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Developments in Landslide Analysis Methodologies

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Abstract. Some of the first analytical tools developed in soil mechanics aimed at evaluating the stability of slopes. In recent years there has been numerous changes proposed on how best to model the behavior of slopes and the mechanisms of instability. "Limit Equilibrium" methods of slices have gone through a series of changes as the computer has provided increased ability to solve complex and nonlinear formulations. In recent years numerous new methods have been proposed for the analysis of slopes. These methods have provided new methodologies for the calculation of the normal forces along any proposed slip surface as well as new search routines that attempt to directly determine the shape and location of the most critical slip surface. Each new method of analysis required testing against a history of experience and previous methods of analysis. Recent developments in the analysis tools used for slope stability have augmented the ability of the practicing engineer to analyze slopes. Three-dimensional (3D) analysis is today easily possible and, therefore, its use in the regulatory environment dominated by 1.3 and 1.5 factors of safety must be understood. The influences of slip shape, new searching methods in 3D, geostrata, loading conditions, anchors, water tables as well as advanced methods of representing surface topology must be considered. How do we now consider the influence of 3D anisotropic bedding planes in rock environments? How do we account for probability? What are new searching methods to determine the slip surface in 3D? How do we determine spatial factors of safety over large land areas? The objective of this paper is to provide the practicing geotechnical engineer with information that allows the assessment of newly proposed methods for determining the factor of safety of soil and rock masses.

Keywords. 3D slope stability, stability, limit equilibrium, factor of safety, rock stability, 3D stability analysis

1. Introduction

Slope stability studies have constituted an important part of geotechnical engineering practice. The ability to analyze a soil or rock mass and calculate a factor of safety has lent considerable credibility to the engineering profession. This analytical ability has also been profitable for geotechnical engineers.

Changes in methodologies for the analysis of slopes have been considerable over a matter of a few decades and this has given rise to concerns over what is the best

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methodology to use in practice. Some slope stability methods emerged in the early years of soil mechanics. More recently new analytical forms of analysis have emerged. This often leaves the practicing geotechnical engineer with questions regarding the significance of the new methodologies. The advent of the computer has proven to be a valuable tool for analysis purposes. At the same time the computer has birthed other more complex computational tools.

The main objective of this paper is to provide a summary of the history of slope stability analysis while at the same time clarifying some strengths and weaknesses associated with the analyses. Another objective is to summarize some of the new methods associated with 3D analysis which have the potential to improve the current state of practice. This paper emphasizes the gradual change that has taken place with the use of limit equilibrium methods. However, changes in slope stability analyses have spread beyond the restrictions of limit equilibrium analyses. This has been largely due to the growth and availability of computing ability.

The geotechnical engineer is left with the task of sorting out the strengths and weaknesses of various slope stability techniques. This paper attempts to not only show the progressive changes that have taken place in slope stability analyses but also illustrate how one methodology has built upon the findings of another technology. This has lead to the place where today it is possible to extend two-dimensional analyses involving one central section through a soil mass into complex three-dimensional analyses. The potential differences with 3D analysis are clarified as well as the new issues that 3D analysis introduces. The methodologies developed to date for improving 3D analysis are presented and clarified. A particular focus of the paper is related to handling probability and the spatial variation of the computed factor of safety in real-world settings. The scope of this paper is limited to sliding earth & rock masses that are quite readily amenable to analysis.

2. History of Limit Equilibrium Analyses

The earliest application of statics to a sliding mass considered a planar sliding surface because of its ease of analysis. The movement of a large earth mass into the Goteborg harbour in Sweden showed the characteristics of a circular shape [1]. The entire sliding mass was considered as a single unit and the tending overturning moment was used to estimate the shear resistance of the soft sediments (Figure 1).

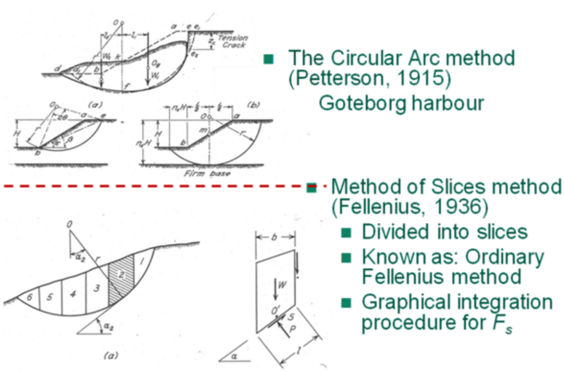

Figure 1. Early history of slope stability analysis by Petterson [1] and Fellenius [2].

Further landslides in the same harbor almost 20 years later resulted in a re-visitation of possible stability analysis that could be performed. Fellenius [2] subdivided the sliding mass into vertical slices, consequently, the name "method of slices". Assumptions were made regarding the inter-slice shear and inter-slice normal forces existing between each of the slices as shown in Figure 2. The assumption was to simply ignore all inter-slice forces. The analysis was considerably simpler to perform but later would become the focus of concerns related to the accuracy of the analysis.

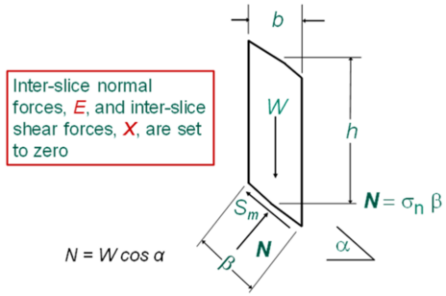

Figure 2. Free-body diagram for Fellenius method.

Little additional research was undertaken to improve the method of slices until the 1950s. In 1955 Bishop (Imperial College, London) published the results of his PhD thesis. Figure 3 shows the free-body diagram of one slice for a sliding mass with a circular shaped slip surface. All inter-slice forces were shown along with a separation of the pore-water force and the force associated with the effective stresses (i.e., an effective stress analysis). Also indicated was the force related to partial submergence of the slope.

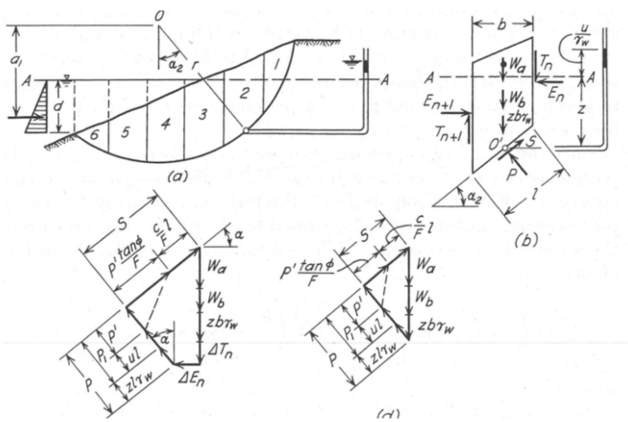

Figure 3. Bishop's Simplified method of slices [3].

Bishop [3] derived three main equations; namely, i.) a moment equilibrium equation for the overall mass with respect to the center of rotation, ii.) a force equilibrium equation for the overall mass in the horizontal direction, and iii.) a vertical force equilibrium equation for each slice comprising the sliding mass. While the equations associated with complete equilibrium of the sliding mass were derived, it was not possible to simultaneously satisfy both the horizontal equilibrium and moment equilibrium equations using longhand calculations. Consequently, it was suggested that the overall horizontal force equilibrium be ignored along with the inter-slice shear forces, for the calculation of the factor of safety, giving rise to the Bishop Simplified method of slices (Figure 4).

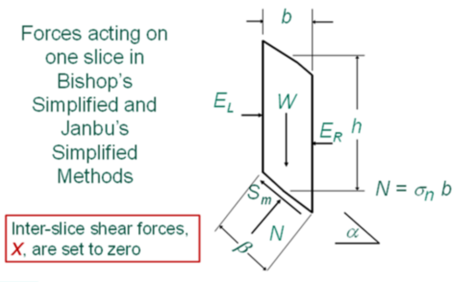

Figure 4. Free-body of a slice for the Bishop's Simplified and Janbu's Simplified methods.

In 1954, Janbu had suggested using the overall force equilibrium equation along with an omission of the inter-slice shear forces and moment equilibrium during the calculation of the factor of safety of a slope. This gave rise to the Janbu Simplified method of slices. Janbu also suggested a more elaborate analysis referred to as the Janbu Generalized method. This method made used of a moment equilibrium for each slice to generate a "line of thrust" to define the point of application of the inter-slice forces [4, 5].

Mainframe digital computers came on the scene in the mid 1960s and with them came additional computing power. Morgenstern and Price [6] were some of the first to take advantage of the increased computational ability. Most importantly it became possible to obtain a factor of safety solution that satisfied both moment and force equilibrium conditions if one additional variable, referred to as Lambda, λ, was introduced into the formulation [6]. It was also suggested that the inter-slice shear and inter-slice normal forces be related through use of an arbitrary but reasonable mathematical function. Morgenstern and Price [6] noted that the slope stability analysis was indeterminate because of a lack of physical understanding of the internal stress state along the sides of each slice. It was also suggested that it might be possible to introduce additional elements of physics into the analysis to render the analysis determinate (Figure 5). While the use of an additional stress analysis might be possible, it was not done until 1983 by Wilson and Fredlund.

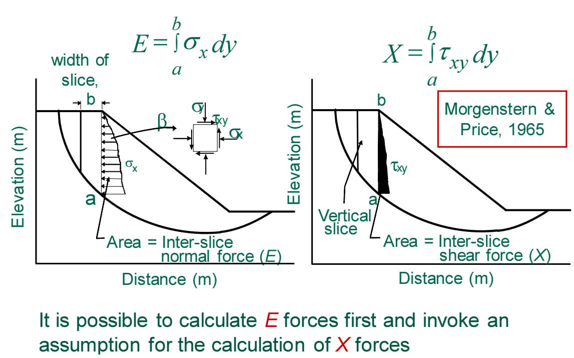

Figure 5. Morgenstern-Price [6] method of analysis.

In 1967 Spencer published a method of slices analysis for calculating the factor of safety of a soil mass [7]. The method satisfied both force and moment equilibrium conditions and assumed that the slope of the inter-slice resultant be maintained at a constant slope. Consequently, the formulation was a special case of the Morgenstern-Price [6] method.

In 1977, Fredlund and Krahn published a general set of force and moment equilibrium equations based on the basic assumptions associated with a limit equilibrium analysis. This did not result in a new method of slope stability analysis; however, it showed the inter-relationship and the limitations associated with each of the methods of analysis that had previously been published. The summary of limit equilibrium analytical methods could be visualized through a common set of Newtonian equilibrium equations and a shear strength criterion. Other suggested limit equilibrium methods of slices were also shown to be related to the common set equilibrium equations [8, 9].

3. Fundamentals of Limit Equilibrium Analysis

The evolution of limit equilibrium methods can be presented in terms of a consistent set of Newtonian equations of statics. Within this context, all limit equilibrium methods of slices can be visualized in terms of the i.) elements of statics that have been satisfied in the formulation, and ii.) the assumption(s) invoked to render the analysis determinate.

A limit equilibrium analysis must start with a designated free-body diagram. Herein lies the first problem since the boundaries of the free-body are not known (Figure 6). The ground surface geometry and the stratigraphy associated with the soil layers may be known but the shape and location of a sliding mass are not known at the start of the analysis. All proposed limit equilibrium methods of slices have made the following recommendations to resolve these problems; namely, i.) the shape of the slip surface is assumed, and ii.) the location of the critical slip surface will be found through a trial and error process.

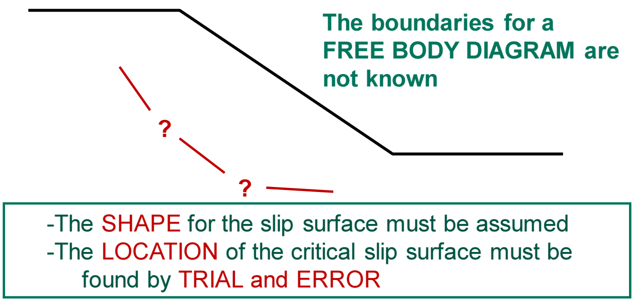

Figure 6. Basic limitations of limit equilibrium methods of analysis.

Figure 7 shows all the forces that need to be applied to an individual slice from the sliding mass. The forces are all shown as total forces without a need to separate the forces between total and effective stress conditions. The selected slip surface was first assumed to be circular but might also deviate from a circular shape.

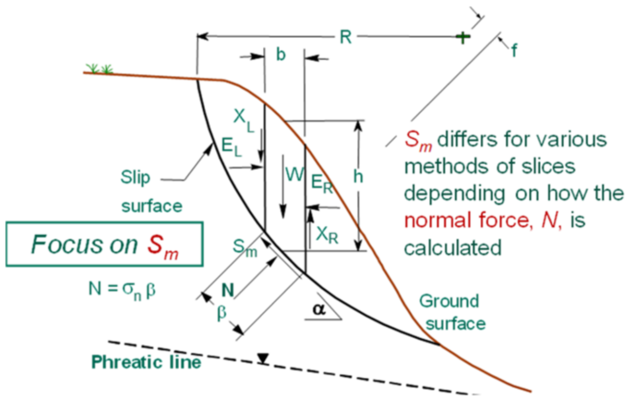

Figure 7. Forces acting on each slice.

The force of greatest interest in Figure 7 is the shear force mobilized, S_m. It is of interest because it is the variable that is calculated by different means in each of the methods of slices. The shear force mobilized changes in the various methods of slices because it is dependent upon the normal force, N, which is calculated differently in the various methods of slices.

All limit equilibrium methods assume that the soils involved behave as Mohr-Coulomb materials. In other words, the soil has a frictional component and a cohesive component. The frictional component is most fundamentally designated as the effective angle of internal friction, ϕ', and the cohesion intercept is designated as the effective cohesion, c'. Another commonality of all limit equilibrium methods is the assumption that the factor of safety for the cohesive component is equal to that of the frictional component. In addition, the factor of safety is assumed to be the same for all slices of the sliding mass.

It is noteworthy that the pore-water pressure does not appear on the free-body diagram because it is internal to the shear strength equation which is part of the shear force mobilized, S_m. The shear force mobilized can be written as follows in terms of the shear strength criteria.

$$S_m = \frac{c'\beta}{F_s} + \frac{\left[(\sigma_n - u_w)\tan\phi'\right]\beta}{F_s} \tag{1}$$

where β = length along the base of a slice; σ_n = normal stress along the base of a slice; u_w = pore-water pressure at the base of a slice; and F_s = factor of safety for the entire sliding mass. The designation of the shear strength mobilized can also be written in a form such that it applies for both saturated and unsaturated materials as shown in Eq. (2).

$$S_m = \frac{c'\beta}{F_s} + \frac{\left[(\sigma_n - u_a)\tan\phi'\right]\beta}{F_s} + \frac{\left[(u_a - u_w)\tan\phi^b\right]\beta}{F_s} \tag{2}$$

where u_a = pore-air pressure at the base of a slice, and u_w = pore-water pressure at the base of a slice; and ϕ^b = friction angle with respect to matric suction. Other variables have been previously defined.

Before attempting to formulate factor of safety equations within the context of a limit equilibrium framework, a list should be made of the known element of statics and physics available for solving the problem. The following is a list of the known elements of physics that are available for the derivation of factor of safety equation.

Equations that can be listed as "knowns" are:

- *n* moment equilibrium equations
- *n* vertical force equilibrium equations
- *n* horizontal equilibrium equations
- *n* Mohr-Coulomb failure criterion equations

The above elements of physics can be applied to each slice giving a total of *4n* "knowns" where *n* is equal to the number of slices into which the sliding mass has been divided. The "unknowns" variables associated with the analysis can be listed as follows:

- *n* total normal forces at the base of a slice
- *n* shears forces at the base of each slice
- *n-1* inter-slice normal forces, *E*
- *n-1* inter-slice shear forces, *X*
- *n-1* points of application of the inter-slice *E* forces
- *n* points of application of the normal force on each slice
- 1 factor of safety, F_s

The tabulation of "knowns" and "unknowns" show that there are a total of (*6n -2*) "unknowns" that need to be computed while there are *4n* equations to use. Since (*6n-2*) is greater than *4n*, the analyst must conclude that the problem is indeterminate. The analyst has a choice to make regarding solving for the factor of safety. Either he/she can attempt to invoke further elements of physics to solve the problem or he/she can estimate (or omit) some of the known forces. An examination of the research literature shows that the choice has been made to assume a relationship between some forces or else simply omit some of the forces.

It is possible to derive a general force and moment equilibrium factor of safety equation even prior to a decision being made with regard to rendering the problem determinate. Eq. (3) shows the moment equilibrium that is applicable for both saturated and unsaturated soils. The pore-air pressure has been assumed to be zero in this equation.

$$F_m = \frac{\sum \left\{ c'\beta R + \left(N - u_w \beta \frac{\tan\phi^b}{\tan\phi'} \right) R \tan\phi' \right\}}{\sum Wx - \sum Nf} \tag{3}$$

Eq. (4) shows the horizontal force equilibrium that is applicable for both saturated and unsaturated soils with pore-air pressure equal to zero.

$$F_f = \frac{\sum \left\{ c'\beta\cos\alpha + \left(N - u_w \beta \frac{\tan\phi^b}{\tan\phi'} \right) \tan\phi' \cos\alpha \right\}}{\sum N \sin\alpha} \tag{4}$$

A general form for vertical force equilibrium on each slice can also be derived as shown in Eq. (5).

$$N = \frac{W - \left(X_R - X_L\right) - \dfrac{c'\beta\sin\alpha}{F} + u_w \dfrac{\beta\sin\alpha\tan\phi^b}{F}}{\cos\alpha + \dfrac{\sin\alpha\tan\phi'}{F}} \tag{5}$$

The normal force equation applies for both saturated and unsaturated soils; however, the pore-air pressure has been set to zero. Various limit equilibrium methods differ with respect to how the inter-slice shear force term, ($X_R - X_L$) will be computed and which overall statics will be satisfied. All methods of slices, except for the Fellenius method, are nonlinear in the sense that the variables being computed by moment and force equilibrium also appears in the equation to compute the normal force at the base of the slice. Consequently, an iteration procedure is required to obtain convergence to the factor of safety.

At this point a decision must be made regarding how to render the analysis determinate. This must be done before proceeding to solve for the factor of safety. Table 1 contains a summary of the equilibrium conditions that have been satisfied and the assumptions that have been made regarding the inter-slice forces for various methods of analysis. The Fellenius method completely ignores all inter-slice forces while the Bishop Simplified and Janbu Simplified methods ignore the inter-slice shear force but keep the inter-slice normal forces. The Morgenstern-Price (GLE) and Spencer methods attempt to calculate the inter-slice shear forces and then use these values in the calculation of the factor of safety that satisfies both moment and force equilibrium.

Table 1. Equilibrium conditions that are satisfied and assumptions that are made for various limit equilibrium methods of slices.

Method of Analysis	Equilibrium Satisfied	Assumptions
Ordinary or Fellenius	Moment, Perpend. to base	E and X = 0
Bishop's Simplified	Vertical, Moment	E is horizontal, X = 0
Janbu's Simplified	Vertical, Horizontal	E is horizontal, X = 0; empirical factor f_0 applied
Janbu's Generalized	Vertical, Horizontal	E is located by an assumed line of thrust
Spencer	Vertical, Horizontal, Moment	X/E constant
Morgenstern-Price & GLE	Vertical, Horizontal, Moment	X/E = λ f(x)
Corps of Engineers	Vertical, Horizontal	X/E is equal to the slope of the ground surface
Lowe & Karafiath	Vertical, Horizontal	X/E is equal to the average of the slopes of the ground surface and base of slice

The Morgenstern-Price [6] method introduced an additional unknown, λ, and a functional distribution as shown in Eq. (6):

$$X = E\lambda f(x) \tag{6}$$

where *f(x)* is a mathematical function that defines the slope of the ratio of *X/E.* Various inter-slice force functions suggested by Morgenstern-Price [6] are shown in Figure 8.

Wilson and Fredlund [10] used a linear elastic analysis in an attempt to compute typical inter-slice force functions.

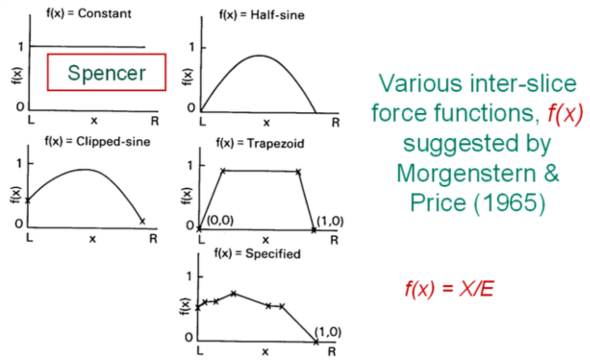

Figure 8. Suggested forms for the inter-slice force function [6].

The results consistently showed that the inter-slice force function took the form of an extended error function as shown in Figure 9 with the peak of the function near the middle of the slope.

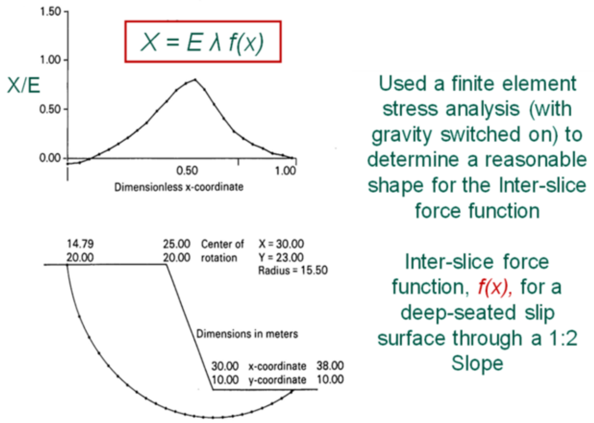

Figure 9. Inter-slice force function based on a linear elastic stress analysis [10].

The interslice force function takes the form of an extended error function [10]. Once the crest and the base of the slope is defined, other variable in the function bear a unique relationship to the steepness of the slope.

$$f(x) = e^{-C^n \omega^n / 2} \tag{7}$$

where: e is the base of the natural log, C is a variable used to define the inflection point along the interslice force function, n is a variable used to describe the steepness of the slope, and ω defines the dimensionless distance along the x-axis.

3.1. Comparison of Moment and Force Equilibrium Factors of Safety

It is possible to compare moment equilibrium and force equilibrium solutions for various geometries (i.e., ground surface profiles) and various shapes of slip surfaces. It is also possible to compare the computed factors of safety from methods that do not satisfy all the elements of statics with methods that satisfy both moment and force equilibrium. Various inter-slice force functions can also be compared through use various percentages of the Lambda, λ, variable.

Figure 10 shows a comparison of the various computed factors of safety when considering a circular slip surface.

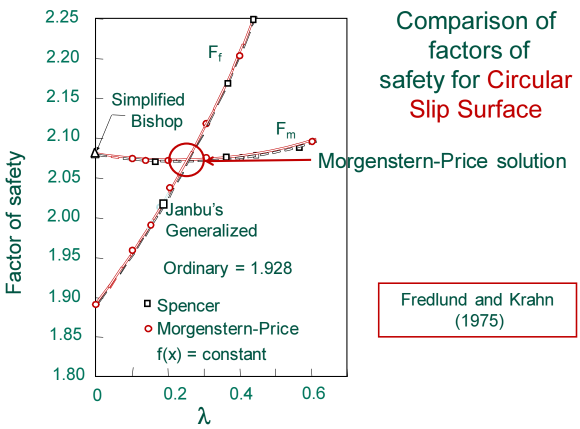

Figure 10. Comparison of moment and force equilibrium factor of safety equations [11].

The following observations can be made from Figure 10 regarding a slope stability analysis where the slip surface is circular.

1. The Morgenstern-Price factor of safety is the cross-over point where moment and force equilibrium are satisfied.
2. The moment equilibrium factors of safety are quite insensitive to the inter-slice force function.
3. Consequently, the Bishop Simplified factor of safety is quite close to the Morgenstern-Price method factor of safety which satisfies both moment and force equilibrium.
4. The Fellenius (or Ordinary) method gives a factor of safety that is lower than the factor of safety satisfying moment and force equilibrium.
5. The force equilibrium factor of safety is quite sensitive to the selected inter-slice force function.
6. The Spencer method is a special case of the Morgenstern-Price method.

Figure 11 shows the results of a comparison of the various computed factors of safety when considering a planar slip surface.

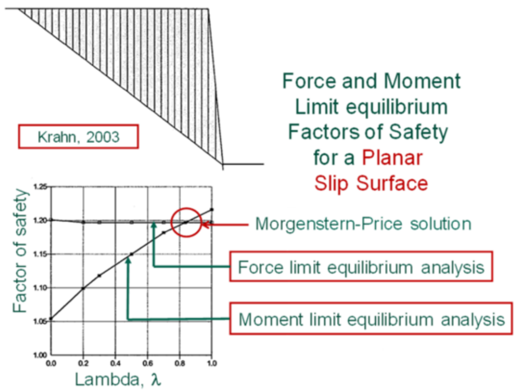

Figure 11. Comparison of moment and force equilibrium factors of safety when the slip surface is planar.

The following observations can be made from Figure 11 regarding a slope stability analysis where the slip surface is planar.

1. The Morgenstern-Price factor of safety is the cross-over point where moment and force equilibrium are satisfied.
2. The moment equilibrium factors of safety are quite sensitive to the inter-slice force function.
3. The Bishop Simplified factor of safety is below the Morgenstern-Price method factor of safety which satisfies both moment and force equilibrium.
4. The moment equilibrium factor of safety is quite sensitive to the selected inter-slice force function.
5. In general, the independent solving of the moment equilibrium and force equilibrium factor of safety equation shows the relationship of the shape of the slip surface to the relationship between moment and force equilibrium.

Figure 12 shows the results of a comparison of the various computed factors of safety when considering a composite slip surface (i.e., part circular and part planar).

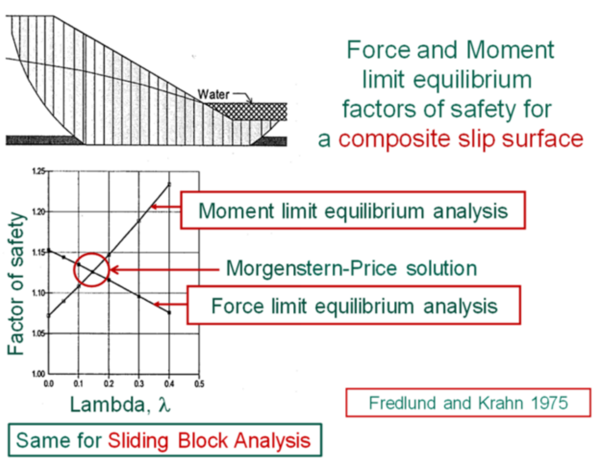

Figure 12. Comparison of moment and force equilibrium factors of safety when the slip surface is composite (i.e., part circular and part planar).

The following observations can be made from Figure 12 regarding a slope stability analysis where the slip surface is composite.

1. The Morgenstern-Price factor of safety is the cross-over point where moment and force equilibrium are satisfied.
2. Both the moment and force equilibrium factors of safety are quite sensitive to the selected inter-slice force function.
3. The Bishop Simplified factor of safety is below the Morgenstern-Price method factor of safety which satisfies both moment and force equilibrium.
4. The independent solving of the moment equilibrium and force equilibrium factor of safety equation shows the relationship of the shape of the slip surface to the relationship between moment and force equilibrium.

The Morgenstern-Price (and GLE) methods of analysis satisfy both moment and force equilibrium conditions and provide considerable flexibility with respect to the selection of an inter-slice force function. The proposed Wilson and Fredlund [10] inter-slice force function is based on a linear elastic analysis of the soil mass. It could be argued that this function is not an accurate reflection of the actual stress state in the soil mass; however, the function has features that are supported by earth pressure theory and appears to always ensure convergence of the factor of safety calculations.

All limit equilibrium methods, (with the exception of the Fellenius method), use Eq. (5) for the calculation of the normal force at the base of the slice. An examination of Eq. (5) shows that under certain conditions it is possible for the denominator to come close to (or exactly) zero and as a result the calculated normal force approaches infinity. This is one of the limitations associated with limit equilibrium methods and in part, has given rise to the consideration of other methodologies.

4. Other Methodologies for Determination of Normal Force on a Slice

One of the first methodologies given consideration for the calculation of the stress state in a soil mass involves the use of a numerical modeling approach such as the finite element method. The search can be put in the form of a question, "Can the stress state within a soil mass be more accurately calculated by switching-on the gravity force (i.e., unit weight of the soil)?" Stated another way, the question can be asked, "Can the complete stress state from a finite element analysis be "imported" into a limit equilibrium framework where the normal stresses and the actuating shear stresses are computed along any selected slip surface?" (See Figure 13).

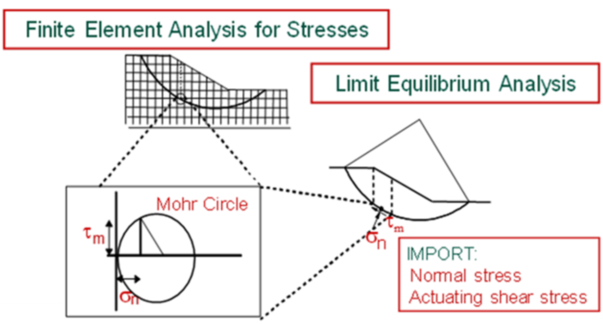

Figure 13. Procedure for "importing stresses" from a finite element analysis into a limit equilibrium analysis.

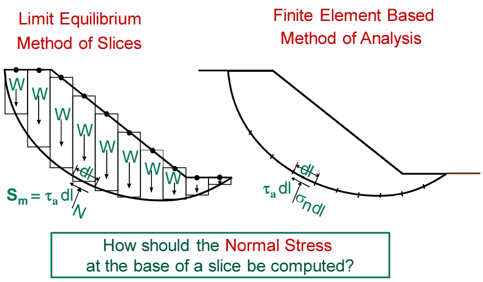

Figure 14. Comparison of the limit equilibrium and stress analysis calculations of the normal stress at the base of a slice.

It can be argued that the stresses computed from “switching-on” gravity are more reasonable than the stresses computed when using a single vertical slice as the free body diagram even though stress-strains conditions might be nonlinear and near failure. Let us compare the free-body diagrams shown in Figure 14. The limit equilibrium free-body diagram shows that the normal force on the base of a slice is largely dependent upon the single slice being analyzed. On the other hand, the finite element simulation also considers overall changes in the ground surface when calculating the stress state at the base of a slice.

The difference in the normal stress at the base of slices can be calculated by comparing the results of a limit equilibrium analysis and a finite element stress analysis. Figure 15 compares the two sets of normal stresses for a slip surface the exits at the toe of the slope. The finite element simulation produces results that reveal that the ground surface geometry goes horizontal after the toe of the slope. However, the limit equilibrium analysis does not show any effect of the geometry past the toe of the slope. In this sense, the finite element analysis would appear to produce a more accurate indication of the normal stresses.

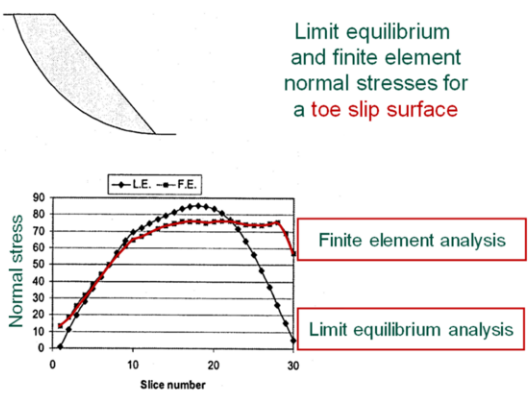

Figure 15. Comparison of limit equilibrium and stress analysis normal forces for slip surface exiting at the toe.

Finite element stress-deformation analyses have been used in different ways for analyzing the stability of a slope. The methodologies can be divided into two broad categories. In the first category there are methods where the stress state information from the stress-deformation analysis are used within the framework of a limit equilibrium analysis. In this case, the normal stress and the actuating shear stresses are "imported" into a limit equilibrium analysis framework. This class of analysis is referred to an "Enhanced Limit" analysis. There are several "Enhanced Limit" analyses that have been proposed and the methods mainly differ in terms of the equation used to define the factor of safety as shown in Figure 18. It appears that the factor of safety definition proposed by Kulhawy [12] seems to have gained the widest acceptance in geotechnical engineering.

4.1. Enhanced Limit Analyses

There are two categories of slope stability analysis that make use of the results from a finite element analysis. The first category, (i.e., Enhanced Limit analyses) where a linear elastic analysis is used to calculate the stress state throughout the soil mass and the second category (i.e., Strength Reduction analyses) where an elasto-plastic analysis is used. There are also two types of Strength Reduction methodologies that have been proposed; namely, i.) methods where the applied load (or gravity forces) are increased until failure occurs or ii.) methods where the strength of all soils is decreased until failure occurs. The two categories of finite element slope stability are first explained followed by an explanation of the relationship between the two analytical procedures.

Figure 16 is a flowchart that shows the types of finite element stress-deformation analyses that have been proposed for slope stability studies. Consideration is first given to the use of an "Enhanced Limit" type of analysis.

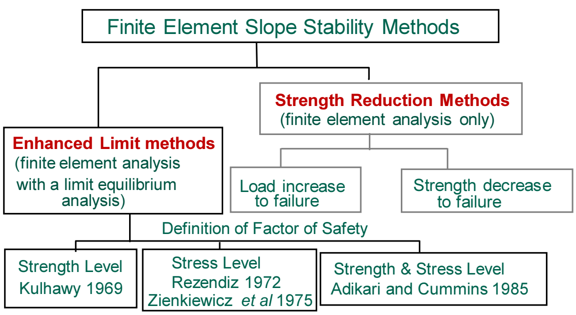

Figure 16. Classification of methodologies that can be used to incorporate stress analysis results.

The factor of safety equation proposed by Kulhawy [12] is shown in Eq. (8) and is most consistent with the equation used in geotechnical engineering.

$$F_K = \frac{\sum\left(c' + \sigma' \tan\phi'\right)\Delta L}{\sum\left(\tau\right)\Delta L} \tag{8}$$

where ΔL represents increments along the slip surface.

5. Use of the "Enhanced Limit" Analysis

There are some distinct differences between a "Limit Equilibrium" analysis and an "Enhanced Limit" analysis. A stress analysis is first performed when undertaking an

Enhanced Limit analysis. The normal stress on any orientation of a slip surface can be calculated because the complete stress state is known. In other words, the Enhanced Limit analysis becomes determinate and the factor of safety equation becomes linear. On the other hand, there are some similarities to a limit equilibrium analysis. It is still necessary to assume a shape for the slip surface and the location of the critical slip surface must be located by trial and error. The Kulhawy [12] definition for the factor of safety is illustrated in Figure 19. For every section along the base of a slip surface, there are two stresses that must be retrieved from the stress-deformation analysis. These are: i.) the normal stress, and ii.) the mobilized shear force. The normal stress is inserted into the shear strength numerator of the factor of safety equation while the shear force mobilized is used in the denominator as shown in Eq. (9).

$$F_{FEM} = \frac{\sum S_r}{\sum S_m} \tag{9}$$

where F_{FEM} is the Kulhawy [12] factor of safety definition, S_m is the shear force mobilized and S_r shear strength resistance.

An example problem illustrates the similarities and the differences between an Enhanced Limit analysis and a limit equilibrium analysis. As part of the Enhanced Limit analysis it is possible to calculate a local factor of safety corresponding to each section along the slip surface, in addition to the global factor of safety. Let us consider a 2:1 slope that is 20 meters high with a piezometric line at 2/3 the height of the slope but exiting at the toe of the slope. Any Young's modulus can be assumed for the analysis. The Poisson's ratio is set to 0.33 since it has some effect of the calculated stresses. Figure 17 shows a plot of the shear strength and the actuating shear stresses at each section across the slope. Note that at all points along the slip surface, the shear strength values are consistently higher than the actuation shear force meaning that all local factors of safety are greater than 1.0.

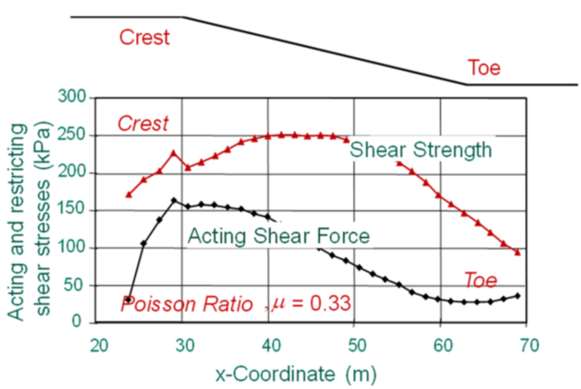

Figure 17. Shear strength and actuating shear force across the slope.

The results from an Enhanced Limit analysis can also be plotted as shown in Figure 18. The results are shown for Poison's ratios of 0.33 and 0.48. The plots of local factor of safety are different for the two Poisson's ratio values; however, the global factors of safety differ by only 0.1%. Also shown are the global factors of safety obtained when the example problem is analyzed using the various limit equilibrium methods of analysis. The global factor of safety using the Morgenstern-Price method was 2.356 or a difference of about 1.0% from the Enhanced Limit analysis. It may seem to be a surprise to find such a similarity in the factors of safety when using two dis-similar methods of analysis.

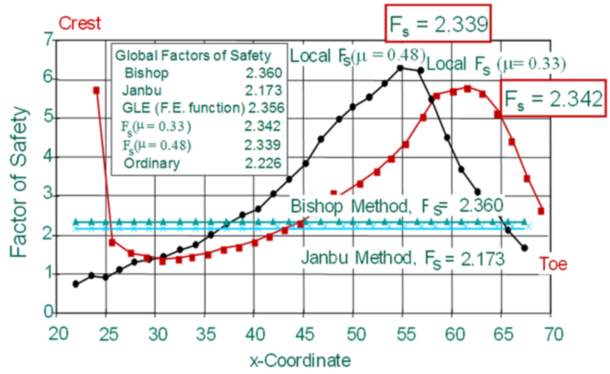

Figure 18. Comparison of Local and Global factors of safety.

The most logical explanation would appear to lie in the fact that both methodologies conserve the potential energy of problem and for this reason are quite similar.

Figure 19 provides a comparison between the results of a parametric study of limit equilibrium analyses and Enhanced Limit analysis. A wide range of cohesion and friction angles were selected and the results are plotted versus the Stability Number. The results are extremely close to each other but it must also be remembered that the problem being analyzed only consists of simple slopes.

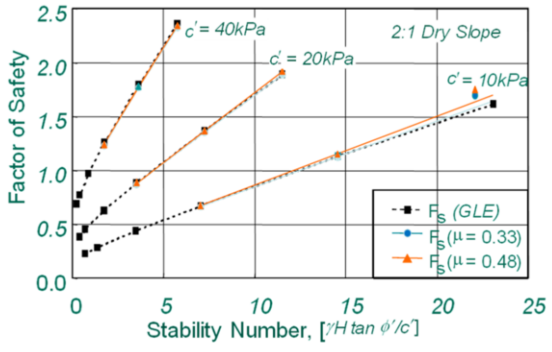

Figure 19. Comparison of factors of safety versus stability number for a range of soil properties.

In general, the global factors of safety appear to be similar between Enhanced Limit and limit equilibrium methods. There is; however, some differences in the location of the critical slip surface locations. This appears to be mainly related to the differences in the calculated normal stresses between the two methodologies. Figure 20 illustrates how the critical slip surface determined by the limit equilibrium method exits at the toe of the slope while the critical slip surface may extend beyond the toe for the Enhanced Limit analysis.

There appears to have been limited interest in geotechnical engineering practice for the use of Enhanced Limit analyses for calculating the factor of safety of a slope.

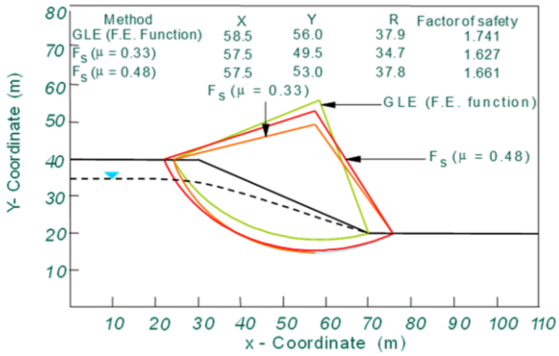

Figure 20. Comparison of the location of the critical slip surfaces.

There may be several reasons for its slow acceptance. First, additional soil properties are required when performing a stress-deformation analysis (e.g., Young's modulus and Poisson's ratio). While the effect of Young's modulus appears to be negligible, there is less confidence in working with deformation soil properties. There is also the perception of inaccuracy in the stresses when performing an analysis corresponding to near failure conditions. There is also the lack of an experience database associated with the Enhanced Limit method. There is the possibility that consideration might be given to use of the Enhanced Limit method when analyzing complex geometries but at present, geotechnical engineering practice is quite committed to using limit equilibrium methods of analysis.

5.1. Comparison Between a Linear Elastic and Elasto-Plastic Stress Analyses

The above comparisons between Enhanced Limit analysis and limit equilibrium analyses were undertaken using a linear elastic stress-deformation analysis. The next question to be addressed is related to whether it might be preferable to use an elasto-plastic analysis rather than using an elastic analysis. This question is examined by comparing the results of a linear elastic and an elasto-plastic analysis when the computed overall factor of safety is; i.) well above 1.0, ii.) near 1.0, and iii.) below 1.0 [13, 14]. The soil properties for the dry slope are shown in Figure 23. The Dynamic Programming technique, described later in this paper, was used to perform the analyses.

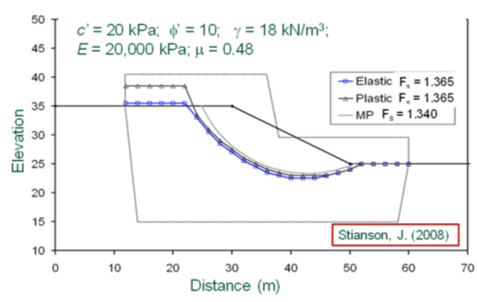

Figure 21. Stress-deformation analysis of a homogeneous dry slope with a factor of safety of approximately 1.3.

Figure 21 shows the computed factors of safety along with the location of the critical slip surfaces for an elastic and elasto-plastic analysis when the factor of safety is approximately 1.3. Also shown are the results of a limit equilibrium analysis (i.e., Morgenstern-Price method). All of the critical slip surfaces have a similar location and the overall factors of safety are the same (i.e., $F_s = 1.365$).

Figure 22 shows a comparison of the calculated local factors of safety across the slip surface. The calculation of the local factors of safety provides an indication of whether there was a re-distribution of stresses along the slip surface (i.e., any plastic zones).

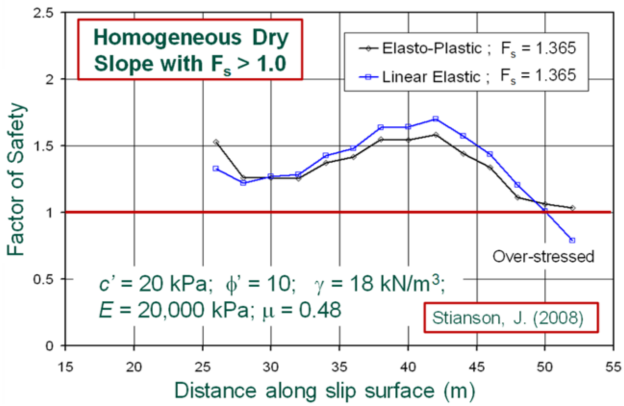

Figure 22. Local factor of safety when the global factor of safety is about 1.3.

The local factors of safety were similar across the slip surface with the exception of some re-distribution of stresses near the toe of the slope, preventing the local factor of safety from going below 1.0.

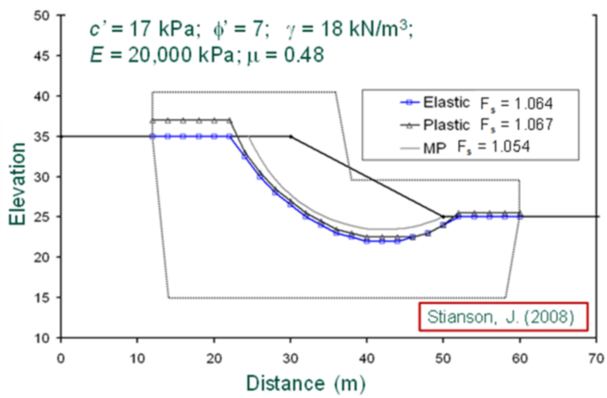

Figure 23. Stress-deformation analysis of a homogeneous dry slope with a factor of safety of approximately 1.0.

Figure 23 shows the computed factors of safety along with the location of the critical slip surfaces for an elastic and elasto-plastic analysis when the overall factor of safety approaches 1.0. The shear strength parameters have been adjusted to bring the overall (global) factors of safety close to 1.0. Also shown are the results of a limit equilibrium analysis (i.e., Morgenstern-Price method with $F_s = 1.054$). The critical slip surfaces have a similar location and the global factors of safety show very slight differences.

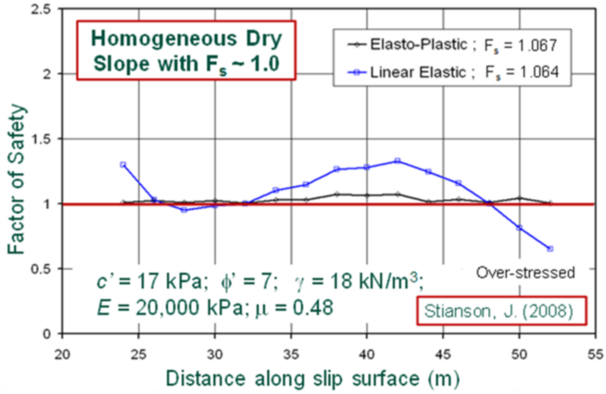

Figure 24. Local factor of safety when the global factor of safety is about 1.0.

Figure 24 shows the computed local factors of safety along the slip surfaces for an elastic analysis and an elasto-plastic analysis. The results correspond to the case where the global factor of safety approaches 1.0. There is clearly a significant redistribution of stresses forcing the shear strength to move into the plastic mode. Consequently, the local factors of safety for the elasto-plastic analysis move towards 1.0. The local factors of safety from the elastic analysis show an over-stressing of the soil near the crest and near the toe of the slope.

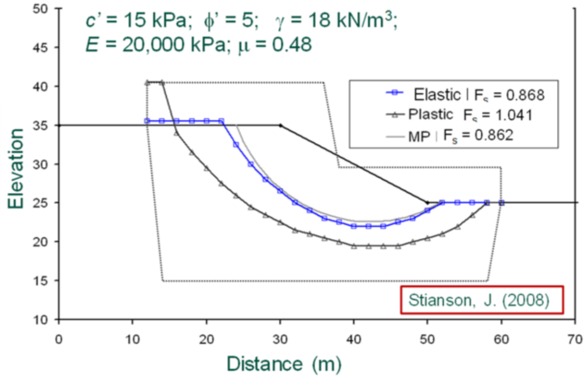

Figure 25. Stress-deformation analysis of a homogeneous dry slope with a factor of safety less than 1.0.

Figure 25 shows the computed factors of safety along with the location of the critical slip surfaces for an elastic and elasto-plastic analysis when the global factor of safety from a limit equilibrium analysis goes less than 1.0. The shear strength parameters have been adjusted to bring the limit equilibrium factor of safety less than 1.0 (i.e., Fs = 0.862). The location of the critical slip surface, and the global factors of safety, for the limit equilibrium analysis and the elastic stress analysis remain essentially the same. However, the elasto-plastic stress analysis undergoes a wide re-distribution of stresses causing the global factor of safety to remain near 1.0. The re-distribution of stresses is clearly shown in Figure 26.

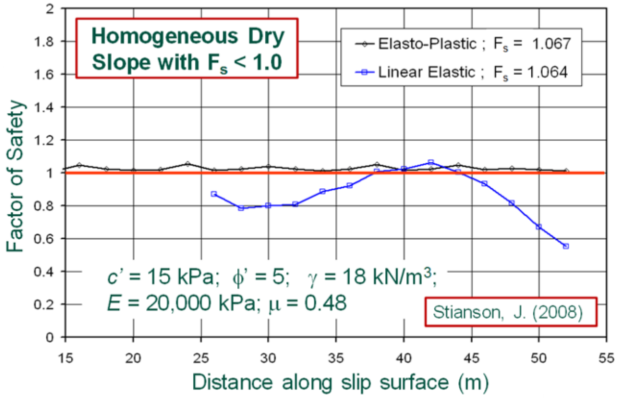

Figure 26. Local factor of safety when the global factor of safety is less than 1.0.

The conclusion from this small comparative study shows that an elastic stress model and an elasto-plastic stress model give similar values in a slope stability analysis as long as the global factor of safety is greater than 1.0. However, an elasto-plastic model should not be used to calculate the *in-situ* stress state when using a finite element stress-based analysis for calculating the factor of safety in a slope stability analysis. Rather, it is an elastic stress-strain model that should be used to estimate the *in-situ* stresses in a soil mass. An elasto-plastic model results in a re-distribution of stresses thereby preventing the factor of safety from ever going below 1.0.

Earlier in this paper (i.e., Figure 16) it was shown that there were two ways in which the results from a finite element stress analysis could be used to assess the factor of safety of a slope; namely, i.) the Enhanced Limit approach and the ii.) Strength Reduction approach. Whereas the Enhanced Limit approach made use of the results from an elastic model, the Strength Reduction approach makes use of an elasto-plastic model of behavior.

5.2. Strength Reduction method

The Strength Reduction method constitutes an approach whereby the results of a finite element analysis can be used as part of a slope stability analysis. There are two ways whereby the strength reduction method can be implemented. For example, the load (e.g., gravity forces) could be increased until the calculated deformation showed that failure was imminent, or the strength of the soil could be decreased until failure was imminent. It is the latter case that has received most attention in geotechnical engineering [15].

The definition for the factor of safety must be changed to be based on a deformation criterion that signifies a "failure state" for the Strength Reduction method. "Failure state" is related to an excessive deformation state such that convergence was not attainable in the finite element stress-deformation model. For this to happen, it is necessary to use a Mohr-Coulomb shear strength type soil model. The factor of safety is equal to the ratio of the actual shear strength parameters (e.g., c' and $\tan \phi'$), to the reduced shears strength parameters corresponding to a non-converged solution. Stated another way, the factor of safety is equal to the maximum reduction in the original shear strength parameters where convergence could still be obtained. One of the driving forces behind the Strength Reduction method is the fact that fewer "a priori" assumptions are required regarding the failure mechanism.

Figure 27 uses a measure the amount of movement on the ordinate versus the factor of safety on the abscissa. Two example analyses are shown. When the full value of the strength parameters is used, the measure of deformation for example 1 was approximately 0.4 (unitless).

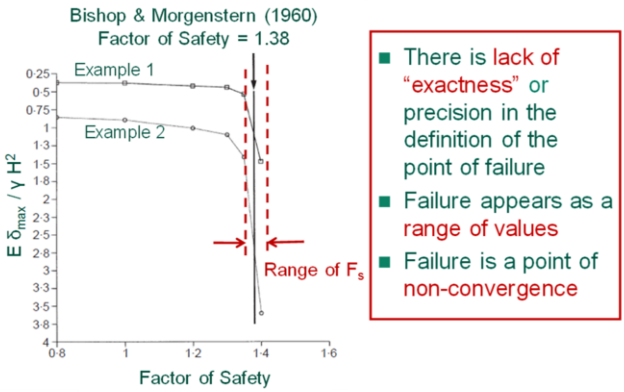

Figure 27. Failure condition specified for the Strength Reduction, method, SRM.

As the shear strength parameters are reduced, the deformation measure slowly decreases to about 0.5 and suddenly the deformation measure rapidly increases to more than 1.5. Further reductions in the shear strength analysis causes the analysis to become unstable (i.e., non-convergence). There is no clear or distinct value for the factor of safety at failure. Rather, failure is defined as a range of values as shown in Figure 29. This behavior is somewhat bothersome to the geotechnical engineer who has been used to working with a precise value for the computed factor of safety when using other methodologies.

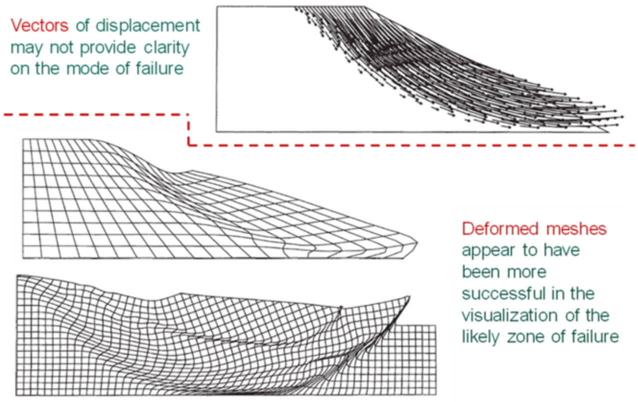

Figure 28. Use of deformed meshes to show zones of excessive deformation.

Figure 28 illustrates how the deformed mesh reveals the zone (or zones) where failure conditions are being approached. In other words, the Strength Reduction method provides the geotechnical engineer with information on the shape and location of failure surfaces.

Additional soils information (e.g., Young's modulus and Poisson's ratio) must be input by the modeler when using an elasto-plastic model. Consequently, the Strength Reduction method require more "know how" than limit equilibrium methods. While this may appear to be a disadvantage for the Strength Reduction methodology, it is noted that the Mohr-Coulomb strength envelope dominates the analysis as failure conditions are approached. The convergence rules (i.e., criterion) may require some user intervention in refining the failure tolerance and this has been viewed as a negative aspect by initial users of the Strength Reduction method. Countering the negative concerns is the fact that the Strength Reduction method provides additional valuable information particularly when analyzing complex geological and geometric conditions.

Figure 29 shows one more example where the results of a Strength Reduction analysis are compares with the results of a limit equilibrium analysis. The values differ slightly but can be considered to be essentially the same from a practicing engineering standpoint.

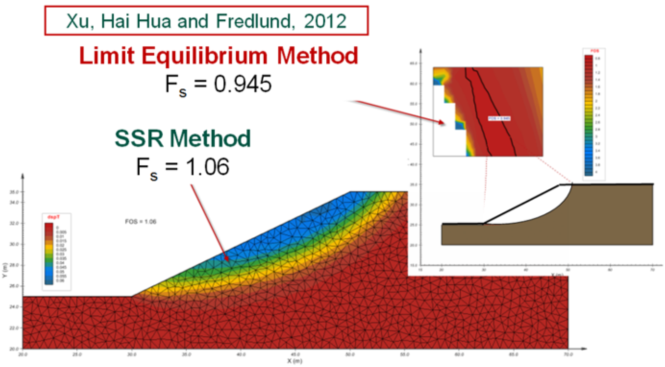

Figure 29. Comparison of the results of a Strength Reduction analysis and a Limit Equilibrium analysis.

The shear strength reduction methodology has also been extended to three-dimensional analyses [16, 17].

6. 3D Analysis and Digital Twins

The traditional application of the limit equilibrium method (LEM) has been in the context of a 2D plane strain analysis. Geotechnical engineers have become complacent with the use of 2D slope stability as it is easy to perform. 2D analysis suffers from fundamental limitations foremost of which are i) the slip shape is assumed to be cylindrical, ii) the slope geometry is assumed to be unchanged in the 3rd dimension, iii) the geo-strata is assumed to not vary in the 3rd dimension, iv) the water table must only vary in the down-slope direction, v) the application of distributed and point surface loads is not considered in the 3rd dimension, and vi) the application of anchors, micropiles, and geomembranes is approximated [18]. Such assumptions have proven 2D analysis to be conservative with respect to the true 3D factor of safety in the amounts between 10-50% [19]. If unsaturated aspects of slope stability are considered then the difference can be as high as 60% [20]. Therefore, significant opportunity exists to optimize existing designs through the application of 3D slope stability analysis. It should be noted that the difference between a 2D and a calculated 3D factor of safety is different in each situation. Therefore, it is impossible to assume a specific 2D/3D difference for a particular scenario.

The theory for the 3D LEM has been in existence for decades and is therefore not new. Early theoretical development efforts are available in research literature [21, 22, 23, 24, 11]. The primary limitation of all presented theory is that the slip direction is assumed to happen exactly along the x-coordinate axis. Such a limitation introduces a significant problem for the practical application of 3D slope stability analysis. This paper presents an extension to the traditional 3D LEM, which allows for the analysis of slips at any direction. Several benchmark examples are presented such that the implementation of the methodology is proven sound. The technique may be applied to Bishop, Morgenstern-Price, GLE, Spencer and other analysis methods.

The application of 3D slope stability LEM in practical geotechnical analysis requires consideration of the aforementioned 3D influences on the factor of safety. This paper focuses on the effects of slip surface shape and ground surface shape. A recommended approach for the application of 3D slope stability analysis in the practice of geotechnical engineering is also presented.

The application of software tools to analyze slope stability in 3D has traditionally been highly limited. Recent developments allow for the easy application of 3D stability analysis in typical problems [20, 25, 26]. The SVSLOPE software developed by SoilVision Systems Ltd. is utilized for the analysis presented in this paper.

6.1. Continuity between 2D and 3D LEM

It is of importance to understand the difference between the slip shape analyzed in a 2D analysis as opposed to a 3D analysis. In a 2D analysis the slip shape is ultimately extended infinitely in the 3rd dimension and shear on the end surfaces is not considered. An example 2D analysis is presented in Figure 30. The equivalent 3D analysis is shown in Figure 31 where there is no shear strength applied to the vertical end surfaces.

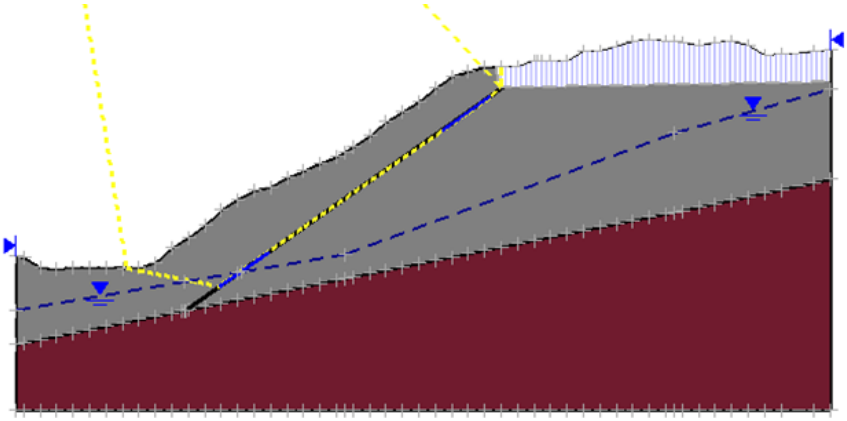

Figure 30. Example 2D stability analysis.

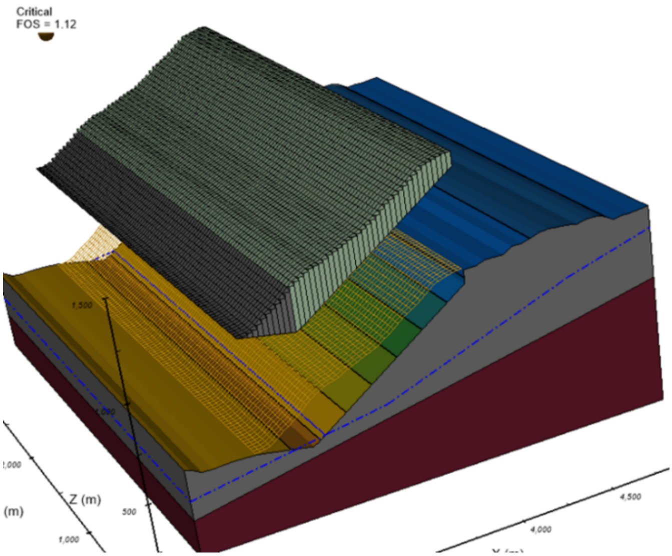

Figure 31. Equivalent 3D analysis (slip surface exploded out of slope).

It should be noted that a simple proof of a 3D analysis can be done by creating a 3D model of a 2D extruded slip surface and applying zero shear strength to the end-walls. This is an easy way to prove the 3D equivalent scenario of any 2D analysis. It also highlights the fundamental limitation of a 2D analysis which i) considers the slip to be of infinite length in the 3rd dimension and ii) does not consider the influence of shear strength on the end surfaces.

7. Directional Three-Dimensional Slope Stability Analysis

Three-dimensional stability analysis of asymmetrical geometries requires the determination of the critical slide direction. Yamagami and Jiang [27, 28] presented a relatively simple solution to this problem by computing the factor of safety using regular one-direction methods and varying the slide direction. The critical slide direction becomes the result of an optimization analysis in search for the minimum factor of safety. Huang and Tsai [29] presented a 3D two-directional moment equilibrium analysis that allow the determination of the critical slide direction. The authors considered semispherical and composite slide surfaces. The developed method is based on the principles of the method of columns and Bishop's assumption regarding the interslice shear forces. The equations for the moment equilibrium factor of safety in two directions is made as a function of the slide direction and the critical slide direction is shown to correspond to equal values of factors of safety. The critical direction is determined in a iterative manner. The method was verified using three different asymmetrical slopes, demonstrating is potential to identify failure mechanisms with lower factors of safety. Later, Cheng and Yip [30] presented a more general approach, based on moment and force equilibrium, the corresponds to extensions of Bishop's simplified, Janbu's simplified, and Morgenstern-Price's methods. The authors explain that past approaches are unstable when the slope is subjected to transverse horizontal forces and that the proposed methods eliminate such instability. A careful examination is presented of the relevance and implications of assuming that all columns move in the same direction or that divergent movement takes place.

Figure 32 presents an illustration of the use of directional 3D slope stability analyses to asymmetrical slopes. The problem consists of a wedge failure with dimension of 2 and 4 m for which the analytical solution is FS = 0.280, a slide direction of 26.5°, and a dip angle of 65.91°. This problem was analyzed by Huang and Tsai [29] and by Cheng and Yip [30] and is analyzed here using SVSLOPE and two implemented approaches: a brute force method similar to what was introduced by Yamagami and Jiang [27, 28] and a one-directional automated method (Gao's method). Both methods produce correct results that match the analytical solution. However, the automated method required approximately 10% of the computation time of the brute force approach.

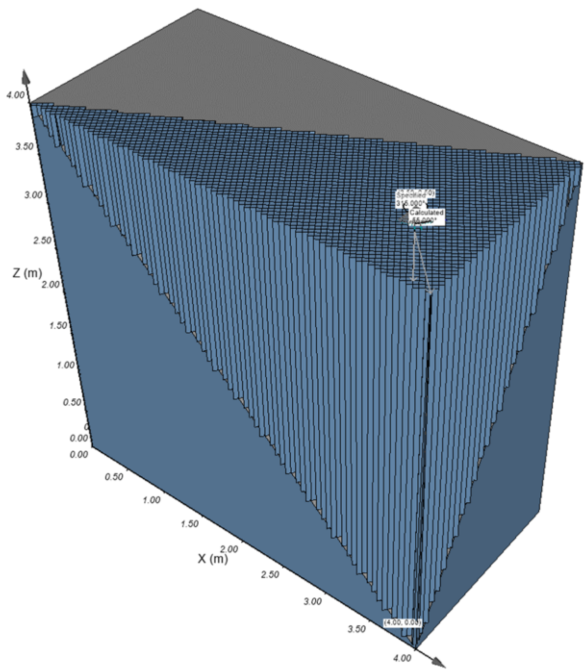

Figure 32. Asymmetrical wedge surface model.

8. Probabilistic approaches for slope stability analysis

From a deterministic standpoint, a slope with an expected factor of safety E[FS] = 1.5 is safer than a slope with E[FS] = 1.2. The two probability density functions (PDFs) presented in Figure 1 show that a PDF with higher E[FS] may also present a higher probability of failure. Therefore, E[FS] alone may provide incomplete or misleading information about the stability of a slope. Reliability-based approaches provide a way of incorporating parameter uncertainty into the slope modeling process. Reliability-based approaches are based on a method of computation of the probability density function (PDF) of the performance variable (e.g., the factor of safety, FS). Measures of reliability are defined based on the central tendency and variability of FS [31].

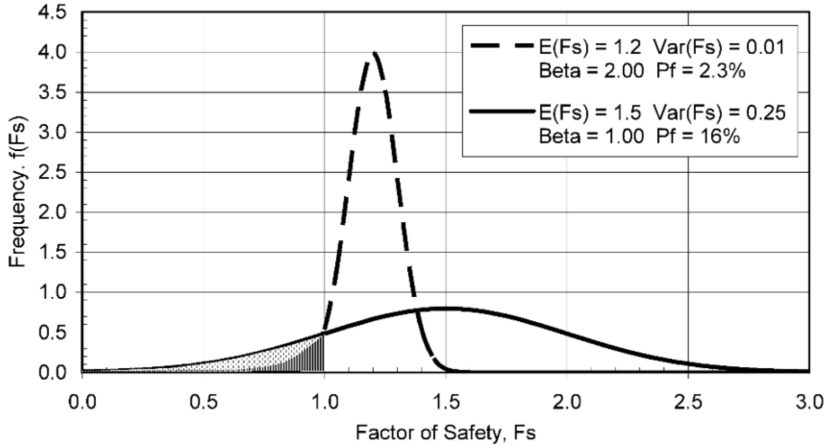

Figure 33. Probability density function of the factor of safety, FS, and probability of failure, Pf.

The design and decision-making process for slope stability requires not only the quantification of geo-hazards, but also the assessment of the vulnerability and risks, the establishment of acceptable risk levels and the selection of managing actions. Several techniques for assessing vulnerability and selecting acceptable risk levels are available and a number of management actions can be taken. Finally, it is important to point out that slope hazard assessment approaches may be implemented at various scales and that most slope hazards occur in the vadose zone and depend on the near ground surface soil characteristics and state. Unsaturated soil mechanics provides the required theoretical background for the quantification of slope hazards because slides are often triggered by the reduction in soil suction and shear strength.

8.1. Overview of reliability analysis methods

A probabilistic method that yields the PDF of the factor of safety is required in order to assess slope stability hazards. The main probabilistic methods available are the Monte Carlo simulation, the Taylor series method, and the Point Estimate methods. The Monte Carlo method was first developed by Hammersley and Handscomb [32]. Sets of input variables are obtained using random number generators. Each randomly generated set must be used to calculate a realization of FS and then used to define the PDF of FS. The Monte Carlo method requires a large number of trials, as shown by Harr [33].

The Taylor series method provides an approximate approach for obtaining the first few moments of the PDF of a function of random variables, such as the factor of safety, FS. Unlike the Monte Carlo method, the Taylor series method does not provide the complete PDF of FS. Therefore, the PDF of FS must be assumed for the computation of the probability of failure, Pf. The use of the first-order, second-moment equation provides a good approximation if FS is a linear function of the input parameters. Unfortunately, the derivatives of FS must be approximated using numerical approaches.

Point Estimate methods (PEM) are probabilistic methods for calculating the moments of the PDF based on the calculation of FS at pre-determined input variable values. The pre-determined values are combined with corresponding discrete probabilities. Evans [34] proposed an early PEM for independent random variables.

However, the first popular PEM approach is generally credited to Rosenblueth (1975)[35]. Several other point estimate methods where proposed (e.g., Li [36]; Panchalingam and Harr [37]). More recently, Gitirana Jr. [38] and Franco et al. [39] introduced a hybrid or alternative PEM based on the combination of the Taylor series approximation and the Rosenblueth two-point estimation for the estimation of the derivatives in Taylor's expansion. The method was show to be as accurate as previous method, and yet required a reduced number of analyses. Figure 34 presents a comparison of the number of evaluations required by different probabilistic approaches.

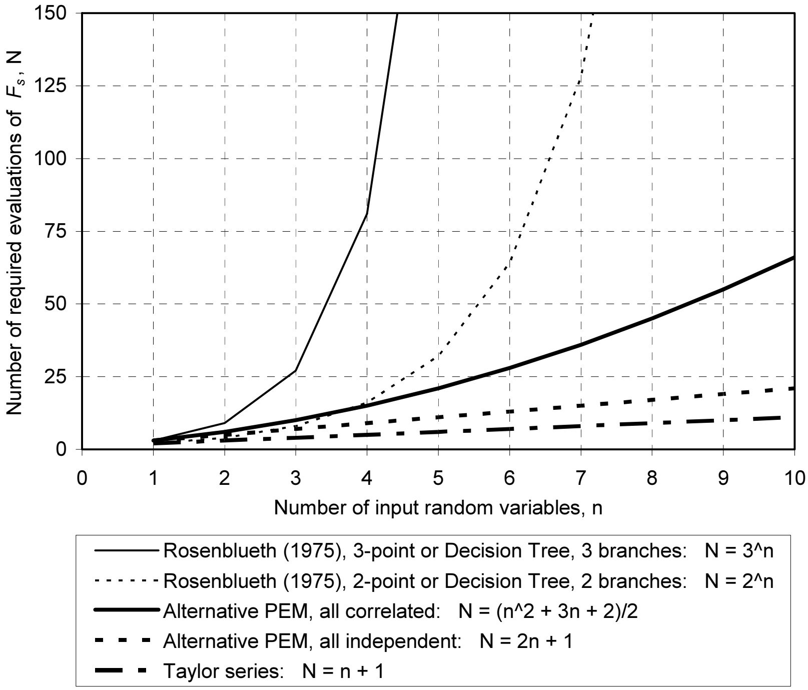

Figure 34. Number of evaluations of FS required by several probabilistic methods.

8.2. Implementation of probabilistic analysis in practice

A variety of issues must be addressed for the use of probabilistic methods in the practice of slope stability analysis and design. Many issues are related to protocols for data collection and the definition of the level of detail required for data collection. Soil properties may be accessed with various levels of detail. Estimates of soil properties may be obtained based on predictive approaches or knowledge-based systems [40]. Site vulnerability and risk levels must be accessed. There are several approaches available for the assessment of vulnerability based on the travel distance of a slide mass and the presence of neighboring structures and facilities. Probability of failure, site vulnerability and consequences must be combined with clearly defined levels of acceptable risks. According to Whitman [31] and Becker [41], the typical range of acceptable probabilities of failure adopted in slope and foundation engineering projects varies from 0.1 to slightly lower than 1%. Acceptable risk levels can be established by compiling observed frequencies and consequences of natural and man-made events and use these values as a comparative basis. Computationally software is now available for computing probabilities of failure in 2D and 3D analysis.

9. Rock Stability Analysis Considering Anisotropy

The analysis of anisotropy is especially relevant in rock masses. Anisotropy can result from the geologic deformation of rock structures. A shear plane failing across a anisotropic structure is typically a strong failure plane while a shear plane that follow parallel to the bedding plane typically demonstrates weak behavior.

The modeling of such anisotropy in slope stability software has encountered two challenges historically. Firstly, the constitutive strength model to represent anisotropy needed to be developed. Pioneering work was performed by Mercer [42] in the Snowden Mining Industry Consultants in 2005 related to development of the Anisotropic Linear Models (ALM).

When the ALM constitutive model was first developed by Snowden Mining Industry Consultants in 2005, it assumed a linear relationship, both upslope and downslope between shear strength and the orientation in the rock mass. This orientation in the rock mass is known as the Angle of Anisotropy, which is measured as the orientation of an arbitrary slice relative to the orientation of the bedding plane (usually the weakness plane). As the ALM strength model evolved, the fourth generation ALM4 now has differences from its predecessors. Most notably, it now includes defined equations to represent the relationship between the Angle of Anisotropy and the shear strength at any orientation. Simply put, this advancement allows the user to determine shear strength at any orientation in the rock mass.

ALM4 presents other significant information, such as shear strength differentials in the upslope and downslope directions; percentages of shear strength differential; and relationships between the Angle of Anisotropy and the percentage of shear strength differential. ALM4 accommodates simple input parameters to provide valuable modeling information in anisotropic environments.

The second challenge of modeling anisotropy is having a methodology to represent anisotropic weak planes in 2D and 3D numerical models. Historical effort has focused on specifying weak surfaces as lines in 2D or as flat planes in 3D. However, this configuration rarely fits the real-world. Therefore a methodology of representing bedding planes in 3D space was developed (Figure 35; Figure 36).

Bedding Guides are surfaces used to conveniently define a series of beddings for rock layers. They are an ordered series of grids or meshes that define two or more known geometries for beddings. Any number of beddings are linearly interpolated between each adjacent pair of guides during model analysis. The effect of a bedding is to modify the shape of the generated trial slip surfaces such that they may follow the weak bedding. This is particularly useful for Anisotropic Linear Model materials, where the slip surface shape must be able to interact with the rock beddings in order to find an accurate factor of safety.

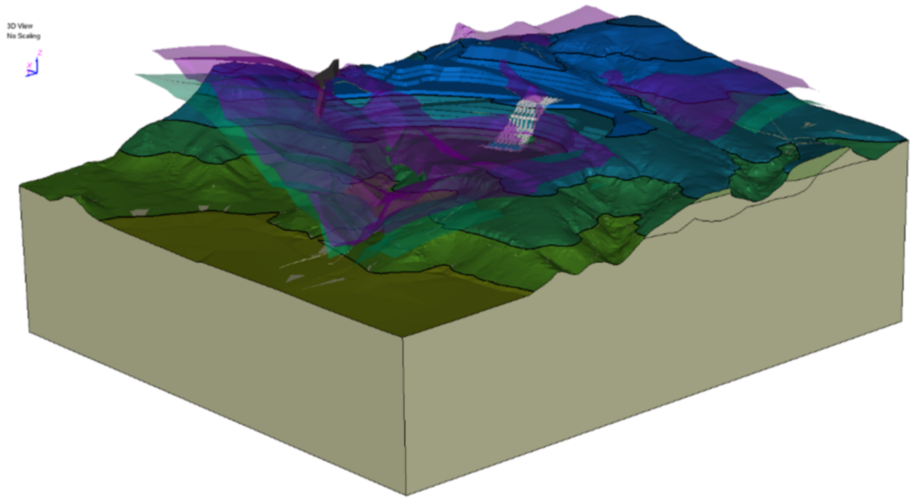

Figure 35. Bedding planes defined in 3D space showing daylighting above the ground.

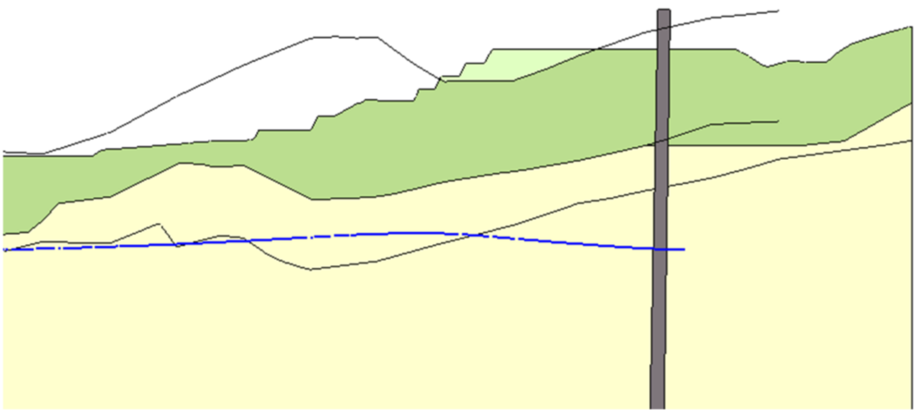

Figure 36. Projection of bedding planes onto 2D space.

Beddings influence a trial slip surface ellipsoid similarly to fully specified wedges or weak surfaces. Where the ellipsoid would cut through and beneath the bedding, instead it follows the bedding (the bedding effectively clips the ellipsoid below the bedding). This allows for parts of the sliding mass to be ellipsoidal shaped while other parts to follow the bedding shape.

Beddings are special in that each ellipsoidal trial will be tested with each individual bedding, one at a time (in addition to only the ellipsoid with no beddings). This means that it's not necessary to know ahead of time which bedding a slip surface should follow. Because only one bedding is active at a time it is not possible for a slip surface to follow one bedding in one area and a different bedding in another area.

10. Spatial Stability Analysis

One of the secondary issues with a standard 2D geotechnical analysis is that the geotechnical engineer may not know the correct location of the 2D plane which produces a critical factor of safety. A classic example of such a problem was presented by Jian [43]. The example presents a simple slope to illustrate two problems with conventional 2D plane-strain stability analysis; namely that the location of the slip as well as the correct factor of safety can be difficult to determine from this relatively straightforward example.

Multi-plane analysis (MPA) allows the engineer to quickly perform hundreds or thousands of analyses around a slope quickly using parallel computations. In the 2D MPA analysis method the slope is sliced into 2D profiles each of which is a full 2D LEM analysis. Each slice is analyzed, and the results are then plotted over top of the original 3D slope therefore giving a spatial perspective to the many 2D slices. Each slice is oriented such that the primary slip direction is the steepest slope. Since the slip may not happen on the steepest slope the MPA analysis method can be specified such that multiple orientations, such as +/- 10 degrees can be analyzed. In this way the most likely slip orientation can also be determined.

The analysis of slope can be seen in Figure 37. From the analysis we can see the critical zone clearly outlined as to the left of the nose of the hill. This is counter-intuitive from the sense that most engineers would feel that the point of the nose of this model would yield the lowest FOS value. Therefore, this illustrates the value of using analytical procedures to determine the correct location of a probable failure zone in an irregular topology. This result must also be noted that it is only the result of 2D slices through differing spatial locations in the 3D model. There is no relation implied or computed between the slices so the 3D lateral effects on the slope are not considered. Therefore, the model must be considered as an indication of a probable failure zone but not a definitive analysis which considers 3D effects.

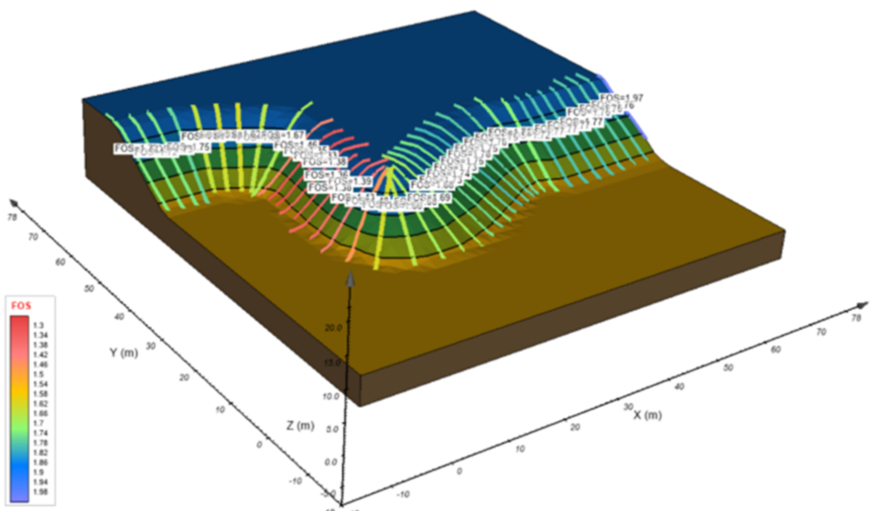

Figure 37. Example of slope difficult to analyze in 2D plane strain [43] – analysis by 2D MPA.

10.1. 3D Spatial Stability Analysis

The benefit of the multi-plane analysis is that it can easily be applied in 3D space using the same slicing planes established for the 2D analysis only with the analysis of a 3D ellipsoid at each slicing plane location. Similar searching techniques such as Entry & Exit, Cuckoo Search, Slope Search, or Auto Refine methods can be applied to the searching technique. The aspect ratio of the ellipsoid as well as faults and fractures can be considered as well in the analysis.

An enormous amount of information is generated with each slice being considered as a full 3D analysis. The 3D results can therefore be presented as a series of individual critical slip surfaces or contoured to produce a contoured map of the relative factor of safety over an area. The results of such an analysis for an open pit are shown in Figure 38. It can be seen that MPA is a useful methodology for i) understanding the relative factor of safety of a large spatial area as well as ii) locating potential zones of instability which may exist. The 3D results also provide a higher and computationally more realistic analysis of the true 3D FOS.

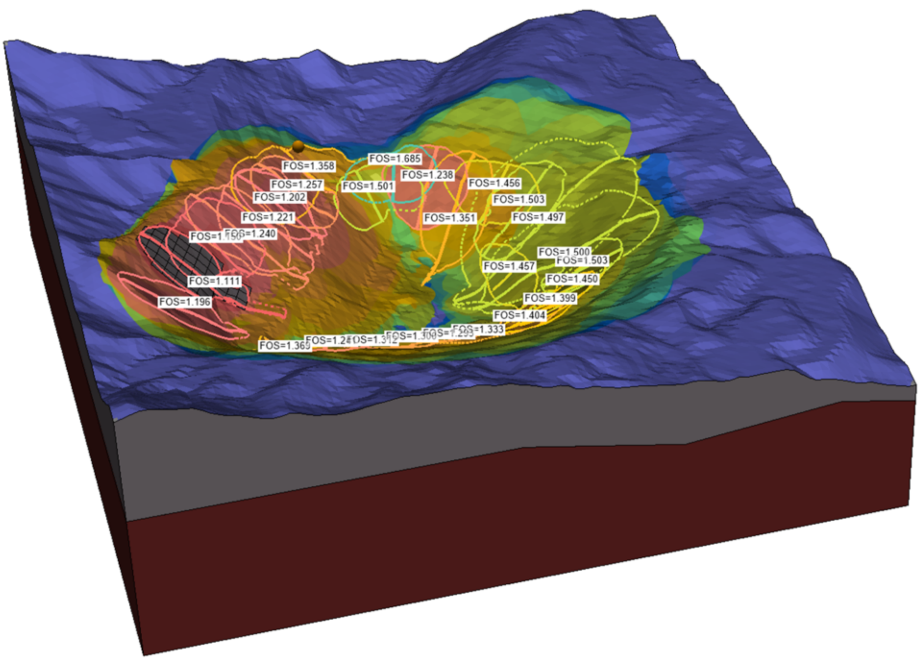

Figure 38. Example of 3D MPA applied to the analysis of an open pit.

11. Summary and Conclusions

Following are a few general conclusions that can be made regarding some of the more recently proposed methods of slope stability analysis and the historically proposed limit equilibrium methods of analysis.

1.) The Morgenstern-Price (or GLE) method of analysis is recommended for use when the limit equilibrium methodology is selected for the calculation of the factor of safety. Other limit equilibrium methods may not give an accurate assessment of the factor of safety depending on the shape of the slip surface (i.e., whether the slip surface is circular or planar). This recommendation holds true for both 2D and 3D analysis.

2.) Most of the more recent alternate methods of analysis appear to give factors of safety quite similar to the Morgenstern-Price (or GLE) method when simple slope geometries are analyzed. The reason for the similarities in the calculated factors of safety appears to be mainly due to the common ground surface geometry which controls the overall potential energy imparted to the slope.

3.) Some general conclusions can be made with regard to the finite element based methods of analysis:

   a.) First, it is recommended that the Kulhawy [12] definition for the factor of safety be used when using the Enhanced Limit method of analysis. Also, the finite element stress-deformation analysis should be performed using a linear elastic model.

   b.) A Mohr-Coulomb model should be used when performing a Strength Reduction finite element stress-deformation analysis. The Strength Reduction method does not provide the engineer with as precise a calculation for the factor of safety; however, the values appear to be essentially the same as those obtained for a limit equilibrium analysis where both moment and force equilibrium are satisfied. The Strength Reduction method can provide the geotechnical engineer with additional concerning the shape and location of the critical slip surface.

There are other aspects associated with the analysis of slopes where consensus would be of value to the geotechnical engineer. Some of these areas are as follows:

4.) There has been considerable attention given to the use of three-dimensional slope stability analyses in recent years [44]. It is commonly concluded that in general, the calculated three-dimensional factor of safety is higher than the two-dimensional factor of safety and as a result it has been brushed aside as being of limited value. However, this does not appear to be a wise conclusion since many slope geometries are truly three-dimensional and cannot be modeled using a two-dimensional analysis. In addition, all slope failure in the field are three-dimensional in character. Two-dimensionand and three-dimensional analysis should be performed in the majority of cases.

5.) Mass movements are often used to back-calculate the shear strength parameters for the soil. While a "forward or design" analysis of a slope involves a search for the minimum factor of safety, the "back-analysis" of a failed slope required a search for the opposite soil parameter conditions.

   Stated another way, the factor of safety from a 3-dimensional "forward or design analysis" may typically be 4 to 20% higher than the F_s obtained from a 2-dimensional analysis. On the other hand, the factor of safety is fixed at 1.0 for a "back-analysis" and the shear strength parameters are computed. As a result, the shear strength parameters (i.e., c' and ϕ') may be 4 to 20% lower in a 3-dimensional analysis than in a 2-dimensional analysis. Or it can be said that the search is for the minimum factor of safety in a "forward analysis" whereas the search is for the maximum required shear strength parameters in a "back analysis". Consequently, the role of the "searches" reverses when considering a "forward analysis" as opposed to a "backward analysis". Understanding the reversed roles is complicated when taking into consideration the differences between a 2-D and a 3-D slope stability analysis.

It should also be noted that a "forward or design" analysis involves a search for one variable (i.e., minimum F_s), whereas a "back-analysis" involves a search for two variables (i.e., maximum required c' and ϕ'). Consequently, further information related to the relationship between the two variables is required to make the analysis determinate.

6.) Geotechnical engineers need to be aware of the limitations associated with so-called three-dimensional slope stability methods. For example, it can be argued that present three-dimensional formulations are not truly three-dimensional in character since all movement must be limited to a single direction. Recent developments where the direction of movement can be searched for critical conditions are welcomed and useful in the assessment of mining pit and riverbank stability studies.

The calculation and importation of pore-water pressures from either an independent saturated-unsaturated seepage analysis is of considerable value in geotechnical engineering. The imported pore-water pressures may also be induced by applied loads under undrained or partial drainage conditions. These interactions between slope stability and pore-water pressures requires the ability to "combine" more than one software package.

As 3D analysis techniques become more common and easily applied the geotechnical engineer must understand the differences between a 2D and a 3D analysis. A 2D equivalent "cylindrical" shape failure can be simulated with a 3D analysis. Probabilistic analysis can provide a more reliable understanding of the risk related to a certain slope and can be performed in 2D and 3D with reasonable computing times. Anisotropy and complex 3D bedding planes can now be accommodated in a 2D or 3D limit equilibrium analysis. Spatial analysis of slopes is possible with multi-plane technologies over larger areas and opens up new analysis possibilities for determining the location of the most critical failure surface.

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