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The Existence of Multiple Positive Solutions of a Riemann-Liouville Fractional q-Difference Equation Under Four-Point Boundary Value Condition with p-Laplacian Operator

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Abstract. This paper mainly studies the existence of multiple positive solutions of a class of Riemann-Liouville fractional q-difference equations under the four-point boundary value condition with p-Laplacian operator. The existence of two positive solutions of the q-difference equation is verified by the monotonic iterative method. Finally, an example is used to prove the validity of the main results obtained.

Keywords. P-Laplacian operator, q-difference equation, multiple positive solutions, the existence.

1. Introduction

The calculus invented by Newton and Leibnitz is the watershed between modern mathematics and ancient mathematics. Fractional calculus is a related theory about differentiation and integration of any order. It is the extension of integer-order calculus. From q-differential calculus and quantum after the calculus was proposed by Jackson, it attracted the attention of many scholars to the q-difference equation. Quantum calculus is called infinite calculus. It replaces the classical derivative with a difference operator and can be used to calculate non-differentiable functions. In addition to the application of q-difference to orthogonal polynomials, combinatorics, hypergeometric functions and other mathematical fields, q-differences are increasingly used in natural sciences and engineering [1].

At present, fractional differential equations with p-Laplacian operators have received widespread attention due to their outstanding applications in viscoelastic mechanics, non-Newtonian mechanics, electrochemistry, fluid mechanics, and materials science. The related theoretical research on the boundary value problem of q-differential equation with p-Laplacian operator is not only the need of the development of differential equation theory, but also the need of social production and

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life. It is hoped that its related theories can play a certain role in practical applications [2, 3].

In recent years, some preliminary results have been achieved:

In 2018, Bai, C. [4] studied the following problems with p-Laplacian operators:

$$\begin{cases} \left(\varphi_p\left(D_{0+}^{\alpha}u(t)\right)\right)' + f(t,u(t)) = 0, \quad 0<t<1, \\ u(0) = D_{0+}^{\alpha}u(0) = 0, \quad {}^{C}D_0^{\beta}u(0) = {}^{C}D_0^{\beta}u(1) = 0, \end{cases}$$

with $0<\beta\le 1, 2<\alpha\le 2+\beta$,and α , β are real number, D_{0+}^{α} and ${}^{C}D_0^{\beta}$ are the standard Riemann-Liouville fractional derivative and Caputo fractional derivative of order α , β respectively, $p>1, f \in C([a,b]\times R, R)$, which used to prove the existence and uniqueness of nontrivial solution with fractional boundary value using the Banach contraction mapping theorem and Guo-Krasnosel'skii fixed point theorem

In 2019, Zhou, B. [5] studied the following high-order fitting fractional boundary value problems with p-Laplacian operators:

$$\begin{cases} T_{\alpha}^{0+}\left(\varphi_p\left(T_{\alpha}^{0+}u(t)\right)\right) = f\left(t,u(t),T_{\alpha}^{0+}u(t)\right), \\ u^{(i)}(0) = 0, \quad \left[\varphi_p\left(T_{\alpha}^{0+}u\right)\right]^{(i)}(0) = 0, \\ \left[T_{\beta}^{0+}u(t)\right]_{t=1} = 0, \quad \left[T_{\beta}^{0+}\left(\varphi_p\left(T_{\alpha}^{0+}u(t)\right)\right)\right]_{t=1} = 0, \end{cases}$$

with $n-1\le\alpha<n$, T_{α}^{0+} is a newly defined fractional derivative called "integrated fractional derivative". Using the Guo-Krasnosel'skii fixed point theorem, sufficient conditions are established to ensure the existence of a positive solution to the above boundary value problem.

For some basic theories and applications of fractional boundary value problems with p-Laplacian operators, please refer to the literature [6-18].

In 2020, Zhou, J. et al. [19] studied the boundary value problem of fractional q-difference equations:

$$\begin{cases} D_q^{\alpha}u(t) + f(t,u(t),v(t)) = 0, \quad t\in(0,1), \\ u(0) = 0, \quad D_q u(0) = D_q u(1) = 0, \end{cases}$$

where $0<q<1, 2<\alpha\le 3$, the function $f(t,u,v)$ may be singular at $v=0$ and $t=0,1$ around. The iterative algorithm is used to obtain the existence and uniqueness of the positive solution of the boundary value problem.

For some basic theories and applications of boundary value problems of fractional q-difference equations, please refer to the literature [20-25].

Inspired by the above literature, we discuss the following equation:

$$\begin{cases} D_q^{\alpha}\left(\phi_p\left(D_q^{\beta}x\right)(t)\right) + h(t,x(t)) = 0, \quad 0<t<1, \\ x(0) = 0, \quad x(1) = aD_q^{\gamma}x(\xi), \quad D_q^{\beta}x(0) = 0, \quad D_q^{\beta}x(1) = bD_q^{\beta}x(\eta), \end{cases} \tag{1}$$

where $h \in C([0,1]\times[0,+\infty),[0,+\infty))$, D_q^{α}, D_q^{β} and D_q^{γ} stand for the Riemann-Liouville fractional q-derivative, φ_p is p-Laplacian operator, $p>1$, $\varphi_p(s)=|s|^{p-1}s$, $\varphi_p^{-1}=\varphi_{q^*}$, $\frac{1}{p}+\frac{1}{q^*}=1$, $1<\alpha,\beta\le 2$, $\gamma=\frac{\beta-1}{2}$, $0<\xi\le\frac{1}{2}$, $0<\eta<1$, $a,b\in[0,+\infty)$, $a\Gamma_q(\beta)\xi^{\frac{\beta-1}{2}}<\Gamma_q\left(\frac{\beta+1}{2}\right)$, $b^{p-1}\eta^{\alpha-1}<1$.

2. Preliminaries

In this section, let $q\in(0,1)$, some related definitions and lemmas are given.

Definition 2.1[26] $[m]_q=\frac{1-q^m}{1-q}$, $m\in R$.

Definition 2.2[26] The q-similar definition power function $(n-m)^k$, $k\in N_0:=\{0,1,2,\ldots\}$ is:

$$(n-m)^{(0)}=1,\ (n-m)^{(k)}=\prod_{i=0}^{k-1}\left(n-mq^i\right),\ k\in N,\ m,n\in R.$$

Generally, $\gamma\in R$,

$$(n-m)^{(\gamma)}=n^{\gamma}\prod_{i-0}^{\infty}\frac{n-mq^i}{n-mq^{\gamma+i}},\ n\ne 0.$$

Particularly, $m=0$, $n^{(\gamma)}=n^{\gamma}$, $[a(n-m)]^{(\gamma)}=a^{\gamma}(n-m)^{(\gamma)}$.

Definition 2.3[26] q-gamma function is defined by

$$\Gamma_q(t)=\frac{(1-q)^{(t-1)}}{(1-q)^{t-1}},\ t\in R\setminus\{0,-1,-2,\ldots\}.$$

Then, $\Gamma_q(t+1)=[t]_q\Gamma_q(t)$.

Definition 2.4[26] The q-derivatives for h is defined by

$$(D_q h)(t)=\frac{h(t)-h(qt)}{(1-q)t},\ t\ne 0.$$

$$(D_q h)(0)=\lim_{t\to 0}(D_q h)(t).$$

Definition 2.5[26] The high order q-derivatives for h is defined by

$$\left(D_q^0 h\right)(t)=h(t),$$

$$\left(D_q^k h\right)(t)=D_q\left(D_q^{k-1}h\right)(t),\ k\in N.$$

Definition 2.6[26] h is defined on the interval $[0,b]$, q-integral for 0 to b is defined by

$$(I_q h)(t)=\int_0^t h(s)d_q s = t(1-q)\sum_{i=0}^{\infty} h(tq^i)q^i \text{, } t\in[0,b].$$

Lemma 2.1[27] If $a\in[0,b]$, and h is defined on the interval $[0,b]$, q-integral for a to b is defined by

$$\int_a^b h(s)d_q s = \int_0^b h(s)d_q s - \int_0^a h(s)d_q s .$$

Lemma 2.2[27] The operator I_q^k obtains

$$(I_q^0 h)(t)=h(t),\quad (I_q^k h)(t)=I_q(I_q^{k-1}h)(t),\quad k\in N .$$

Lemma 2.3[27] $(D_q I_q h)(t)=h(t)$; if function h is continue at $t=0$, $(I_q D_q h)(t)=h(t)-h(0)$.

Definition 2.7[26] Let $v\geq 0$ and h be a real function defined on a certain interval $[0,T]$. The Riemann-Liouville fractional q-integral of order v is defined by $(I_q^0 h)(t)=h(t)$ and

$$(I_q^v h)(t)=\frac{1}{\Gamma_q(v)}\int_0^t (t-qs)^{(v-1)}h(s)d_q s ,\quad v>0 ,\quad t\in[0,T].$$

Definition 2.8[26] Let $v>0$. The Riemann-Liouville fractional q-derivative of order v is defined by Riemann-Liouville $(D_q^0 h)(t)=h(t)$ and

$$(D_q^v h)(t)=(D_q^l I_q^{l-v} h)(t),\ v>0 .$$

Where l is the smallest integer greater than or equal to v .

Lemma 2.4[27] Let $\alpha,\beta\geq 0$, f be a function defined on a certain interval $[0,T]$, Then the following formulas hold:

(1) $(I_q^\beta I_q^\alpha f)(t)=(I_q^{\alpha+\beta} f)(t)$,

(2) $(D_q^\alpha I_q^\alpha f)(t)=f(t)$.

Lemma 2.5[27] Let $\alpha>0$, n is positive integer. Then the following equality hold:

$$(I_q^\alpha D_q^n f)(t)=(D_q^n I_q^\alpha f)(t)-\sum_{i=0}^{n-1}\frac{t^{\alpha-n+i}}{\Gamma_q(\alpha+i-n-1)}(D_q^i f)(0).$$

Lemma 2.6[28] Let $v>0$, $\alpha\in R$, for $t\in[a,b]$, Then the following equality hold:

$$D_q^v (t-a)^\alpha = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-v+1)}(t-a)^{\alpha-v} .$$

Definition 2.9[29] Let $p>1$, the p-Laplacian operator is defined by

$$\varphi_p(x) = |x|^{p-2} x .$$

Obviously, φ_p is the continuously increasing invertible operator, its inverse operator is φ_{q^*}, $q^* > 1$, $\frac{1}{p} + \frac{1}{q^*} = 1$.

Lemma 2.7 For any $y \in C[0,1]$, the problem

$$\begin{cases} D_q^\alpha\left(\phi_p\left(D_q^\beta x\right)(t)\right) = y(t), \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = aD_q^\gamma x(\xi), \quad D_q^\beta x(0) = 0, \quad D_q^\beta x(1) = bD_q^\beta x(\eta), \end{cases} \tag{2}$$

has the unique solution

$$x(t) = \int_0^1 G(t,qz)\phi_q\left(\int_0^1 M(z,qr)y(r)d_q r\right)d_q z .$$

With

$$B_1 = b^{p-1}\eta^{\alpha-1} \neq 1,\ B_2 = a\Gamma_q(\beta)\xi^{\frac{\beta-1}{2}} \neq \Gamma_q\left(\frac{\beta+1}{2}\right),$$

$$M(z,qr) = \begin{cases} \dfrac{z^{\alpha-1}(1-qr)^{(\alpha-1)} - b^{p-1}z^{\alpha-1}(\eta-qr)^{(\alpha-1)} - (1-B_1)(z-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}, & 0 \le qr \le z \le 1, qr \le \eta, \\ \dfrac{z^{\alpha-1}(1-qr)^{(\alpha-1)} - (1-B_1)(z-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}, & 0 \le \eta \le qr \le z \le 1, \\ \dfrac{z^{\alpha-1}(1-qr)^{(\alpha-1)} - b^{p-1}z^{\alpha-1}(\eta-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}, & 0 \le z \le qr \le \eta \le 1, \\ \dfrac{z^{\alpha-1}(1-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}, & 0 \le z \le qr \le 1, \eta \le qr, \end{cases} \tag{3}$$

$$G(t,qz) = \begin{cases} \dfrac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)} - \left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)(t-qz)^{(\beta-1)} - a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)}, & 0 \le qz \le t \le 1, qz \le \xi, \\ \dfrac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)} - \left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)(t-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)}, & 0 < \xi \le qz \le t \le 1, \\ \dfrac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)} - a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)}, & 0 \le t \le qz \le \xi < 1, \\ \dfrac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)}, & 0 \le t \le qz \le 1, \xi \le qz. \end{cases} \tag{4}$$

Proof By Lemma 2.5, one has

$$\begin{aligned} I_q^\alpha D_q^\alpha\left(\phi_p\left(D_q^\beta x(t)\right)\right) &= \phi_p\left(D_q^\beta x(t)\right) - A_1 t^{\alpha-1} - A_2 t^{\alpha-2} \\ &= I_q^\alpha y(t) \\ &= \frac{1}{\Gamma_q(\alpha)}\int_0^1 (t-qr)^{(\alpha-1)} y(r) d_q r , \end{aligned}$$

where $A_1, A_2 \in R$. Combining (2) with $D_q^\beta x(0) = 0$, we have $A_2 = 0$. Then

$$\phi_p\left(D_q^\beta x(t)\right)=\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qr)^{(\alpha-1)}y(r)d_qr+A_1t^{\alpha-1}.$$

Hence

$$\phi_p\left(D_q^\beta x(1)\right)=\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qr)^{(\alpha-1)}y(r)d_qr+A_1,$$

$$\phi_p\left(D_q^\beta x(\eta)\right)=\frac{1}{\Gamma_q(\alpha)}\int_0^\eta (\eta-qr)^{(\alpha-1)}y(r)d_qr+A_1\eta^{\alpha-1},$$

combining with $D_q^\beta x(1)=bD_q^\beta x(\eta)$, one has

$$A_1=-\int_0^1\frac{(1-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)\left(1-b^{p-1}\eta^{\alpha-1}\right)}y(r)dr+\int_0^\eta\frac{b^{p-1}(\eta-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)\left(1-b^{p-1}\eta^{\alpha-1}\right)}y(r)dr,$$

$$\begin{aligned}\phi\left(D_q^\beta x(t)\right)&=\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qr)^{(\alpha-1)}y(r)d_qr-\int_0^1\frac{t^{\alpha-1}(1-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)\left(1-b^{p-1}\eta^{\alpha-1}\right)}y(r)dr\\&\quad+\int_0^\eta\frac{b^{p-1}t^{\alpha-1}(\eta-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)\left(1-b^{p-1}\eta^{\alpha-1}\right)}y(r)dr\\&=-\int_0^1 M(t,qr)y(r)d_qr.\end{aligned}$$

So

$$D_q^\beta x(t)=-\phi_{q^*}\left(\int_0^1 M(t,qr)y(r)d_qr\right). \tag{5}$$

Applying Lemma 2.5 to (5), we have

$$\begin{aligned}I_q^\beta D_q^\beta x(t)&=x(t)-C_1t^{\beta-1}-C_2t^{\beta-2}\\&=-I_q^\beta\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right),\end{aligned}$$

where $C_1,C_2\in R$. Since $x(0)=0$, we have $C_2=0$. Therefore,

$$x(t)=-I_q^\beta\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)+C_1t^{\beta-1}. \tag{6}$$

Applying D_q^γ to both sides of (6), and by Lemma 2.6, we have

$$\begin{aligned}D_q^\gamma x(t)&=-D_q^\gamma I_q^\beta\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)+C_1D_q^\gamma t^{\beta-1}\\&=-\int_0^t\frac{(t-qz)^{\left(\frac{\beta+1}{2}\right)}}{\Gamma_q\left(\frac{\beta+1}{2}\right)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz+C_1\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\gamma)}t^{\beta-\gamma-1}.\end{aligned}$$

So

$$x(1)=-\int_0^1\frac{(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz+C_1,$$

$$D_q^\gamma x(\xi)=-\int_0^\xi\frac{(\xi-qz)^{\left(\frac{\beta+1}{2}\right)}}{\Gamma_q\left(\frac{\beta+1}{2}\right)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz+C_1\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\gamma)}\xi^{\beta-\gamma-1},$$

combining with $x(1)=aD_q^{\gamma}x(\xi)$, we have

$$C_1=\frac{\Gamma_q\left(\frac{\beta+1}{2}\right)}{\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2}\left\{\int_0^1\frac{(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz\right.$$
$$\left.-a\int_0^{\xi}\frac{(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q\left(\frac{\beta+1}{2}\right)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz\right\}.$$

Thus, we obtain the unique solution of problem (2):

$$x(t)=-\int_0^t\frac{(t-qz)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz$$
$$+\frac{t^{\beta-1}\Gamma_q\left(\frac{\beta+1}{2}\right)}{\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2}\left\{\int_0^1\frac{(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz\right.$$
$$\left.-a\int_0^{\xi}\frac{(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q\left(\frac{\beta+1}{2}\right)}\phi_{q^*}\left(\int_0^1 M(z,qr)y(r)d_qr\right)d_qz\right\}.$$

The proof of Lemma 2.7 is complete. □

Lemma 2.8 Let G and M be defined by (3) and (4), respectively. If $a\Gamma_q(\beta)\xi^{\frac{\beta-1}{2}}<\Gamma_q\left(\frac{\beta+1}{2}\right)$ and $b^{p-1}\eta^{\alpha-1}<1$, then:

(a) $G,M\in C([0,1]\times[0,1])$;

(b) $G(t,qz)>0$, $M(t,qz)>0$ for all $t,z\in(0,1)$;

(c) there exist two positive functions $\mu,\nu\in C((0,1),(0,+\infty))$, so that for all $z\in(0,1)$, one has

$$\mu(qz)\ge\max_{0\le t\le 1}G(t,qz),\quad \nu(qz)\ge\max_{0\le t\le 1}M(t,qz).$$

Proof (a) Obviously, $G(t,qz)$ and $M(t,qz)$ is a continuous function of above. $[0,1]\times[0,1]$.

(b) To prove that $G(t,qz)>0$ for all $t,z\in(0,1)$, put

$$g_1(t,qz)=\frac{t^{\beta-1}(1-qz)^{(\beta-1)}-(t-qz)^{(\beta-1)}}{\Gamma_q(\beta)} \tag{7}$$

$$= \frac{t^{\beta-1}(1-qz)^{(\beta-1)} - t^{\beta-1}\left(1-\frac{qz}{t}\right)^{(\beta-1)}}{\Gamma_q(\beta)}$$

$$> 0,$$

$$g_2(t,qz) = \frac{t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)}$$

$$> 0.$$

$\forall \xi \in \left(0, \frac{1}{2}\right]$, $qz \in [0,\xi]$, we have

$$g^*(\xi,qz) = \xi^{\frac{\beta-1}{2}}(1-qz)^{(\beta-1)} - (\xi - qz)^{\left(\frac{\beta-1}{2}\right)}$$

$$= \xi^{\frac{\beta-1}{2}}(1-qz)^{(\beta-1)} - \xi^{\frac{\beta-1}{2}}\left(1-\frac{qz}{\xi}\right)^{\left(\frac{\beta-1}{2}\right)}$$

$$\geq \xi^{\frac{\beta-1}{2}}(1-qz)^{\left(\frac{\beta-1}{2}\right)} - \xi^{\frac{\beta-1}{2}}\left(1-\frac{qz}{\xi}\right)^{\left(\frac{\beta-1}{2}\right)}$$

$$\geq 0.$$

the case $0 \leq qz \leq t \leq 1, z \leq \xi$, then

$$G(t,qz) = \frac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)} - \left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)(t-qz)^{(\beta-1)} - a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)}$$

$$= \frac{t^{\beta-1}(1-qz)^{(\beta-1)} - (t-qz)^{(\beta-1)}}{\Gamma_q(\beta)} + \frac{B_2 t^{\beta-1}(1-qz)^{(\beta-1)} - a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)}$$

$$= g_1(t,qz) + \frac{at^{\beta-1}}{\left(\Gamma_q\left(\frac{\beta+1}{2}\right) - B_2\right)} g^*(\xi,qz)$$

$$> 0.$$

The remaining three cases $0 < \xi \leq z \leq t \leq 1$ or $0 \leq t \leq z \leq \xi < 1$ or $0 \leq t \leq z \leq 1, \xi \leq z$ can be handled in a similar way, so that we omit the obvious modification. Thus, $G(t,qz) > 0$ for all $t, z \in (0,1)$.

Similarly, to prove that $M(t,qz) > 0$, for all $t, z \in (0,1)$.Put

$$m_1(t,qz) = \frac{t^{\alpha-1}(1-qz)^{(\alpha-1)} - (t-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)} \tag{8}$$

$$= \frac{t^{\alpha-1}(1-qz)^{(\alpha-1)} - t^{\alpha-1}\left(1-\frac{qz}{t}\right)^{(\alpha-1)}}{\Gamma_q(\alpha)}$$

$$>0.$$

$$m_2(t,qz)=\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)}$$

$$>0.$$

The case $0\le qz\le t\le 1, qz\le\eta$, then

$$M(t,qz)=\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)}-b^{p-1}t^{\alpha-1}(\eta-qz)^{(\alpha-1)}-(1-B_1)(t-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}$$

$$=\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)}-(t-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{b^{p-1}t^{\alpha-1}\left[\eta^{\alpha-1}(1-qz)^{(\alpha-1)}-(\eta-qz)^{(\alpha-1)}\right]}{\Gamma_q(\alpha)(1-B_1)}$$

$$=m_1(t,qz)+\frac{b^{p-1}t^{\alpha-1}}{1-B_1}m_1(\eta,qz)$$

$$>0.$$

One can apply a similar argument in order to treat the remaining three cases $0\le\eta\le qz\le t\le 1$ or $0\le t\le qz\le\eta\le 1$ or $0\le t\le qz\le 1,\eta\le qz$. Thus, that $M(t,qz)>0$, for all $t,z\in(0,1)$.

(c) For a fixed qz, the functions given by (7) and (8), respectively, are increasing in t for $t\le qz$ and decreasing in t for $t\ge qz$. Therefore,

$$\max_{0\le t\le 1}\{g_1(t,qz),g_2(t,qz)\}=g_1(qz,qz)=\frac{z^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)},\quad qz\in(0,1);$$

$$\max_{0\le t\le 1}\{m_1(t,qz),m_2(t,qz)\}=m_1(qz,qz)=\frac{z^{\beta-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)},\quad qz\in(0,1).$$

Put

$$\mu(qz)=g_1(qz,qz)+\frac{B_2(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)},\quad qz\in(0,1);$$

$$\nu(qz)=m_1(qz,qz)+\frac{B_2(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)\left(\Gamma_q\left(\frac{\alpha+1}{2}\right)-B_2\right)},\quad qz\in(0,1).$$

It is certain that $\mu,\nu\in C((0,1),(0,+\infty))$.

Consider the four cases.

If $0\le qz\le t\le 1, z\le\xi$, then

$$\max_{0\le t\le 1}G(t,qz)=\max_{0\le t\le 1}\left(g_1(t,qz)+\frac{B_2t^{\beta-1}(1-qz)^{(\beta-1)}-a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}\right)$$

$$\leq g_1(qz,qz)+\frac{B_2(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$=\mu(qz).$$

If $0<\xi\leq z\leq t\leq 1$, then

$$\max_{0\leq t\leq 1}G(t,qz)=\max_{0\leq t\leq 1}\frac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)}-\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)(t-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$=\max_{0\leq t\leq 1}\left(\frac{t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)}+\frac{B_2t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2}-\frac{(t-qz)^{(\beta-1)}}{\Gamma_q(\beta)}\right)$$

$$=g_1(qz,qz)+\frac{B_2(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$=\mu(qz).$$

If $0\leq t\leq z\leq \xi<1$, then

$$\max_{0\leq t\leq 1}G(t,qz)=\max_{0\leq t\leq 1}\frac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)}-a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$=\max_{0\leq t\leq 1}\left(\frac{t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)}+\frac{B_2t^{\beta-1}(1-qz)^{(\beta-1)}-a\Gamma_q(\beta)t^{\beta-1}(\xi-qz)^{\left(\frac{\beta-1}{2}\right)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}\right)$$

$$=g_1(qz,qz)+\frac{B_2(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$=\mu(qz).$$

If $0\leq t\leq z\leq 1,\xi\leq z$, then

$$\max_{0\leq t\leq 1}G(t,qz)=\max_{0\leq t\leq 1}\frac{\Gamma_q\left(\frac{\beta+1}{2}\right)t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$= \max_{0\le t\le 1}\left(\frac{t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)} + \frac{B_2 t^{\beta-1}(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2}\right)$$

$$= g_1(qz,qz) + \frac{B_2(1-qz)^{(\beta-1)}}{\Gamma_q(\beta)\left(\Gamma_q\left(\frac{\beta+1}{2}\right)-B_2\right)}$$

$$= \mu(qz).$$

Thus,

$$\max_{0\le t\le 1} G(t,qz) \le \mu(qz), \quad qz\in(0,1).$$

Similarly, consider the four cases of the function ν.

If $0\le qz\le t\le 1, qz\le \eta$, then

$$\max_{0\le t\le 1} M(t,qz) = \max_{0\le t\le 1}\left(m_1(t,qz) + \frac{b^{p-1}t^{\alpha-1}\left[\eta^{\alpha-1}(1-qz)^{(\alpha-1)} - (\eta-qz)^{(\alpha-1)}\right]}{\Gamma_q(\alpha)(1-B_1)}\right)$$

$$\le m_1(qz,qz) + \frac{B_2(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}$$

$$= \nu(qz).$$

If $0\le \eta\le qz\le t\le 1$, then

$$\max_{0\le t\le 1} M(t,qz) = \max_{0\le t\le 1}\left(\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)} - (1-B_1)(t-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}\right)$$

$$\le \max_{0\le t\le 1}\left(m_1(t,qz) + \frac{B_1 t^{\alpha-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}\right)$$

$$\le m_1(qz,qz) + \frac{B_2(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}$$

$$= \nu(qz).$$

If $0\le t\le qz\le \eta\le 1$, then

$$\max_{0\le t\le 1} M(t,qz) = \max_{0\le t\le 1}\left(\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)} - b^{p-1}t^{\alpha-1}(\eta-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}\right)$$

$$= \max_{0\le t\le 1}\left(\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{b^{p-1}t^{\alpha-1}\left[\eta^{\alpha-1}(1-qz)^{(\alpha-1)} - (\eta-qz)^{(\alpha-1)}\right]}{\Gamma_q(\alpha)(1-B_1)}\right)$$

$$\le m_1(qz,qz) + \frac{B_2(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}$$

$$= \nu(qz).$$

If $0\le t\le qz\le 1, \eta\le qz$, then

$$\max_{0\le t\le 1} M(t,qz) = \max_{0\le t\le 1} \frac{t^{\alpha-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}$$
$$= \max_{0\le t\le 1}\left(\frac{t^{\alpha-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{B_1 t^{\alpha-1}(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}\right)$$
$$\le m_1(qz,qz) + \frac{B_2(1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-B_1)}$$
$$= \nu(qz).$$

Thus,

$$\max_{0\le t\le 1} M(t,qz) \le \nu(qz), \quad qz \in (0,1).$$

The proof is complete.

Lemma 2.9 Let $E = C[0,1]$ be a continuous function space equipped with standard sup-norm $\|x\| = \max_{0\le t\le 1}|x(t)|$ and denote by $P = \{x \in E | x(t) \ge 0, 0 \le t \le 1\}$ the corresponding cone. Let $T : P \to E$ be given by

$$Tx(t) = \int_0^1 G(t,qz)\phi_{q^*}\left(\int_0^1 M(z,qr)h(r,x(r))d_q r\right)d_q z,$$

where $h \in C([0,1]\times[0,+\infty),[0,+\infty))$ and $G(t,qz)$ and $M(z,qr)$ are defined by (3) and (4), respectively. Then T takes P into itself, and as such is completely continuous.

Proof Since G, M and h are nonnegative and continuous, one has $T(P) \subset P$ and T is continuous. In order to prove the complete continuity of T, it is necessary to use a standard argument based on the Arzela-Ascoli theorem and the Lebesgue rule of convergence theorem.

3. Main result

We can now formulate our main results. To this end, denote

$$J^{-1} = \int_0^1 \mu(qz)\phi_{q^*}\left(\int_0^1 \nu(qz)d_q r\right)d_q z,$$

where μ, ν defined in Lemma 2.8.

Theorem 3.1 Let $h \in C([0,1]\times[0,+\infty),[0,+\infty))$, suppose there is a positive number k satisfying the following:

(S_1) if $0 \le t \le 1$, $0 \le s_1 \le s_2 \le k$, establishment $h(t,s_1) \le h(t,s_2)$;

(S_2) $\max_{0\le t\le 1} h(t,k) \le \phi_p(kJ)$;

(S_3) $0 \le t \le 1$ for all $h(t,0) = 0$.

Then the problem (1) has two positive solutions x^* and y^* makes

(i) $0 < \|x^*\| \leq k$, $\lim_{n\to\infty} T^n x_0 = x^*$, where $x_0(t) = k$ for $0 \leq t \leq 1$;

(ii) $0 < \|y^*\| \leq k$, $\lim_{n\to\infty} T^n y_0 = y^*$, where $y_0(t) = 0$ for $0 \leq t \leq 1$.

Proof Let $\Omega = \{x \in P \| \|x\| \leq k\}$. Suppose, $x \in \Omega$. Obviously, $0 \leq x(t) \leq \|x\| \leq k$.
From the assumptions (S_1) and (S_2) :

$$0 \leq h(t, x(t)) \leq h(t,k) \leq \max_{0\leq t\leq 1} h(t,k) \leq \phi_p(kJ).$$

We claim that $T(\Omega) \subseteq \Omega$. Actually, for $\forall x \in \Omega$, we have $Tx \in P$, and by Lemma 2.8, we know

$$\begin{aligned}\|Tx\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,qz)\phi_{q^*}\left(\int_0^1 M(s,qr)h(r,x(r))d_q r\right)d_q z\right| \\ &\leq \int_0^1 \mu(qz)\phi_{q^*}\left(\int_0^1 \nu(qr)\phi_p(kJ)d_q r\right)d_q z \\ &= kJ\int_0^1 \mu(qz)\phi_{q^*}\left(\int_0^1 \nu(qr)d_q r\right)d_q z \\ &= k.\end{aligned}$$

Thus, $Tx \in \Omega$. The next proofs the existence of x^*.Take the function $x_0 \equiv k$ on $0 \leq t \leq 1$, then $\|x_0\| = k$, and $x_1(t) = Tx_0(t)$ with $x_0 \in \Omega$.Define

$$x_{n+1} = Tx_n = T^{n+1}x_0,\ n = 0,1,2,\ldots.$$

Then, for all $n = 0,1,2,\ldots$,one has $x_n \in \Omega$.
From the assumptions (S_2) and Lemma 2.8, we can get that for $\forall t \in [0,1]$:

$$\begin{aligned}x_1(t) = Tx_0(t) &= \int_0^1 G(t,qz)\phi_{q^*}\left(\int_0^1 M(z,qr)h(r,x_0(r))d_q r\right)d_q z \\ &\leq \int_0^1 \mu(qz)\phi_{q^*}\left(\int_0^1 \nu(qr)\phi_p(kJ)d_q r\right)d_q z \\ &\leq kJ\int_0^1 \mu(qz)\phi_{q^*}\left(\int_0^1 \nu(qr)d_q r\right)d_q z \\ &= k = x_0(t).\end{aligned}$$

Hence,

$$x_2(t) = Tx_1(t) \leq Tx_0(t) = x_1(t),\ 0 \leq t \leq 1.$$

By mathematical induction, we know

$$x_{n+1}(t) \leq x_n(t),\ 0 \leq t \leq 1,\ n = 0,1,2,\ldots.$$

From Lemma 2.9, we know that the operator is a completely continuous operator, so $\overline{T(\Omega)}$ is a compact set.

Therefore, $\{x_n\}_{n=1}^{\infty}$ exists the sub-sequence converges $\{x_{n_i}\}_{i=0}^{\infty}$ to $x^* \in \Omega$. Because $\{x_{n_i}\}_{i=0}^{\infty}$ is monotonic, and $x_n \to x^*$, again from the continuity of the operator T, it can be known that $Tx_n = x_{n+1} \to x^*$, that is $Tx^* = x^*$.

It can be proved $Ty^* = y^*$ by the same method. Take the function $y_0 \equiv 0$, identically on $0 \le t \le 1$, Clearly,then $\|y_0\| = 0$,and $y_0 \in \Omega$.Also, $y_1(t) = Ty_0(t)$.Define

$$y_{n+1} = Ty_n = T^{n+1}y_0,\ n = 0,1,2,\ldots$$

Then, for all $n = 0,1,2,\ldots$, one has $y_n \in \Omega$. By the same computation as above,

From the assumptions (S_2) and Lemma 2.8, we can see that for lemma 2.8, for $\forall t \in [0,1]$,

$$\begin{aligned} y_1(t) = Ty_0(t) &= \int_0^1 G(t,qz)\phi_{q^*}\left(\int_0^1 M(z,qr)h(r,y_0(r))d_q r\right)d_q z \\ &= \int_0^1 G(t,qz)\phi_{q^*}\left(\int_0^1 M(z,qr)h(r,0)d_q r\right)d_q z \\ &= 0 = y_0(t). \end{aligned}$$

Hence,

$$y_2(t) = Ty_1(t) \le Ty_0(t) = y_1(t),\ 0 \le t \le 1.$$

By mathematical induction, there is

$$y_{n+1}(t) \le y_n(t),\ 0 \le t \le 1,\ n = 0,1,2,\ldots.$$

Therefore, $\{y_n\}_{n=1}^{\infty}$ exists the sub-sequence converges $\{y_{n_i}\}_{i=0}^{\infty}$ to $y^* \in \Omega$. Because $\{y_{n_i}\}_{i=0}^{\infty}$ is monotonic, and $y_n \to y^*$, again from the continuity of the operator T, it can be known that $Ty_n = y_{n+1} \to y^*$, that is $Ty^* = y^*$. It remains to be seen is that, the problem (3.1) has two positive solutions x^* and y^*.So $\|x^*\| > 0$ and $\|y^*\| > 0$. The proof is complete.

4. Example

Consider the following problem:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{3}{2}}\left(\phi_{\frac{3}{2}}\left(D_{\frac{1}{2}}^{\frac{3}{2}}x(t)\right)\right)=\dfrac{x^2}{15}+\dfrac{tx}{12}, \quad 0<t<1 \\ x(0)=0, \quad x(1)=\dfrac{1}{4}D_{\frac{1}{2}}^{\frac{1}{4}}x\left(\dfrac{1}{2}\right), \quad D_{\frac{1}{2}}^{\frac{3}{2}}x(0)=0, \quad D_{\frac{1}{2}}^{\frac{3}{2}}x(1)=\dfrac{1}{2}D_{\frac{1}{2}}^{\frac{3}{2}}x\left(\dfrac{1}{2}\right). \end{cases} \tag{9}$$

Where $q=\frac{1}{2}$, $p=\frac{3}{2}$, $h(y,x(t))=\frac{x^2}{15}+\frac{tx}{12}$, $\alpha=\beta=\frac{3}{2}$, $\gamma=\frac{1}{4}$, $\xi=\frac{1}{2}$, $\eta=\frac{1}{2}$, $a=\frac{1}{4}$, $b=\frac{1}{2}$ .Then

$$\Gamma_q\left(\frac{\beta+1}{2}\right)-a\Gamma_q(\beta)\xi^{\frac{\beta-1}{2}}=\Gamma_{\frac{1}{2}}\left(\frac{5}{4}\right)-\frac{1}{4}\times\Gamma_{\frac{1}{2}}\left(\frac{3}{2}\right)\times\left(\frac{1}{2}\right)^{\frac{1}{4}}>0,$$

$$1-b^{p-1}\eta^{\alpha-1}=1-\left(\frac{1}{2}\right)^{\frac{1}{2}}\times\left(\frac{1}{2}\right)^{\frac{1}{2}}=0.5>0,$$

$$J=\left(\int_0^1\mu(qz)\phi_{q^*}\left(\int_0^1 v(qr)d_qr\right)d_qz\right)^{-1}\approx 4.3367.$$

Let $k=8$, we have

(1) For $\forall 0\le t\le 1$, one has $0\le s_1\le s_2\le 8$, $h(t,s_1)\le h(t,s_2)$;

(2) $\max\limits_{0\le t\le 1}h(t,k)=h(1,8)\approx 4.9334<\phi_p(kJ)\approx 5.8901$;

(3) $h(t,0)=0$, for $0\le t\le 1$.

The problem (9) has two positive solutions x^* and y^*:

(i) $0<\|x^*\|\le 8$ and $\lim\limits_{n\to\infty}T^n x_0=x^*$,where $x_0(t)=8$;

(ii) $0<\|y^*\|\le 9$ and $\lim\limits_{n\to\infty}T^n y_0=y^*$, where $y_0(t)=0$.

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