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The Research of G-Almost Periodic Point

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Abstract: Firstly, it is introduced that the concepts of G-almost periodic point and G-sequence shadowing property. Then, we discuss the dynamical relationship between sequence map $\{g_k\}_{k=1}^{\infty}$ and limit map g under G-strongly uniform convergence of topological group action. We can get that (1) Let sequence map $\{g_k\}_{k=1}^{\infty}$ be G-strongly uniform converge to the map g where g is equicontinuous and the point sequence $\{y_k\}_{k=1}^{\infty}$ be the G-almost periodic point of sequence map $\{g_k\}_{k=1}^{\infty}$. If $\lim_{k\to\infty} y_k = y$, then the point y is an G-almost periodic point of the map g; (2) If sequence map $\{g_k\}_{k=1}^{\infty}$ are G-strongly uniform converge to the map g where g is equicontinuous, then $\limsup AP_G(g_k) \subset AP_G(g)$; (3) Let sequence map $\{g_k\}_{k=1}^{\infty}$ be G-strongly uniform converge to the map g. If every map g_k has G-fine sequence shadowing property, the map g has G-sequence shadowing property. These results generalize the corresponding results given in Ji and Zhang [1] and make up for the lack of theory under G-strongly uniform convergence of group action.

Keywords. Metric G-space, G-strongly uniform, G-sequence shadowing property, G-almost periodic point

1. Introduction

Strongly uniform convergence is an important concept in topological dynamical system and an important tool for us to study sequence function $\{f_n\}_{n=1}^{\infty}$ and limit function f. The research results of strongly uniform convergence are shown in [1-15]. For example, Ji and Zhang [1] proved that if sequence map $\{f_n\}_{n=1}^{\infty}$ is strongly uniform converge to the map f where f is equicontinuous, then $\limsup AP(f_n) \subset AP(f)$; Deng and Jin [2] showed that the sequence function $\{f_n\}_{n=1}^{\infty}$ are asymptotically periodic implies its limit function f is asymptotically periodic.

This paper discuss the dynamical property of $G-$almost periodic point set and $G-$sequence tracking property in metric $G-$space. Then, we can get these results.

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Theorem2.3 Let Y be compact $G-$ metric space, sequence map $\{f_n\}_{n=1}^{\infty}$ be $G-$strongly uniform converge to f where f is equicontinuous and the point sequence $\{x_k\}_{k=1}^{\infty}$ be the $G-$almost periodic point of $\{f_n\}_{n=1}^{\infty}$. If $\lim_{k\to\infty} x_k = x$, then we have $x \in AP_G(f)$

Theorem 2.4 Let Y be compact $G-$metric space. If $\{f_n\}_{n=1}^{\infty}$ is $G-$strongly uniform converge to the map f where f is equicontinuous, then we can obtain $\limsup AP_G(f_n) \subset AP_G(f)$.

Theorem 3.1 Let Y be compact G- metric space and $\{f_n\}_{n=1}^{\infty}$ be G- strongly uniform converge to f. Then, for any $n \geq 1$, f_n has $G-$ fine sequence shadowing property implies f has $G-$sequence shadowing property.

Now we will give the proof of the above theorem in sections 2 and 3.

2. G-Almost Periodic Point Set

The definition of equicontinuous can be found in [1]. The concept of $G-$Strongly uniform converge is seen in [15]. The concept of metric $G-$space can be found in [16]. The definition of $G-$almost periodic point set is seen in [17].

Lemma2.1 (see[18]) Let (Y, d') be metric $G-$ space and G be a compact topological group. Then for all $\eta > 0$,there exists $0 < \delta' < \eta$ satisfying $d'(u,v) < \delta'$ implies

$$d'(su, sv) < \eta \text{ for all } s \in G.$$

Lemma2.2 Let (Y, d') be metric $G-$ space and $\{g_k\}_{k=1}^{\infty}$ be $G-$ strongly uniform converge to g. If y is an $G-$almost periodic point of every map g_k, then we have $y \in AP_G(g)$.

Proof: Since sequence map $\{g_k\}_{k=1}^{\infty}$ are $G-$strongly uniform converge to the map g, for any $\eta > 0$ there exist $N_1 > 0$ such that $n \geq N_1$ implies

$$d'(pg_n^{\,l}(x), sg^l(x)) < \frac{\eta}{2} \text{ for any } x \in X,\ p, s \in G \text{ and } l \geq 0 \tag{1}$$

Let $m \in N_+$ satisfying $m \geq N_1$. Then, we can get

$$y \in AP_G(g_m).$$

Hence for above $\frac{\eta}{2} > 0$, there exists $N_2 > 0$ such that for any $l > 0$ there exists $r' \in (l, l + N_2]$ and $s_l \in G$ satisfying

$$d'(s_l g_m^{r'}(y), y) < \frac{\eta}{2}.$$

By equation (1), we can get

$$d'(s_l g_m^{r'}(y), s_l g^{r'}(y)) < \frac{\eta}{2}.$$

Then, we have

$$d'(s_l g^{r'}(y), y) < d'(s_l g^{r'}(y), s_l g_m^{r'}(y)) + d'(s_l g_m^{r'}(y), y)$$

$$< \frac{\eta}{2} + \frac{\eta}{2} < \eta.$$

Hence $y \in AP_G(g)$.

Theorem2.3 Let (X, d') be compact $G-$ metric space, $\{f_n\}_{n=1}^{\infty}$ be $G-$ strongly uniform converge to the map f where f is equicontinuous and $\{x_k\}_{k=1}^{\infty}$ be the $G-$almost periodic point of $\{f_n\}_{n=1}^{\infty}$. If $\lim_{k\to\infty} x_k = x$, then x is an $G-$almost periodic point of f .

Proof: According to Lemma 2.1, for all $\eta > 0$ there exists $0 < \delta_1 < \frac{\varepsilon}{3}$ such that when $d'(z_1, z_2) < \delta_1$, we have

$$d'(gz_1, gz_2) < \frac{\eta}{3} \text{ for all } g \in G. \tag{2}$$

Since f is equicontinuous, for the above $\delta_1 > 0$ there exists $0 < \delta_2 < \delta_1$ such that for any nonnegative integer $l \geq 0$, $d'(z_1, z_2) < \delta_2$ implies

$$d'(f^l(z_1), f^l(z_2)) < \delta_1. \tag{3}$$

According to $\lim_{k\to\infty} x_k = x$, for above $\delta_2 > 0$ there exists $k > 0$ satisfying

$$d'(x_k, x) < \delta_2. \tag{4}$$

By Lemma2.2, we can obtain

$$x_k \in AP_G(f).$$

Then for above $\frac{\eta}{3} > 0$, there exists $m' > 0$ such that for any $l > 0$ there exists $r' \in (l, l+m]$ and $s_l \in G$ satisfying

$$d'(s_l f^{r'}(x_k), x_k) < \frac{\eta}{3}. \tag{5}$$

By equations (2)-(4), we can obtain

$$d'(s_l f^{r'}(x_k), s_l f^{r'}(x)) < \frac{\eta}{3} \tag{6}$$

By equations (4)(5)(6), we can get

$$d'(s_l f^{r'}(x), x) < d'(s_l f^{r'}(x), s_l f^{r'}(x_k)) + d'(s_l f^{r'}(x_k), x_k) + d'(x_k, x) < \eta.$$

So $x \in Ap_G(f)$.

Theorem2.4 Let (Y, d') be compact $G-$ metric space. If $\{f_n\}_{n=1}^{\infty}$ is $G-$ strongly uniform converge to the map f where f is equicontinuous, then,

$$\limsup AP_G(f_n) \subset AP_G(f).$$

Proof: According to Lemma2.1, for all $\eta > 0$ there exists $0 < \delta_1 < \frac{\eta}{4}$ such that when $d'(z_1, z_2) < \delta_1$, we have that

$$d'(gz_1, gz_2) < \frac{\eta}{4} \text{ for any } g \in G. \tag{7}$$

Since f is equicontinuous, for the above $\delta_1 > 0$ there exists $0 < \delta_2 < \delta_1$ such that for any $l \geq 0$, $d(z_1, z_2) < \delta_2$ implies

$$d'(f^l(z_1), f^l(z_2)) < \delta_1. \tag{8}$$

Since $\{f_n\}_{n=1}^{\infty}$ are $G-$strongly uniform converge to the map f, for given $\eta > 0$ there exist $N_1 > 0$ such that $n \geq N_1$ implies

$$d'(pf_n^{\ l}(y), sf^{\ l}(y)) < \frac{\eta}{4} \text{ for any } y \in X,\ p,s \in G \text{ and any } l \geq 0 \tag{9}$$

Suppose $z \in \limsup AP_G(f_n)$. Then there exists positive integer $m > N_1$ such that

$$AP_G(f_m) \bigcap B(z, \delta_2) \neq \varnothing$$

Let $y \in AP(f_m) \bigcap B(z, \delta_2)$. According to $y \in AP_G(f_m)$, for the above $\frac{\eta}{4} > 0$ there exists $N_2 > 0$ such that for any $l > 0$ there exists $r' \in (l, l + N_2]$ and $s_l \in G$ such that

$$d'(s_l f_m^{\ r'}(y), y) < \frac{\eta}{4} \tag{10}$$

According to $y \in B(z, \delta_2)$ and equations (7)-(8), we have

$$d'(s_l f^{r'}(y), s_l f^{r'}(z)) < \frac{\eta}{4} \tag{11}$$

By equation (9), we can get

$$d'(s_l f_m^{\ r'}(y), s_l f^{r'}(y)) < \frac{\eta}{4}. \tag{12}$$

According to triangle inequality and equations (10)-(12), we have

$$d(s_l f^{r'}(z), z) < d(s_l f^{r'}(z), s_l f^{r'}(y)) + d(s_l f^{r'}(y), s_l f_m^{\ r'}(y)) +$$

$$d(s_l f_m^{\ r'}(y), y) + d(y, z) < \eta$$

So $z \in AP_G(f)$. Hence, $\limsup AP_G(f_n) \subset AP_G(f)$.

3. G-sequence Shadowing Property

The concept of $G-$sequence shadowing property and $G-$fine sequence shadowing property can be found in [19].

Theorem 3.1 Let (Y,d') be compact G- metric space and $\{f_n\}_{n=1}^{\infty}$ be G- strongly uniform converge to f . Then, for any $n \geq 1$, f_n has $G-$ fine sequence shadowing property implies f has $G-$sequence shadowing property.

Proof: For any $\varepsilon > 0$, let $0 < \delta < \frac{\varepsilon}{3}$. Suppose that $\{x_i\}_{i\geq 0}$ is $(G,\delta)-$pseudo orbit of f . Then, for all $j \geq 0$, there exists $g_j \in G$ satisfying

$$d'(g_j f(x_j), x_{j+1}) < \delta \tag{14}$$

Since $\{f_n\}_{n=1}^{\infty}$ are $G-$ strongly uniform converge to the map f , for given $\frac{\varepsilon}{3} > 0$ there exist a positive integer N_1 such that $n \geq N_1$ implies

$$d(pf_n^{\,l}(y), sf^{\,l}(y)) < \frac{\varepsilon}{3} \text{ for any } y \in X \text{ , } p,s \in G \text{ and } l \in N_+ \tag{15}$$

Let $m \in N_+$ satisfying $m \geq N_1$. By (15), we can get that

$$d'(g_j f_m(x_j), g_j f(x_j)) < \frac{\varepsilon}{3} \tag{16}$$

By (14) and (16), When $j \geq 0$, we can get that

$$d'(g_j f_m(x_j), x_{j+1}) < \frac{2\varepsilon}{3}$$

So $\{x_j\}_{j\geq 0}$ are $(G,\frac{2\varepsilon}{3})-$pseudo orbit of f_m . Hence there exists $y \in Y$, $t_j \in G$ and nonnegative increasing integer sequence $\{n_j\}_{j=0}^{\infty}$ such that $j \geq 0$ implies

$$d'(f_m^{\,n_j}(y), t_j x_{n_j}) < \frac{2\varepsilon}{3}$$

By equation (15), when $p = s = e$, we have

$$d'(f_m^{\,n_j}(y), f^{n_j}(y)) < \frac{\varepsilon}{3}$$

Hence for all $j \geq 0$, we can obtain that

$$d(f^{n_j}(y), t_j x_{n_j}) < \varepsilon$$

So the map f has $G-$sequence shadowing property.

4. Conclusion

In this paper, the dynamical property of $G-$almost periodic points and $G-$sequence shadowing property is discussed under $G-$ strong uniform convergence in metric $G-$space. The obtained results generalize the corresponding conclusions given in Ji and Zhang [1] and make up for the lack of theory under $G-$strongly uniform convergence of group action. It provides a theoretical basis for its application in real life.

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