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Partial Overlapping Order Problems in a Strip

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Abstract. In this paper, we provide an analysis on the partial overlapping order problem in a strip, i.e., whether a given partial order involving only part of the squares of the strip corresponds to a valid flat-folded state of the strip or not. On the contrary to the general intractability of partial orders, we investigated the partial orders onto some particular sets to obtain tractable results. To rapidly get access to the solution, our methodology is based on the abstracted visualized folded states rather than a mathematical explanation by matrix. In conclusion, a strip having at least three disordered squares aligning on the strip between any two of its ordered squares always corresponds to a final flat-folded state.

Keywords. Partial overlapping order, strip folding

1. Introduction

A variation of the *strip folding problem* called the partial overlapping order problem in a strip, is investigated in this research. It is expressed like this: A strip of size $1 \times n$ with all its squares labeled in order from one end to the other is supposed to be completely folded to a flat state R_t (folded to a plane) of size 1×1. Given a partial order O on a set P of squares, does there exist such a R_t that the squares in P are ordered just as O from bottom to top?

An instance is illustrated in figure 1. For the clarity, layers from bottom to top in the folded states are aligned from left to right in all the illustrations. P is composed of the shadowed squares in (a). When the input O is given as (6, 5, 3, 7, 9, 10), a corresponding valid R_t can be achieved by a folding illustrated in (b). Following this folding, we can achieve R_t with all the squares ordered as (1, 6, 5, 2, 3, 4, 7, 11, 8, 9, 10) from bottom to top. This R_t respects the partial order O. As a counterexample, when given an O as (5, 3, 7, 9, 6, 10), there exists no valid R_t corresponding to it because this O causes a self-penetration as illustrated in (c). It forces the continuous pairs (5, 6) and (9, 10) to cross each other and thus is invalid.

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Even though some work on the enumeration and characterization of the total overlapping orders of a strip was done, there seems no research considering the partial orders. In [1], it is proved that the amount of valid total overlapping orders increases regularly (exponential in n), bringing the general decision on their validity. On the opposite, the increase of the number of partial orders depends on the order itself, thus may correspond to much more intricate cases.

In this paper, we consider the cases when O is given on some particular sorts of Ps, where the length of the part of strip between every two squares ordered in O is restricted. Based on the results of these particular cases, we give the proof for the validity of O in the cases where O is given on a P that at least three disordered squares exist between any two ordered squares. In other words, any such O corresponds to a valid R_t.

1.1. Some Basic Definitions and Notations

The strip folding problem can be viewed as a restriction of the *map folding problem*. The original map folding problem was proposed by Jack Edmonds in 1997. It asks the computational complexity of the decision on whether or not a rectangle grid pattern composed of $m \times n$ squares can be folded flatly to the shape of a square of size 1×1. The grid pattern is called a *map*. It has two sides, a front and a back. Every non-boundary edge of the map, which is incident to two squares, is called a *crease*. Each crease is assigned as either a *Mountain* ("*M*") or a *Valley* ("*V*"). The assignment of a crease represents the side of its two incident squares supposed to face each other after folding it. *A Mountain-Valley assignment* (*An MV assignment*) is composed of the assignment on all the creases. When $m = 1$, the map folding problem is also called the strip folding problem, and the map is called the *strip*. An available flat state of the map after all the creases are folded is called *a final flat-folded state*. We use R_t to denote a final flat-folded state. Correspondingly, an available flat state of the map after folding only part of the creases is called a *partly flat-folded state*.

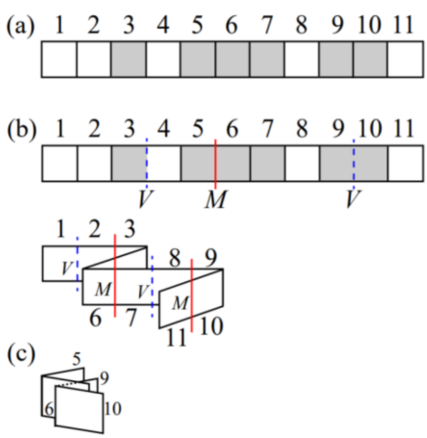

Figure 1. An instance of the partial overlapping order problem in a strip.

Unlike the decision problem based on the MV assignment, one of the variations concerns whether or not a given order on a set of squares corresponds to a valid R_t with or without a given MV assignment. If the answer is positive, then the given order is called *a valid overlapping order*. This variation is called *the overlapping order problem*. In this paper, the discussion is of the case where $m = 1$. We call this restricted case *the overlapping order problem in a strip*. Especially, when the given order is on a set includes all the squares of the map, the given order is called *the total overlapping order*. On the other hand, when the given order is on a subset of the squares, the order is called *the partial overlapping order*. This paper concerns the partial overlapping order in a strip with no MV assignment as the input.

1.2. Related Work

In the following, we list some remarkable results about the map folding problem and its variations. The original map folding problem remains unsolved when $m, n \geq 3$, i.e., the computational complexity of the decision on whether an $m \times n$ map can be folded flatly to the size 1×1 remains unknown when $m, n \geq 3$. For $2 \times n$ maps, Morgan proposed an $O(n^9)$ time algorithm to decide the foldability [2]. Arkin et al. proved that the strip folding problem is solvable in time linear in the length of the strip [3].

As a general solution of the total overlapping order problem, Nishat provided an $O(mn)$ time algorithm to decide the validity of a given overlapping order of $m \times n$ maps. He also provided an $O(mn^{mn+1})$ time algorithm to enumerate all the possible overlapping orders corresponding to valid R_ts [4]. When the target is a strip, another $O(n)$ time approach is provided in [5] to solve the decision problem. Furthermore, corresponding to a positive answer, a concrete folding process is also provided in their approach. However, the time complexity of the enumeration of the valid overlapping orders cannot be reduced to a polynomial scale even for the strips [1, 5].

In the original map folding problem, the only limit on the folding operation is no tearing, stretching, or self-penetrating. Every part of the map is flexible and bendable. As a result, the entire folding process may not be able to be divided into phases such that the partly folded states are flat at the end of every phase. This brings the difficulty to solve the problem by a reduction. To reduce the difficulty, Arkin et al. considered a restricted folding operation, which is called the *simple folding* [3]. The difference between a simple folding and a general folding is that a simple folding ensures the rigidity of every square. Concretely, in a simple folding, (1) the states before and after folding along a line are always flat; (2) only some continuously adjacent layers whose surfaces touch each other in pairs are folded along the line. Under this context, they provided linear-time algorithms to determine if an MV assignment can be folded to a size of 1×1 by simple folds for both strips and $m \times n$ maps. In fact, for the strips, any valid overlapping order achievable by a general folding is also achievable by a simple folding [5]. There are also researches concerning the general folding and the simple folding on generalized patterns instead of grid patterns [6, 7, 8, 9].

Another research direction considers the enumerative combinatorics about the strip folding. Koehler [1] and Lunnon [10] counted the possible final flat-folded states of a strip without mountains or valleys assigned to the creases. Legendre et al. discussed the relationship between foldings and the flat-folded states [11].

In an earlier study, we have investigated some partial overlapping orders in an $m \times n$ map. This paper focuses on the validity of some partial overlapping orders in a strip, which is essentially the decision on the existence of the valid total overlapping order corresponding to an input partial overlapping order. Generally, without any restriction or context, the decision problems on the total orders with partial orders given as inputs are usually intractable. These orders are usually modeled as graphs so that the corresponding decision problem can be solved using the graph theory [12, 13]. In this paper, instead of deciding the tractability of general cases, we intend to find some tractable results of the partial overlapping order problem in a strip by restricting the input partial orders onto some particular sets.

In the following sections, we will investigate the validity of the partial orders on some particular sets of squares. In Section 2.1, the condition for a valid total overlapping order will be introduced. In Section 2.2, we will show that the condition to the total overlapping orders is not feasible to the partial overlapping orders. In Section 2.3, we will define four sorts of particular Ps as the targets. In Section 3, 4, and 5, the validity of the partial order given on each P is respectively investigated. Then, as the most crucial result of this paper, for an O having at least three disordered squares aligning on the strip between any two of its ordered squares, we will prove that O is always valid, i.e., it always corresponds to a final flat-folded state.

2. Preliminaries

2.1. The Total Overlapping Order Problem in a Strip

As introduced in Section 1.1, the total overlapping order problem in a strip asks whether a given order of all the squares of the strip corresponds to the overlapping of the squares in a final flat-folded state or not. A pair of squares labeled $i, i+1$ in a strip are called *neighbors*. We say a pair of squares whose surfaces touch each other in R_t *adjacent*. Adjacent squares are supposed to be adjacent in the total overlapping order. For example, in the instance introduced at the beginning of the paper, 5 and 6 are neighbors. 1 and 6 are adjacent in R_t according to the total overlapping order (1, 6, 5, 2, 3, 4, 7, 11, 8, 9, 10). The counterexample in figure 1(c) shows that, when two pairs of neighbors penetrate each other, the corresponding overlapping order must be invalid. A precise description is given as Lemma 2.1. In [1], Koehler proved that the no penetration condition is in fact both the sufficient and necessary condition for a total overlapping order in a strip to be valid.

Lemma 2.1. A total overlapping order is valid if and only if for any two pairs of neighbors respectively labeled $i, i+1$ and $j, j+1$, any of the following four orders (a) to (d) does not occur when i and j are either both odd or both even.

(a) $i < j < i+1 < j+1$,
(b) $j < i+1 < j+1 < i$,
(c) $i+1 < j+1 < i < j$,
(d) $j+1 < i < j < i+1$,

The orders (b) to (d) can be viewed as a circular order of (a). For convenience, we say that two pairs of neighbors ordered as any of these cases form a *penetrating pair* and use *a penetrating meander* to refer to a pair of neighbors involved in a penetrating pair in the following.

2.2. Even a Partial Order Without a Penetrating Pair Could Be Invalid

According to the necessary and sufficient condition for a valid total overlapping order, it is natural to think of this question: Is the condition in Lemma 2.1 also the necessary and sufficient condition for a partial overlapping order to be valid? The answer is negative. A counterexample is proposed as follows. Considering a given partial order O without any obvious penetration itself as (6, 1, 5, 3, 4) in a 1×6 strip. Corresponding to O, six possible all permutations of the six squares can be listed as:

1. (2, 6, 1, 5, 3, 4) with penetrating pairs {1, 2} and {5, 6};
2. (6, 2, 1, 5, 3, 4) with penetrating pairs {2, 3} and {4, 5};
3. (6, 1, 2, 5, 3, 4) with penetrating pairs {2, 3} and {4, 5};
4. (6, 1, 5, 2, 3, 4) with penetrating pairs {1, 2} and {5, 6};
5. (6, 1, 5, 3, 2, 4) with penetrating pairs {1, 2} and {5, 6};
6. (6, 1, 5, 3, 4, 2) with penetrating pairs {1, 2} and {3, 4}, as well as penetrating pairs {1, 2} and {5, 6}.

Because penetrating pairs exist in every possible total order, O cannot correspond to a final flat-folded state of the strip. This result reflects that the partial overlapping order problem in a strip is not trivial. There may exist intractable cases for the decision problem. First paragraph.

Since the objective of this paper is to find some tractable results of the partial overlapping order problem, in the following, we will define four different kinds of squares sets and determine the validity of the partial orders on them.

2.3. Definitions of the Partial Overlapping Order Problem in a Strip

We consider the partial order overlapping problem in a strip whose squares are labeled from 1 at one end to n at the other end. For convenience, in the following, we use these labels to indicate the squares. In this paper, we study the partial orders on four sets of squares P_1, P_2, P_3 and P_4. We will prove the tractability of the decision problems on the validity of these partial orders. These sets are illustrated in figure 2 and defined as follows. The elements in P_1, P_2 and P_3 are certain whereas P_4 is much more generalized.

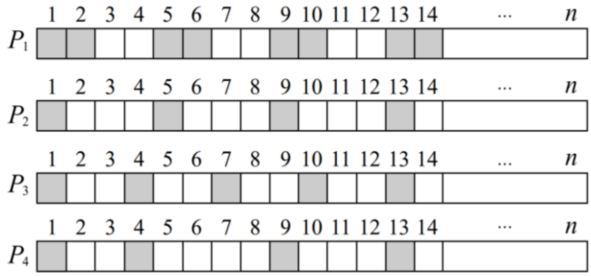

Figure 2. Four sorts of partial orders investigated in this paper.

(1) $P_1 = \{4i+1, 4i+2 \mid 0 < i < n/4\}$;
(2) $P_2 = \{4i+1 \mid 0 < i < n/4\}$;
(3) $P_3 = \{3i+1 \mid 0 < i < n/3\}$;
(4) $P_4 = \{a_1, a_2,\ldots,a_k \mid \forall a_i, a_{i+1}, a_{i+1} - a_i \geq 3, a_k \leq \mathrm{n}\}$.

O_1, O_2, O_3 and O_4 are arbitrary orders given onto P_1, P_2, P_3 and P_4, respectively. In the following, we will prove that (a) an O_1 is valid if and only if it involves no penetrating pair and (b) any O_2, O_3 and O_4 are always valid.

In the following, we will give the results of the partial overlapping order problem in a strip corresponding to the inputs O_1, O_2, O_3 and O_4, respectively. For the clearness, in some of the following expressions where whether squares appearing in the given order or not is emphasized, we use the indication $[a]$ to indicate a square labeled a appears in the given order (red ones in the illustrations in figure 3, figure 8, figure 10, figure 12 and figure 15) and use the indication $\_a\_$ (black ones in the same illustrations) if not.

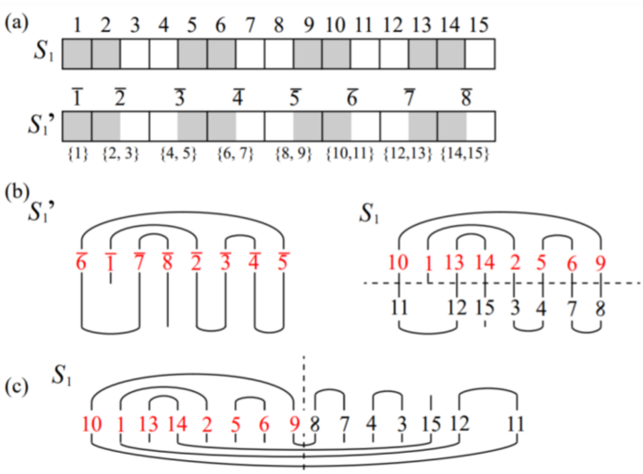

Figure 3. An instance of the folding process corresponding to an O_1 without a penetrating pair. (a) illustrates the ordered squares and how we separate the squares into groups; (b) illustrates the initial folded state corresponding to the groups and the folding process to fold this state to a final flat-folded state while respecting the given order.

3. Solution of O_1 and the Validity of an Arbitrary O_2

The squares labeled $4i$+1 and $4i$+2 comprise a pair of neighbors and may form penetrating pairs with other pairs of neighbors. Thus, penetrations may occur in every total order respecting O_1. By Lemma 2.1, we should first check the existence of the four orders (a) to (d) in O_1. If any of the four orders exist, O_1 should be decided as an invalid overlapping order. For the remaining O_1s, we will prove their validity by proving Theorem 3.1.

Theorem 3.1 O_1 is always a valid overlapping partial order given that it includes no penetrating pair.

Proof. The proof for the validity of an O_1 with no penetrating pair is given by finding an available folding corresponding to such an O_1. We first discuss a given $1 \times n_1$ strip S_1 with an odd n_1. The folding can be found through the following steps.

Step 1. Suppose there is another strip S_1' with n_1'=$(n_1 + 1)/2$, whose square are labeled as $\overline{1}$ to $\overline{(\mathrm{n}_1 + 1)/2}$ from one end to the other end. We define a mapping ϕ from S_1 to S_1'. Under this mapping, $\{1\}$ is mapped to $\overline{1}$ in S_1'. The set comprised by every pair of elements $2k$ and $2k + 1$ $(1 \leq 2k \leq n_1 - 1)$ is mapped to $\overline{k + 1}$ in S_1'.

Step 2. Under the mapping defined in Step 1, for every integer $i \geq 0$, $\overline{2\imath + 1}$ in S_1' corresponds to $4i + 1$ in O_1 and $\overline{2\imath + 2}$ in S_1' corresponds to $4i + 2$ in O_1. Therefore, the mapping induces a total overlapping order O_1' of S_1' corresponding to O_1 by mapping the order of the squares in O_1 to an order of the layers of S_1'. There exists no penetrating pair in O_1' by the condition that O_1 includes no penetrating pair.

Step 3. By tracing the mapping ϕ back, the final folded state of S_1' respecting O_1' can also be considered as a valid partly flat-folded state of S_1. In this state, the crease between every pair of elements $2k$ and $2k+1$ remains unfolded whereas all the other creases are folded. The unfolded creases align on the same line (*centerline*) so that we can view the partly folded state as a reduced strip of size 1×2 and fold its crease as a valley. The state after this fold is of size 1×1 and respect O_1.

The three steps above lead to a valid folding corresponding to an O_1 with no penetrating pair. For a strip S_1 with an even n_1, we only have to append an additional square adjacent to its neighbor square $n_1 - 1$ in the overlapping order and then implement the same steps. Theorem 3.1 is proven.

To make these steps easy to understand, we use an example as illustrated in figure 3 to show a concrete folding process. This instance involves a strip S_1 of size 1×15 with its $P_1 = \{1, 2, 5, 6, 9, 10, 13, 14\}$ as shadowed. The valid folding process is supposed to correspond to an $O_1 = (10, 1, 13, 14, 2, 5, 6, 9)$.

As illustrated in figure 3(a), corresponding to S_1, we have a strip S_1' with $n_1' = 8$. The mapping ϕ from S_1 to S_1' maps the squares sets indicated below S_1' in the figure to the singletons indicated over S_1'. Each singleton is a rectangle formed by two continuous squares in S_1 and considered as a single layer in S_1'. The total overlapping order of S_1' induced by O_1 is $O_1' = (6, 1, 7, 8, 2, 3, 4, 5)$. The left figure in figure 3(b) gives a valid flat-folded state of S_1' respecting O_1'. The right figure in figure 3(b) shows the corresponding partly flat-folded state of S_1. The unfolded creases align on the dashed line. We view these creases as a single crease and view this state as a strip of size 1×2. It is clear that in such a partly flat-folded state, the squares over the dashed line (which can be viewed as a single square of the 1×2 strip) comprise P_1 and respect O_1. Then, we fold the squares below the dashed line to the right of the squares over the line. The folded state is shown in figure 3(c). It indicates a valid total overlapping order of S_1 as $([4], [6], [13], [14], [7], [9], [1], [2], \lfloor 8 \rfloor, \lfloor 7 \rfloor, \lfloor 4 \rfloor, \lfloor 3 \rfloor, \lfloor 15 \rfloor, \lfloor 12 \rfloor, \lfloor 11 \rfloor)$ corresponding to the input O_1.From Step 2, it is also clear that for an O_1 including no penetrating pair, we can always find a corresponding folding in time linear in the size of the strip. Next, we give Theorem 3.2 for the validity of an arbitrary O_2 in a strip.

Theorem 3.2 Any arbitrary O_2 corresponds to at least one final flat-folded state of the strip.

Proof. By Theorem 3.1, we only have to prove the existence of an O_1= ($4a_1$+1, $4a_1$+2, $4a_2$+1, $4a_2$+2,..., $4a_k$+1, $4a_k$+2) with no penetrating pair corresponding to any given O_2 = ($4a_1$+1, $4a_2$+1,..., $4a_k$+1). For every square $4i+1$ $(0 < i < (n-2)/4)$ in O_2, we append its neighbor square $4i+2$ as its adjacent square over it in the overlapping order. In this way, every pair of neighbor squares in O_1 is adjacent to each other. Therefore, no penetrating pair would occur in O_1. Theorem 3.2 is concluded.

4. Validity of an Arbitrary O_3

After proving the validity of the partial order given onto the set of squares labeled $\{4i + 1\}$, we aim to prove the validity of an arbitrary O_3 by a different method in this section with the assumption that O_3 is the partial order given onto the set of squares labeled $\{3i + 1\}$ on the strip S_3 of size $1 \times n_3$.

4.1. A Mathematical Analysis Using Matrices

Any given order on the set of squares can be considered as a poset $(A, \overline{<})$, where A is the set of squares in O_3 and $\overline{<}$ denotes the "*adjacent under-over*" relation. If $a \overline{<} b$ holds, then a should be directly adjacent and below b in the final overlapping order. The final total order on the set of all squares forms a *chain* with respect to $\overline{<}$. Moreover, when generalized to the ``under-over" but not necessarily adjacent relation $<$. $<$ on A can be computed as the transition closure of $\overline{<}$. Our objective is to extend the relation $<$ to the set of all squares while avoiding penetration.

The matrix expression of $<$ ($\overline{<}$) is described as: Create a $n \times n$ logical matrix (each element is either 0 or 1) M for the given n squares. Its element m_{ij}, i.e., the element at the ith row and jth column denotes the truth value of the relation "Square i is below (and adjacent to) Square j". m_{ij} is definitely different from m_{ji}, endowing the matrix with a "skew-symmetric" property.

We now focus on the matrix of $<$, which is a power sum of the matrix of $\overline{<}$ according to the transition closure correspondence [14].

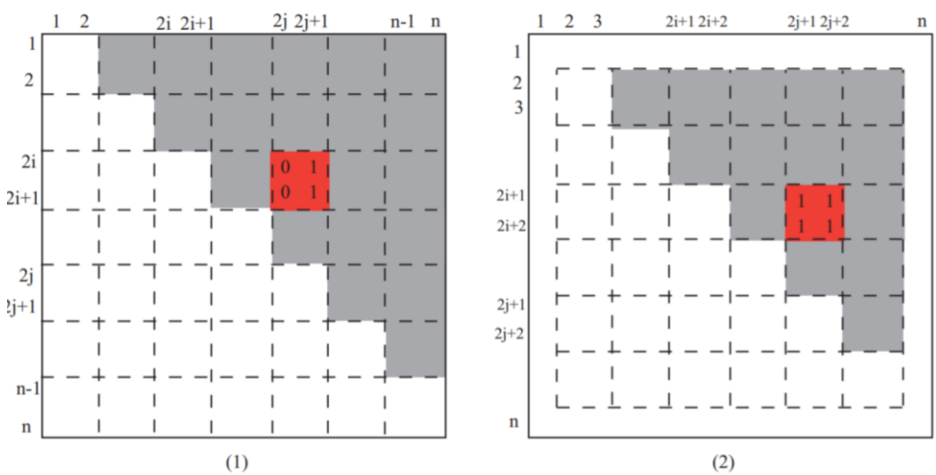

Figure 4. Detecting pairs form 2×2 sub-matrices in the representation matrix. Matrices denoted by the red squares in the two illustrations correspond to whether $\{(2i, 2i + 1), (2j, 2j + 1)\}$ and $\{(2i + 1, 2i + 2), (2j + 1, 2j + 2)\}$ form penetrations or not, respectively. Both assignments have even number of 0s, defining two non-penetration cases.

The non-penetration constraint is expressed as: for all the pairs $\{(2i, 2i + 1), (2j, 2j + 1)\}$ and $\{(2i + 1, 2i + 2), (2j + 1, 2j + 2)\}$, if their corresponding 2×2 sub-matrices, i.e., all the grey enclosed areas of size 2×2 illustrated in figure 4 (where n is assumed to be even), do not have an odd number of 0s, then there is no penetration.

Following this expression, the task becomes: other than the elements fixed by O_3, assign the other elements such that there is no odd number of 0s in every sub-matrix. The assigning should also make sure that the matrix actually corresponds to an overlapping order, which corresponds to this constraint: the matrix should be a power sum of a matrix with $n-1$ elements assigned 1.

Since O_3 only occupies one in every two sub-matrices and only one element in every four elements of a sub-matrix, the best strategy is to give elements reasonable initial values and then locally revise the values of the elements in the sub-matrix with odd 0s. However, the revision simultaneously influences the order. Thus, each revision involves a group of small local revisions.

The following method can be considered as the visualized version of the revisions to the matrix. In such a manner, we can always fix the type of penetrations and handle them differently. This enables us to find a fine way to do local revisions and rapidly get to the global solution.

4.2. Workflow of Our Solution

In all the following illustrations, the overlapping rectangles of size 1×2 are indicated by parallel vertical line segments in the length 2. The horizontal line segment passing through all the midpoints of these line segments is called the *centerline*.

O_3 would not include any penetrating pair itself because no neighbor squares are involved in O_3. Without loss of generality, we assume that n_3 is even. A strip whose n_3 is odd can be folded in the same way as the strip of $1 \times (n_3+1)$ size with the last square removed.

The strategy to obtain a valid folded state of S_3 respecting O_3 can be concluded as four steps. The concrete handling in each step will be respectively introduced in the following sections.

Step 1. Build a mapping like we used for O_1 from S_3 to S_3'. This mapping induces a partial order O_3' on S_3'.

Step 2. Choose a *pseudo folded state* of S_3' such that in this state, no *spiral* exists, and only two sorts of specialized penetrating pairs exist. “pseudo” refers to the fact that some penetrations may exist in such a state. The definition of spirals will be introduced in Section 4.4.

Step 3. Use split planes to separate the pseudo folded state obtained in Step 2 to sub-parts (as illustrated in figure 9(a)), so that each sub-part has the penetrating pairs all aligning either above or below the centerline l. Respectively find the valid flat-folded state of each sub-part. For convenience, following we say that each sub-part is supposed to have the penetrating pairs all aligning on the same *side*.

Step 4. Join the valid flat folded states of the sub-parts to get the valid final folded state of O_3.

4.3. The Mapping from S_3 to S_3'

Using a similar mapping as we introduced for O_1 in the last section, S_3 is firstly mapped to another strip S_3' whose every layer is comprised of two neighbor squares. S_3' is of size $1 \times \mathrm{n}_3'$ where $\mathrm{n}_3' = n_3/2$. Its squares are labeled as $\overline{1}$ to $\overline{n_3/2}$ from one end to the other. There exists a mapping ψ from S_3 to S_3'. Under this mapping, every $\{2k-1, 2k\}$ $(1 \le 2k \le n_3)$ is mapped to $\bar{k}$ in S_3'. An instance is illustrated in figure 15 to help the explanation. O_3 is given as (1, 7, 13, 4, 10, 16). The sets of the squares in S_3 indicated below S_3' are respectively mapped to the elements of S_3' (single layers in the shape of a rectangle of size 1×2) indicated over S_3'.

Similar to the induced total overlapping order mentioned in Section 3.1, the mapping from S_3 to S_3' induces a partial overlapping order O_3' of S_3', which is an order onto the set $\{\overline{3j+1}, \overline{3j+2} \mid 0 \leq j \leq n_3/3\}$. Since $\overline{3j+1}$ and $\overline{3j+2}$ are a pair of neighbors, two such pairs may form a penetrating pair in a pseudo folded state of S_3'. For example, the mapping of the instance in figure 15 induces a partial order $O_3' = (\overline{1}, \overline{4}, \overline{7}, \overline{2}, \overline{5}, \overline{8})$ of S_3'. In any of its flat state, two pairs of neighbors $\{(\overline{1}, \overline{2}\}$ and $\{(\overline{7}, \overline{8})\}$ must form a penetrating pair.

4.4. Choose a Pseudo Folded State of S_3'

We first choose a pseudo folded state of S_3'. The choice of the state respects the principle that no *spiral* formed by more than 4 layers exist. The description of spirals can be detailed by the overlaps of spiral states, which satisfy that: either (a) the pair (i+2, i+3) forms an adjacent pair with i, i+1 respectively folded to the two sides of the adjacent pair, as shown in figure 5, or the other half, (b) the pair (i, i+1) exchanges the role with (i+2, i+3) in (a).

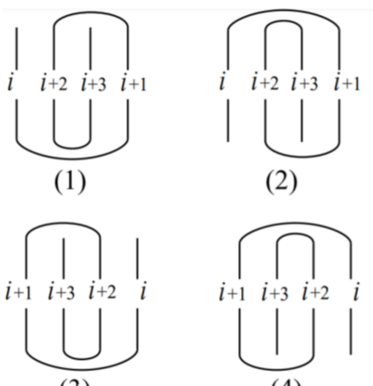

Figure 5. Half of the possible spiral states. The other half is achieved by exchanging (i, i+1) and (i+2, i+3).

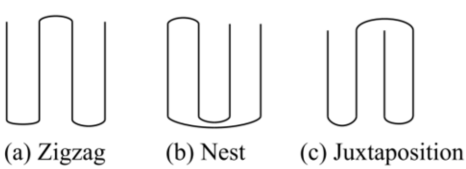

Figure 6. The desired flat-folded states of four continuous layers.

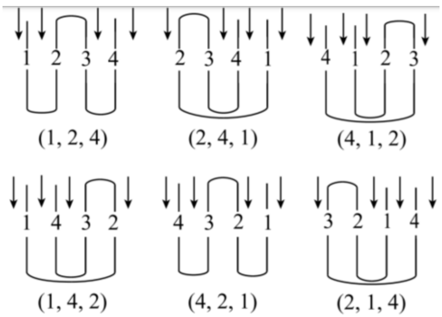

Figure 7. The folded states without spiral.

This non-spiral principle can be realized by locally restricting the folded state of every four continuous layers. Concretely, every four continuous layers of S_3' are desired to be folded to a *zigzag*, a *nest*, or a *juxtaposition*, as shown in figure 6. The realization satisfying this principle is illustrated in figure 7, showing some possible choices of the folded states of $\{1, 2, 3, 4\}$ corresponding to the partial orders given on $\{1, 2, 4\}$. In these illustrations, except for two zigzags (1, 2, 4) and (4, 2, 1), each of the rests forms a nest. Then, for the folded state of {2, 3, 4, 5} along the other side, we indicate the possible arrangements of 5 using arrows, showing that in some cases, juxtapositions may be produced as well. Referencing the mapping defined from S_3 to S_3', any four continuous layers $\{\overline{i}, \overline{i+1}, \overline{i+2}, \overline{i+3}\}$ in S_3' must have one layer or two layers not involved in O_3'. Hence, every four continuous layers of S_3' can be arranged following the illustration, i.e., to a zigzag, a nest or a juxtaposition.

Based on the local arrangement, we have the following lemma for the pseudo folded state of S_3'.

Lemma 4.1 There always exists a non-spiral state of S_3' respecting O_3'.

Proof. Such a pseudo folded state can be achieved by a step-by-step process where each time the next four layers are appended to the folded state. Since in figure 6, we have shown that whatever the given order is, the first five layers have a corresponding arrangement. Since the next layer, 6, is disordered, we can just choose one of the possible positions referencing the arrangement of 3, 4, 5 to avoid producing a spiral. Since we do not mind the penetrations, the possible choices for the ordered 4, 5, 7 are the same as 1, 2, 4. By a mathematical induction, along such a process, every continuous four layers can be arranged without a spiral because a disordered layer must exist in the four-layers-tuple. This lemma is concluded.

A pseudo folded state of S_3' defined by Lemma 4.1 is chosen in Step 2. In a nest, a neighbor pair locates inside another neighbor pair. We use *inner pair* and *outer pair* to refer to the neighbor pairs, respectively.

It should be remarked that an appropriate pseudo folded state matching S_3' is not trivial, i.e., other than the unavoidable penetrations (the neighbor pairs violating Lemma 2.1 in O_3'), an induced S_3' sometimes must involve other penetrations. For example, we consider the four layers respectively formed by {[11], _8_}, {[13], _14_}, {_5_, _6_} and {_3_, [8]} in the instance illustrated on the right of figure 15(b). Only the disordered layer of 5 and 6 can be moved, and once it is moved to the left of {13, 14} to avoid the penetration of {5, 6, 7, 8} and {13, 14, 15, 16}, then on the other side, {3, 4, 5, 6} and {11, 12, 13, 14} would form a penetrating pair, as shown in figure 8. Such penetrating pairs have the property that even though they can be avoided by moving some disordered layers, the moving must cause other penetrations.

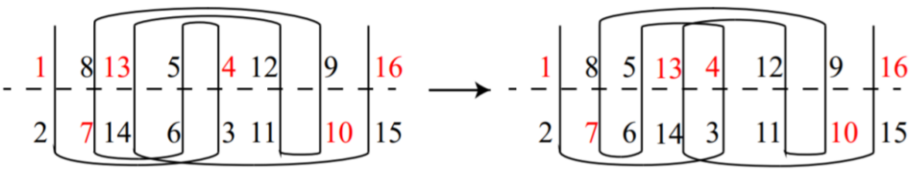

Figure 8. Moving the nested penetrating pair causes penetration on the other side of the centerline.

4.5. *The Folding for a State Whose Penetrating Pairs Aligning on the Same Side*

The next objective is to prove that all the penetrating pairs can be taken away while respecting O_3 by folding some layers along the centerline l. O_3' can always be separated to sub-parts. The principle is, each sub-part corresponds to a folded state with penetrating pairs aligning on only one side and with no spiral. For example, one of the possible two sub-part-separations of a pseudo folded state is illustrated in figure 9. Note that the layers in a sub-part can be disconnected because the separating line may intersect at multiple points with the folded state of O_3.

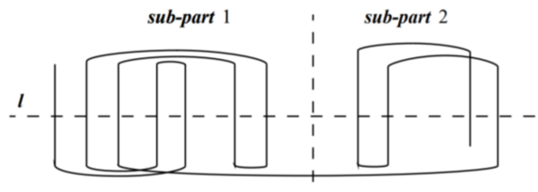

Figure 9. Separate the folded state obtained in Step 2 to sub-parts.

To simplify the entire proof of the existence of flat-folded states for any O_3, we first provide the folding process to fold each sub-part to a valid folded state in this section.

For convenience, we use A to indicate a sub-part. Without loss of generality, the penetrations in A is assumed to be under the centerline. In Section 4.6, we will show that we can connect the folded sub-parts together without new penetration, so that we can achieve a final folded state of S_3 respecting O_3.

A is handled by the following process, which is supposed to be repeated until no penetrating pair exists any more.

1. Find the innermost penetrating pair. A penetrating pair formed by neighbor pairs p_i_1 and p_i_2 is called *innermost* if there exist no penetrating pair formed by p_j_1 and p_j_2 where either p_j_1 is nested in p_j_1 or p_j_2 is nested in p_i_2.For example, $\{1,2\}$ and $\{3,4\}$ both form penetrating pairs with $\{7,8\}$ in the left figure in figure 15(b) while the one formed by $\{3,4\}$ and $\{7,8\}$ is treated as the innermost one.

2. Fold either neighbor pair of the innermost penetrating pair along the centerline with obstacles handled at the same time. By obstacles we mean the layers which would form new penetrations during the fold.

Now putting the handling of obstacles aside, our basic handling of the penetrating pairs is divided into three cases as illustrated in figure 10, where the moved layers are indicated by the segments colored red. The division is with respect to the ordered layers in the neighbor pairs. When the order matters, the ordered layers are indicated by colored numbers. These three cases exhaust all the possibilities. The left column lists the penetrating states and the right column gives corresponding solutions. The penetrating pairs p_i_1 (layers labeled from i_1) and p_i_2 (layers labeled from i_2) are supposed to be handled. Without loss of generality, we fold the layers of p_i_2 and assume the non-existence of obstacles. The first case (a) and the second case (b) correspond to the same folded state where $i_1 + 4$ is located on the right side of $i_2 + 3$ but with different squares ordered. Case (a) is the illustration with $i_2 + 2$ ordered (indicated by the red script), while in (b), either $\{i_2, i_2 + 3\}$ (indicated by the red script) or $i_2 + 1$ (indicated by the blue script) is ordered. In the case (b), we only show the solution when the layer $\{i_2 - 2, i_2 - 1\}$ locates on the right side of $\{i_2, i_2 + 1\}$. The converse case corresponds to a solution where i_2 and $i_2 + 1$ exchange their positions, so as to avoid penetrations formed by $\{i_2 - 1, i_2\}$ and $\{i_2 + 1, i_2 + 2\}$. In the third case (c), no matter which square in i_2 appears in the order, the handling is always the same. The illustration explains the invariance of the order of the squares in O_3. The feasibility to apply these operations is based on the non-spiral restriction, which will be detailed together with the handling of obstacles in the proof of Lemma 4.3.

To show an example, in figure 15(c) and (d), the penetrating pairs formed by $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$ and $\{13, 14, 15, 16\}$ are differently handled. (c) gives the handling when $\{13,14,15,16\}$ is folded while the two figures in (d) give the handlings when $\{5, 6, 7, 8\}$ and $\{1, 2, 3, 4\}$ are successively folded to avoid the penetration with $\{13, 14, 15, 16\}$.

The rest cases which cannot be handled with only the basic handlings are taken as cases with *obstacles*.

Concretely, the obstacles refer to the pairs forming new penetrating pairs with other tuples when we intend to use the basic handling. In such cases, the obstacles are supposed to be handled before the handling of penetrating pairs. When obstacles exist, the obstacles have to be removed before the basic handling. For single obstacles, we use *left folds* or *right folds* illustrated in figure 11 to remove them.

As an example, when folding $\{14, 15\}$ as illustrated in the left figure in figure 15(c), it certainly intersects with the layers formed by $\{9, 10, 11, 12\}$. In this occasion, the four-tuple $\{9, 10, 11, 12\}$ forms a single obstacle and is supposed to be folded by a right fold as in the right figure in figure 15(c).

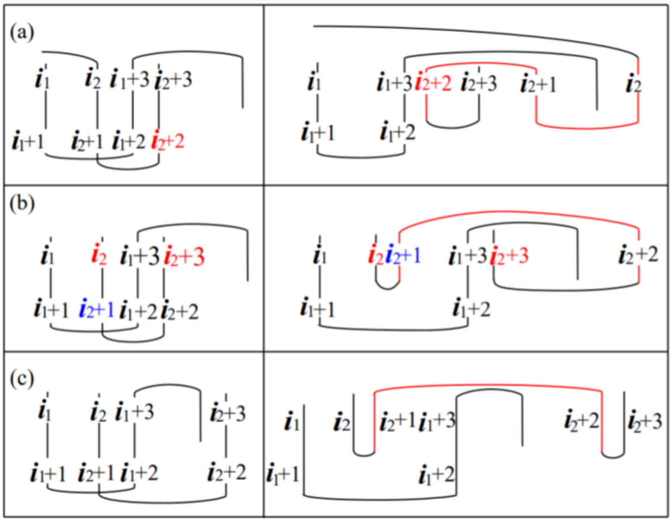

Figure 10. The basic operations to remove the penetration.

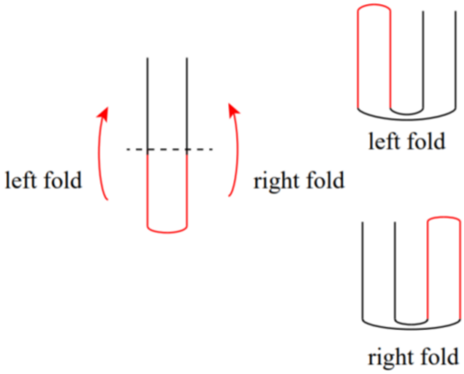

Figure 11. The left fold and the right fold.

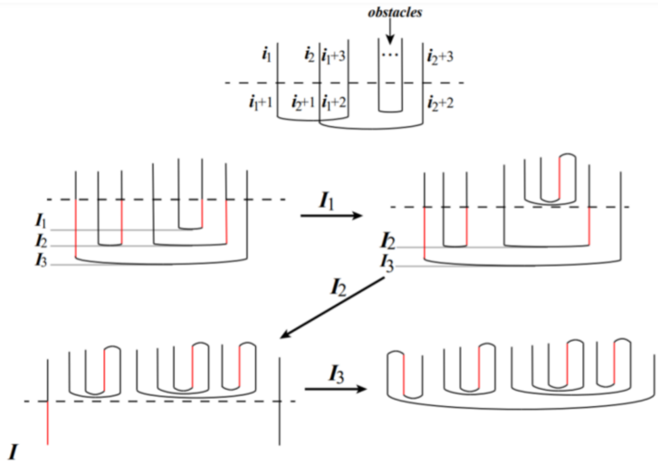

Figure 12. Multiple obstacles handled at the same time.

When multiple obstacles exist, the concrete operation to remove all the obstacles is explained by the process illustrated figure 12. This example details the possible combinations of non-spiral obstacles, which follows the pseudo initial state. The only possible position of obstacles is shown in the first illustration. All the obstacles must be surrounded by either neighbor pair of the penetrating pair because a neighbor pair formed by an outside layer and an inside layer of the neighbor pairs would form a penetration with the penetrating pair. Based on our assumption that the currently handled penetrating pair is always the innermost one, any two obstacles would not penetrate with each other. Thus, the obstacles can only be combined in a nested way or in a juxtaposed way. For example, in figure 12, the obstacles are classified to the sets (I_1, I_2, I_3). I_1 is nested in an obstacle in I_2 and the two obstacles in I_2 are juxtaposed together. The ordered layers are colored red and supposed to keep their order after the handling. For an obstacle with a square on the left side and below the centerline involved in the given order, a left fold is supposed to be applied. Similarly, when such a square on the right side is involved in the given order, a right fold is supposed to be applied. When the ordered squares are over the centerline, both the left fold and the right fold are available. This instance shows that such handling of obstacles would not change the given order. Moreover, because the handling of every obstacle only locally changes the folded state around itself, no new penetration would be produced. After all the obstacles are handled, the penetrating pairs are removed by the basic handlings. Sometimes after the handling of an inner penetrating pair, the outer penetrating pairs may have their penetration moved to another place, as illustrated in figure 13. In such cases, the handling of them keeps unchanged, i.e., they are handled as if the penetrations are not moved. The entire handling of A is a repeating process of the above handlings.

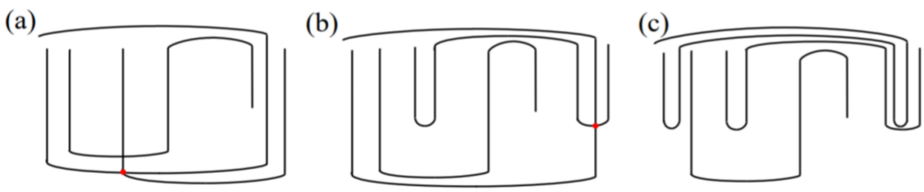

Figure 13. An example of the moving of the outer penetrating pairs after an inner one is handled. (a) indicates the initial positions of two penetrating pairs. (b) indicates the state after the inner penetrating pair is handled. The penetrating points of outer penetrating pairs are painted red. (c) shows a possible handling of the outer penetrating pair, which still follows the initial penetrating.

Lemma 4.2 The above process leads to a valid folded state of A without penetration.

Proof. To prove Lemma 4.2, it is sufficient to prove that the process of handling the obstacles and removing the penetrating pairs is always feasible without producing new penetrations. Because all the obstacles are shown to be dealt with left folds and right folds along the centerline which only changes itself, no new penetration would be produced. Then, as illustrated in figure 10, the step-by-step removal of penetrating pairs also would not form any penetration along the centerline. It only remains to prove that such a process would not produce penetrations on the above side, which is the same with the next sub-part connected with A (if exists). By our assumption of non-spiral, the analysis of the penetration on the upper side can be induced from the arrangement on a gadget of length 1 (the ones of length 2 can be dealt as obstacles) with the connected layer from different directions, as illustrated in figure 14. We can only talk about the gadget because in a trivial case as illustrated, all the basic handlings can all be applied below it. All the possible positions of such a gadget are checked one by one for the three basic cases (a), (b) and (c), where the locally trivial cases are omitted. A folding is feasible if it keeps the gadget not penetrated with the removed squares and the relative position of its direction lines unchanged. There are two possibilities for both (b) and (c) with respected to the square ordered in O_3 in the gadget. All the other sophisticated cases can be achieved by finitely combining such gadgets together. The non-spiral condition permits us to adjust the preceding or the succeeding two squares of an ordered square. Along this way, all the foldings can be proven feasible. Lemma 4.2 is then concluded.

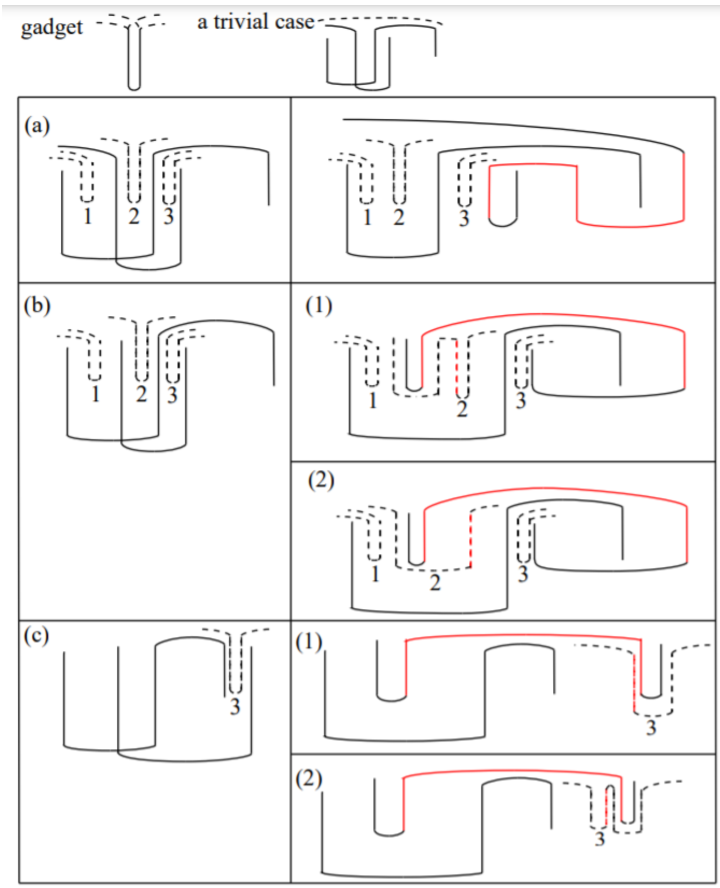

Figure 14. An instance of the folding process corresponding to an O_3.

As an example, in figure 15(c) and (d), the above process is realized by folding {14, 15} and {2, 3}, {6, 7}, respectively. The process shown in (c) leads to the folded state in the left figure in (e). The figure on the right side gives the final folded state.

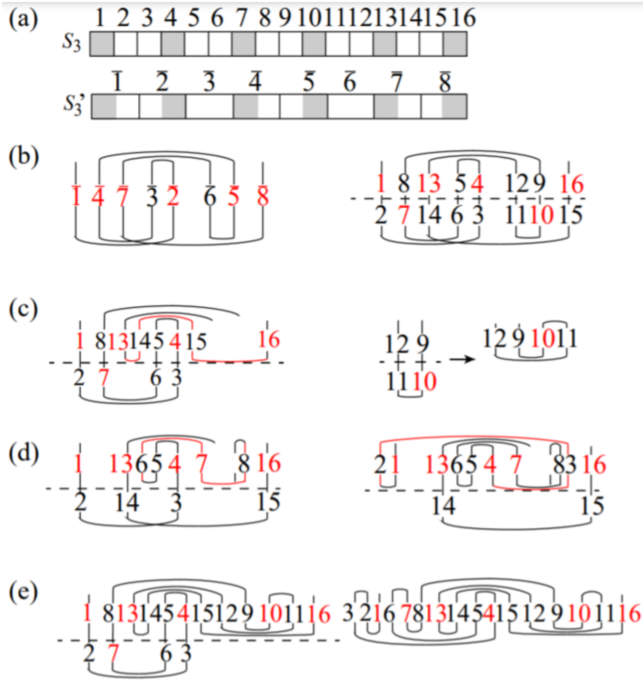

Figure 15. An instance of the folding process corresponding to an O_3.

4.6. Joining the Sub-parts

After all the sub-parts respectively reach the valid folded states (even though some layers may be still of size 1×2), they are supposed to be joined together.

We first consider the case that the separating points remain visible after the penetrations in every sub-part are removed. Because we always have the choice to fold either pair of layers of a penetration pair, we can find a folding after which all the separating points keep their positions relatively unchanged. In other words, the order of the separating points on the separating line is kept. This invariance ensures the connectability of the folded sub-parts.

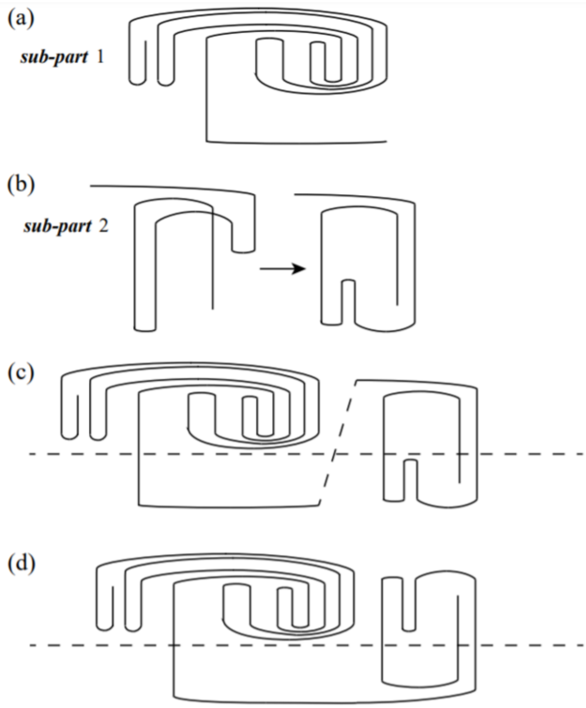

Figure 16. The joining process with respect to a single separating point.

When two parts are supposed to be joined together, one part is supposed to be reflected with respect to the centerline, so as to achieve a folded state where handled layers (when removing the penetrations) locate on the same side of the centerline. When there is only one separating point, two parts can be simply joined by first moving the separating point to the other side, then symmetrically reflecting this part with respect to the centerline, and finally gluing the two parts together. A joining for the instance given in figure 9 is illustrated in figure 16, where the sub-part 2 goes through the reflection. To avoid the possible penetration induced by moving the separating point, we can always move the separating point before removing all the penetrations in a sub-part, so that the penetration related to the separating point can be handled with other penetrations at the same time.

When multiple separating points exist, the reflection may cause new penetrations. To deal with such cases, in the following we give the handlings corresponding to whether two separating points are aligning on the same side of the centerline or on different sides. The cases where more separating points exist can then be dealt with the same method, with the feasibility given by the mathematical induction.

(a) When two separating points align on the same side of the centerline, in one of the two separated sub-parts, the positions of the two separating points would be exchanged before the handling of the penetration. By choosing the sub-part supposed to have squares exchanged, we can keep the local exchange of separating points either inducing a new penetrating pair on the same side with the other penetrations of the same sub-part, or leading to a remove of an exist penetrating pair. In either case, it would not influence the other penetrations and the non-spiral state because this penetration is supposed to happen at the end of this sub-part. The handling of penetrations still follows the discussion in the last section.

(b) When two separating points align on different sides of the centerline, also, we first exchange the positions of separating points of one sub-part and then deal with the penetrations. The exchange would not influence the other obstacles and thus the handling of the second part still follows the discussion in the last section. An example is illustrated in figure 17 and 18. The positions of the separating points a and b are first exchanged in the second part. Then, the two parts are respectively handled and joined. If the separating points become invisible after the handling of penetrations, we first join them following the above method. The result of such a joining may still have penetrations of the two layers of size 1×2 connected by the separating point. These penetrations are supposed to be handled after all the other layers of size 1×2 are folded to 1×1, which will be introduced later in this section. In such a state, since a neighbour pair has at least two freely movable squares and the non-spiral condition is kept during the entire folding, the penetration can then be removed by the basic operations introduced in the last section. According to the fact that all the other layers are folded to size 1×1, no obstacle would exist.

Using the above method, all the folded sub-parts can be joined while respecting their total order in O_3. After the joining, all the remaining 1×2 layers should align on the same side. Conversely, on one side of the current state, except for the layers at the separating point, all the other layers should be of size 1×2 (non-handled layers). Similar to the handling of obstacles, these remaining 1×2 layers are folded by left folds and right folds to keep O_3, as illustrated in figure 15(e). The 1×1 layers at the separating points can be handled in a similar way, only with an additional twist if penetrations happen, as illustrated in figure 17.

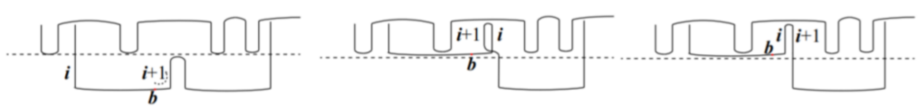

Figure 17. An instance of a right-fold at a separating point.

After such folds, the valid folded state of the entire strip can be obtained. The feasibility of the folding process is concluded in Lemma 4.3, its correctness follows by the connectability of sub-parts and Lemma 4.2.

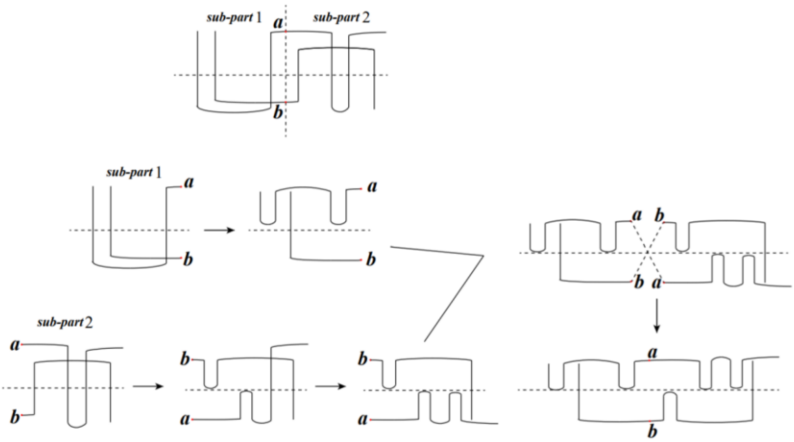

Figure 18. The joining process when multiple separating points exist.

Lemma 4.3. Any arbitrary O_3 can correspond to a valid flat-folded state of the strip.

5. Validity of an Arbitrary O_4

O_4 is a generalized version of O_2 and O_3. By our definition, O_4 is onto the set $P_4 = \{a_1, a_2, \cdots, a_k | \forall a_i, a_{i+1}, a_{i+1} - a_i \geq 3, a_k \leq n\}$. When $\forall a_i, a_{i+1}, a_{i+1} - a_i = 4$, deciding the validity of O_4 is reduced to deciding the validity of O_2. Also, when $\forall a_i, a_{i+1}, a_{i+1} - a_i = 3$, deciding the validity of O_4 is reduced to deciding the validity of O_3.

We will first prove the validity of an O with an additional condition that $\forall a_i, a_{i+1}, a_{i+1} - a_i = 3$ or 4, as presented in Lemma 5.1. Based on this result, we then prove the validity of an arbitrary O_4 without any additional condition.

Lemma 5.1. An arbitrary O given onto the set $\{a_1, a_2, \cdots a_k | \forall a_i, a_{i+1}, a_{i+1} - a_i = 3 \text{ or } 4, a_k \leq n\}$ is a valid overlapping partial order.

Proof. O can be folded following a similar process as the handling for O_3. Each two continuous squares in S_4 (assumed with even number of squares) are mapped to a layer in with $n'_4 = n_4/2$. Under this mapping, every pair of elements $2k - 1$ and $2k(1 \leq 2k \leq n_4)$ is mapped to s_k in S'_4, which is formed by n'_4. Then, S'_4 is folded following the steps we fold S'_3. During the folding process, once a penetrating pair appears, it can be handled in the same way as we introduced for S_3 because there always exist the same number of squares that can be freely arranged with S_3. Thus, O always corresponds to a valid folded state of the map.

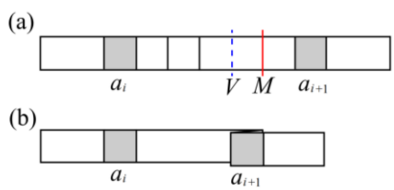

Figure 19. The zigzag formed by $\{a_{i+1} - 2, a_{i+1} - 1, a_{i+1}\}$.

Based on Lemma 5.1, we can give the conclusion as Theorem 5.2 on the generalized set O_4.

Theorem 5.2 An arbitrary O_4 given onto the set $\{a_1, a_2, \cdots a_k | \forall a_i, a_{i+1}, a_{i+1} - a_i \geq 3, a_k \leq n\}$ is a valid overlapping partial order.

For every pair of a_i, a_{i+1} with $a_{i+1} - a_i > 4$ in O_4, consider a zigzag formed by $\{a_{i+1} - 2, a_{i+1} - 1, a_{i+1}\}$ as a_{i+1} as illustrated in figure 19. Then O_4 can be folded using the same method as the folding of O in Lemma 5.1.

6. Conclusion and Future Work

The topic is the partial overlapping order problem in a strip, which is a variation of the well-known strip folding problem. Unlike the total overlapping order problem in a strip, the validity of the partial overlapping order cannot be decided by whether or not penetrating pairs exist in the order. We first explained this conclusion by giving a counterexample. Then, we investigated four sorts of partial orders and their validity. The first three are on typical sets of squares, and the fourth is a generalized partial order inspired by them. We showed the tractability of the partial overlapping order problem for these partial orders.

The partial overlapping order problem of other sorts of partial orders would be considered as an open question. Furthermore, are the results for the partial overlapping order problem in a strip able to be generalized to the partial overlapping order problem in a map of size $m \times n$ would be considered as another interesting topic.

Acknowledgments

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