}_{i=0}^{\infty}$ such that for any $i\ge 0$ there exists $g_i\in G$ such that

$$\#(\{r: g_i f_m^{\,r}(x)\in B(x,\frac{\varepsilon}{2}), 0\le r<n_iN_2\})\ge n_i.$$

Let

$$A_i=\{r: g_i f_m^{\,r}(x)\in B(x,\frac{\varepsilon}{2}), 0\le r<n_iN_2\}$$

$$B_i = \{r : g_i f^r(x) \in B(x,\varepsilon), 0 \le r < n_i N_2\}.$$

Suppose $r \in A_i$. Then,

$$d'(g_i {f_m}^r(x), x) < \frac{\varepsilon}{2}. \tag{2}$$

By equation (1), we can get that

$$d'(g_i {f_m}^r(x), g_i f^r(x)) < \frac{\varepsilon}{2}. \tag{3}$$

According to equation (2) and (3),

$$d'(g_i f^r(x), x) < d'(g_i f^r(x), g_i {f_m}^r(x)) + d'(g_i {f_m}^r(x), x) < \varepsilon.$$

Hence $r \in B_i$. Thus we have that

$$\#(\{l : g_i f^l(x) \in B(x,\varepsilon), 0 \le l < n_i N_2\}) > n_i.$$

So $x \in QW_G(f)$.

Theorem2.3 Let (Y, d') be a compact G-metric space and $f : X \to X$ be equicontinuous. Let $\{f_n\}_{n=1}^{\infty}$ be G-strongly uniform converge to the map f and $x_k \in QW_G(f_n)$. If $\lim_{k\to\infty} x_k = x$, then $x \in QW_G(f)$.

Proof: According to Lemma 2.1, for any $\varepsilon > 0$ there exists $0 < \delta_1 < \frac{\varepsilon}{3}$ such that $d(z_1, z_2) < \delta_1$ implies

$$d'(gx, gy) < \frac{\varepsilon}{3} \text{ for any } g \in G. \tag{4}$$

Since f is equicontinuous, for above $\delta_1 > 0$, let $0 < \delta_2 < \delta_1$. If $d'(z_1, z_2) < \delta_2$ then,

$$d'(f^l(z_1), f^l(z_2)) < \delta_1 \text{ for any } l \ge 0. \tag{5}$$

According to $\lim_{k\to\infty} x_k = x$, for above $\delta_2 > 0$ there exists $m > 0$ such that

$$d'(x_m, x) < \delta_2. \tag{6}$$

By Lemma 2.2, we can get that

$$x_m \in QW_G(f).$$

Hence for the above $\frac{\varepsilon}{3} > 0$ there exists $q > 0$ and nonnegative increasing integer sequence $\{n_i\}$ such that there exists $g_i \in G$ satisfying

$$\#(\{r : g_i f^r(x_m) \in B(x_m, \frac{\varepsilon}{3}), 0 \le r < n_i q\}) \ge n_i.$$

Let

$$A_i = \{r : g_i f^r(x_m) \in B(x_m, \frac{\varepsilon}{3}), 0 \le r < n_i q\}$$

$$B_i = \{r : g_i f^r(x) \in B(x, \varepsilon), 0 \le r < n_i q\}.$$

Suppose $r \in A_i$. Then we have that

$$d'(g_i f^r(x_m), x_m) < \frac{\varepsilon}{3}. \tag{7}$$

According to equations (4)-(6),

$$d'(g_i f^r(x_m), g_i f^r(x)) < \frac{\varepsilon}{3}. \tag{8}$$

By equations (6)-(8), we can obtain

$$d'(g_i f^r(x), x) <$$

$$d'(g_i f^r(x), g_i f^r(x_m)) + d'(g_i f^r(x_m), x_m) + d'(x_m, x) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \delta_2 < \varepsilon.$$

So $r \in B_{n_i}$. Thus for any $i \ge 0$, we can get that

$$\#(\{l : g_i f^l(x) \in B(x, \varepsilon), 0 \le l < n_i q\}) \ge n_i.$$

So $x \in QW_G(f)$.

Theorem 2.4 Let (X, d) be a compact $G-$metric space and f be equicontinuous from X to X. If $\{f_n\}_{n=1}^{\infty}$ is G-strong uniform converge to f, then we have

$$\limsup QW_G(f_n) \subset QW_G(f).$$

Proof: According to Lemma2.1, for any $\varepsilon > 0$ there exists $0 < \delta_1 < \frac{\varepsilon}{4}$ such that $d(z_1, z_2) < \delta_1$ implies

$$d(gz_1, gz_2) < \frac{\varepsilon}{4} \text{ for any } g \in G. \tag{9}$$

Since f is equicontinuous, for the above $\delta_1 > 0$, let $0 < \delta_2 < \delta_1$. If $d(z_1, z_2) < \delta_2$ then,

$$d(f^n(z_1), f^n(z_2)) < \delta_1 \text{ for any } n \geq 0. \tag{10}$$

Since $\{f_n\}_{n=1}^{\infty}$ are G-strongly uniform converge to f, for the above $\varepsilon > 0$ there exist a positive integer $N_1 \in N_+$ such that $n \geq N_1$ implies

$$d(pf_n^{\,l}(x), sf^{\,l}(x)) < \frac{\varepsilon}{4} \text{ for any } x \in X,\ p, s \in G,\ l \in N_+. \tag{11}$$

Suppose $z \in \limsup QW_G(f_n)$. Then there exists positive integer $m > N_1$ such that

$$QW_G(f_m) \cap B(z, \delta) \neq \varnothing.$$

Let

$$y \in QW_G(f_m) \cap B(z, \delta_2).$$

According to $y \in QW_G(f_m)$, for above $\frac{\varepsilon}{4} > 0$ there exists $q > 0$ and $\{n_i\}_{i=0}^{\infty}$ such that for any $i \geq 0$ there exists $g_i \in G$ such that

$$\#(\{r : g_i f_m^{\,r}(y) \in B(y, \frac{\varepsilon}{4}), 0 \leq r < n_i q\}) \geq n_i.$$

Let

$$A_i = \{r : g_i f_m^{\,r}(y) \in B(y, \frac{\varepsilon}{4}), 0 \leq r < n_i q\}$$

$$B_i = \{r : g_i f^{r}(z) \in B(z, \varepsilon), 0 \leq r < n_i q\}.$$

Suppose $r \in A_i$. Then we have that

$$d(g_i f_m^{\,r}(y), y) < \frac{\varepsilon}{4}. \tag{12}$$

According to $y \in B(z, \delta_2)$ and (9), (10), we get that

$$d(g_i f^r(y), g_i f^r(z)) < \frac{\varepsilon}{4}. \tag{13}$$

By equation (11), then,

$$d(g_i f^r(y), g_i f_m^{\,r}(y)) < \frac{\varepsilon}{4}. \tag{14}$$

By equations (12)-(14), we can get that

$$d(g_i f^r(z), z) <$$

$$d(g_i f^r(z), g_i f^r(y)) + d(g_i f^r(y), g_i f_m^{\,r}(y)) + d(g_i f_m^{\,r}(y), y) + d(y, z)$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \delta < \varepsilon.$$

Thus $r \in B_i$. Hence for any $i \geq 0$ we have that

$$\#(\{r : g_i f^r(z) \in B(z, \varepsilon), 0 \leq r < n_i q\}) \geq n_i$$

So

$$z \in QW_G(f).$$

Hence we have

$$\limsup QW_G(f_n) \subset QW_G(f).$$

This completes the proof.

3. Conclusion

We study topological structure of $QW_G(f)$ in metric G-space. The results obtained generalize the corresponding conclusions and make up for the lack of theory under G-strongly uniform convergence. In addition, it provides a theoretical basis for its application in real life.

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