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On the Consistency of Conjectural Variations as the Solution of a Two-Stage Game

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Abstract. In our previous research publications, we established a significant connection between consistent conjectures and those of Cournot–Nash. We developed a new mathematical model based on a two-stage game specifically for the oligopolistic market. In this two-stage game, the role of the players correspond to the agents operating within the oligopoly, and the role of their game strategies is played by the conjectures accepted by these agents. We successfully demonstrated that if a Nash equilibrium exists within this two-stage game, it aligns with one of the consistent equilibria observed in the oligopolistic market. In our proposed study, we aim to identify the specific conditions through which we can substantiate the consistency of the equilibrium within the oligopoly by means of the Nash equilibrium in this two-stage game. This finding is of particular significance as it provides a pathway to expand the notion of consistency for the conjectural variations to mathematical models that differ from the oligopolistic market. Examples of such models include financial markets, the natural gas industry, the oil industry, and various others.

Keywords. consistent conjectures, two-stage game, Cournot–Nash equilibrium

1. Introduction

Bowley [1] and Frisch [2] introduced the notion of *conjectural variations*, which extends the equilibrium concept established by Cournot and Nash for non-coperative games.

The papers [3,4] and the book [5] present an absolutely novel framework for conjectural variations. In this proposed framework, the conjectures of each agent are referred to as the *influence coefficients*, which have enabled the authors mentioned above to unify various existing oligopolistic models under a single framework.

The book [5] provides a comprehensive analysis of the conjectural variations equilibrium (CVE) under broad conditions concerning the conjectures, cost functions, and

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price function. Theorems regarding the existence and uniqueness of the CVE are established. The publications [6,7] further explore the concepts described above.

In [8], the authors provide a comparison between the equilibrium frameworks proposed by Bertrand and Cournot–Nash. The Bertrand model assumes perfect competition, with production and demand aligning and price competition taking place. In their work [8], the authors demonstrate the essentiality of both these assumptions in the context of a two-stage oligopoly. Specifically, they show that all products are simultaneously produced, and price competition occurs at the time the sale of the products begins. By considering general assumptions about demand, the paper establishes that the Cournot–Nash equilibrium is the unique equilibrium in this model.

In [9], a distinct two-stage game, different from the one discussed previously, is analyzed. The paper explore an electricity system that involves three investment models for generation capacity. The first model assumes perfect competition, while the second model employs the Cournot–Nash model. The third model incorporates a two-stage game, where decisions regarding investment and sales are made in the first stage, followed by decisions related to sales (i.e., the spot market) in the second stage. A notable outcome presented in the aforementioned paper is that, in the third game, when a solution exists, the quantities produced and the prices fall between those observed in the second game and the competitive equilibrium.

The research conducted in [10] builds upon previous investigations. The authors examine how players' conjectures regarding knowledge impact the outcomes of a game where the results align with their accurate conjectures. A key finding of this study is that for an oligopolistic equilibrium to mimic perfect competition, the conjectural variations must be consistent. Furthermore, the authors obtain results akin to those in [11] regarding forward markets. However, this is achieved by considering different assumptions regarding knowledge and consistent conjectures.

Every idea inevitably experiences ups and downs, and the CVE concept is no exception. For a comprehensive account of the "ups and downs" of CVE, please refer to the publications by [12] and [13]. To narrow our scope, we will direct our attention to the study presented in [13].

The concept CVE has been extensively discussed from various perspectives, as discussed by Figuières et al. [14] and Giocoli [15]. A notable publication exploring this concept is Lindh [16]. Despite the prevalence of CVE usage among economists, it is unfortunate that the papers focusing on conjectural variations primarily consider two-player games (*cf.*, [14]). In general, the conjectures determine the response of player j, as anticipated by player i, to a small change in player i's strategy, leading us to introduce the concept of a "conjectured reaction function" for the rival player. Under the assumption of these conjectured responses from rivals, each agent can optimize their payoff, resulting in the notion of a "conjectural best response function." A natural equilibrium arises when all of the players do not have any incentive to deviate from their strategy. In this scenario, the conjectures of each player align with those of their rivals, thus termed a "consistent" equilibrium, with the conjectures themselves also being considered consistent.

A challenge arises when examining consistency in scenarios involving numerous agents, i.e., beyond two players (*see*, again, [13] and [14]).

Consider n agents, resulting in n best response functions and $n(n-1)$ conjectured reaction functions. The existence of equilibrium requires the best response functions to match the conjectured reaction functions, a condition that can only be realistically explored in two-player games. This concept is explicitly employed by Başar and Olsder [17] and Fershtman and Kamien [18]. Consequently, papers on static n-player games with conjectural variations typically assume player as identical [19,20,21]. In the case of oligopoly [22], public products [23,24], and games involving more than two players, the problem is resolved by introducing virtual players. Each agent is associated with a unique virtual player whose payoff function is the sum of all payoffs related to the agent's rivals [22,23,24].

The problem concerning many-player games was effectively resolved in [25]. In this study, each player formulates conjectures regarding the relationship between output variations and market price variations, known as influence coefficients. By taking into account the adversaries' conjectures and employing a verification procedure, each agent can determine if their conjecture aligns with those of their adversaries. A conjecture is deemed consistent when it aligns with the conjectures of all players, leading to an equilibrium state. The consistency criterion, formulated in [25], was subsequently derived in [26] after a decade, albeit under more stringent assumptions and employing a more complex optimal control technique (similar ideas were employed in [27]).

An application of the findings from [25] in the context of mixed oligopolies was explored in [12]. Similarly to [3,4], the paper [12] focused on an oligopoly model where the level of influence exerted by each agent is quantified using specific parameters known as influence coefficients.

However, unlike the models discussed in [3,4] and [6,7], the models analyzed in [12] and [13] employ the concepts introduced in [25] but with a different observable variable. Instead of using the total output volume, these models utilize the market price as the observable variable.

In [12], it was shown that the consistent conjectures differ from the Cournot–Nash conjecture. In our study [13], we developed a two-stage game where the role of the players is taken agents in the oligopolistic market, and their strategies correspond to the conjectures adopted by these agents. A significant finding of our study is that the consistent conjectures for the original oligopoly align with the optimal Nash strategies in the two-stage game introduced (referred to as meta-game or meta-model). Similar results were briefly mentioned in the publication [28] for quadratic cost functions, although without providing a proof.

Given that the feasible set of players' strategies in the meta-game is the nonnegative orthant in $\mathbb{R}^n$, which is not compact, ensuring the existence of a Nash equilibrium requires additional assumptions. In this study, we introduce such assumptions, and under these assumptions, we establish the relationship between the consistent conjectural variations equilibrium (CCVE) in the original oligopoly model and the Nash equilibrium in the meta-game. This result is of significant importance as it paves the way for extending the concept of consistency of the conjectures to mathematical models that differ from traditional oligopolistic models.

2. Model Specification

Let us consider an oligopoly consisting of multiple producers (indexed by i) who sell a homogeneous good. We denote the set of all producers by I. Each producer's cost function is denoted as $c_i = c_i(s_i)$, where $s_i \geq 0$ represents the supply quantity of producer

i. The demand function is denoted as $D = D(r)$, where r is the market price. We define W as the active demand, which remains constant regardless of the price and is nonnegative. Furthermore, the following balance equality is valid:

$$\sum_{i=1}^{n} s_i = D(r) + W. \tag{1}$$

In addition, we accept the following assumptions for the mathematical model:

Assumption A1 *The demand function $D = D(r) > 0$, for $r > 0$, is continuously differentiable and decreasing.*

Assumption A2 *The function $c_i = c_i(s_i)$, $i \in I$, exists for any $s_i \geq 0$, has a continuous second order derivative, and satisfies the next relationship:*

$$c_i'(0) > 0 \textit{ and } c_i''(s_i) > 0,\ \forall s_i \geq 0. \tag{2}$$

All agents $i \in I$ maximizes their own net profit function:

$$f_i(r, s_i) = rs_i - c_i(s_i). \tag{3}$$

We assume that the agents believe that they can influence the price r by adjusting their production volumes. In this case, the first order maximum condition can be written in the following way:

$$\frac{\partial f_i}{\partial s_i} = r + s_i \frac{\partial r}{\partial s_i} - c_i'(s_i) \begin{cases} = 0, & \text{if } s_i > 0, \\ \leq 0, & \text{if } s_i = 0, \end{cases} \quad i \in I. \tag{4}$$

To describe the actions of agent i, the significance lies more in the partial derivative $\partial r / \partial s_i \equiv -\sigma_i$ than in the dependence of r on s_i. To work with nonnegative values of σ_i, we introduced a negative sign in the latter identity. We assume the coefficient σ_i (referred to as the i-th agent's "influence coefficient") to be a nonnegative constant. The relationship between the changes in the profit and production volume ξ_i conjectured by agent i, is expressed as follows:

$$\widehat{f}_i(\xi_i) = [r - \sigma_i(\xi_i - s_i)]\xi_i - c_i(\xi_i). \tag{5}$$

At $\xi_i = s_i$ the maximum necessary condition can be expressed as follows:

$$\begin{cases} r = \sigma_i s_i + c_i'(s_i), & \text{if } s_i > 0, \\ r \leq c_i'(0), & \text{if } s_i = 0, \end{cases} \quad i \in I, \tag{6}$$

which (in this case) is sufficient.

In this work, we will not assign the influence coefficients in an exogenous manner. Instead, we will introduce them as parameters of the equilibrium, analogous to the price and output volumes of the agents. Such equilibrium will be described by the vector $(r; s_1, \ldots, s_n; \sigma_1, \ldots, \sigma_n)$ and called **interior**. But before that, we will introduce an auxiliary concept of **exterior** equilibrium.

3. Conjectural variations equilibrium

Definition 1 *The vector $(r; s_1, \dots, s_n)$ is referred to as the* ***exterior equilibrium*** *for the given $\sigma_i \geq 0$, $i \in I$, when equation (1) is satisfied, and the maximum conditions (6) hold for each $i \in I$.*

We introduce a new assumption

Assumption A3 *Let $r_0 = \max\limits_{1 \leq i \leq n} \{c_i'(0)\}$ and $i \in I$. There exists a unique supply volume $s_i^0 \geq 0$ satisfying the condition $r_0 = c_i'(s_i^0)$. Moreover, the following inequality holds:*

$$\sum_{i=1}^{n} s_i^0 < D(r_0). \tag{7}$$

Lemma 1 *Let A1, A2 and A3 be valid. Then, for any selection of $\sigma_i \geq 0$, $i \in I$, the supply values s_i in the exterior equilibrium are strictly positive if and only if $r > r_0$.*

The proof of existence and uniqueness for the conjectural variations equilibrium (referred to as "exterior equilibrium") has been demonstrated in [25]. However, there are certain additional results that are needed and not covered in the aforementioned work.

Theorem 1 *Let us assume that A1, A2, and A3 are valid. There exists a unique exterior equilibrium $(r; s_1, \dots, s_n)$ for $W \geq 0$ and $\sigma_i \geq 0$, $i \in I$. This equilibrium is a continuous function of $W \geq 0$, $\sigma_i \geq 0$, $i \in I$. The equilibrium price r satisfies $r(W, \sigma_1, \dots, \sigma_n) > r_0$ and has first order derivatives with respect to W and σ_i, $i \in I$. These derivatives are given by:*

$$\frac{\partial r}{\partial W} = \frac{1}{\sum\limits_{i=1}^{n} \dfrac{1}{\sigma_i + c_i''(s_i)} - D'(r)}, \tag{8}$$

and

$$\frac{\partial r}{\partial \sigma_i} = \frac{\dfrac{s_i}{\sigma_i + c_i''(s_i)}}{\sum\limits_{k=1}^{n} \dfrac{1}{\sigma_k + c_k''(s_k)} - D'(r)} > 0, \ i \in I. \tag{9}$$

The equilibrium outputs s_i also have first order derivatives with respect to σ_k, $k \in I$, and these derivatives are:

$$\frac{\partial s_i}{\partial \sigma_i} = -\frac{s_i}{\sigma_i + c_i''(s_i)} \left[\frac{\sum\limits_{\substack{k=1 \\ k \neq i}}^{n} \dfrac{1}{\sigma_k + c_k''(s_k)} - D'(r)}{\sum\limits_{k=1}^{n} \dfrac{1}{\sigma_k + c_k''(s_k)} - D'(r)} \right] < 0, \ i \in I, \tag{10}$$

and

$$\frac{\partial s_i}{\partial \sigma_j} = \frac{1}{\sigma_i + c_i''(s_i)} \left[\frac{\dfrac{s_j}{\sigma_j + c_j''(s_j)}}{\sum\limits_{k=1}^{n} \dfrac{1}{\sigma_k + c_k''(s_k)} - D'(r)} \right] > 0, \; i, j \in I, \; j \neq i. \tag{11}$$

4. Consistent conjectural variations equilibrium

To verify the consistency of the influence coefficients, we follow the procedure outlined in [25]. Let $(r; s_1, \ldots, s_n)$ be the exterior equilibrium for given W and σ_i, $i \in I$. Suppose that agent k temporarily leaves the market and makes small changes to his equilibrium output s_k, we can consider the remaining agents in $I_{-k} := I \setminus \{k\}$ and the active demand adjusted accordingly. This change corresponds to $dW_k := d(W - s_k) = -ds_k$, where dW_k represents the change in active demand, which is equal to the change in production output by agent k but with the opposite sign.

Treating these changes as infinitesimal, we can use the theoretical results obtained earlier to evaluate the influence coefficient of agent k by considering the derivative of the equilibrium price with respect to the active demand. From formula (8), we must exclude the terms corresponding to $i = k$ from the summation. Based on this procedure, we can determine the consistency of the influence coefficients.

Definition 2 (Consistency) *The influence coefficients $\sigma_i \geq 0$, $i \in I$, of the exterior equilibrium $(r; s_1, \ldots, s_n)$ are considered **consistent** if the equalities stated below are satisfied:*

$$\sigma_i = \frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \dfrac{1}{\sigma_j + c_j''(s_j)} - D'(r)}, \; i \in I. \tag{12}$$

Building upon the concept of consistency, we introduce a new definition as follows.

Definition 3 *A vector $(r; s_1 \ldots, s_n; \sigma_1, \ldots, \sigma_n)$ is called **interior equilibrium** if it satisfies the following conditions: $(r; s_1 \ldots, s_n)$ is the exterior equilibrium and the consistency criterion holds for all σ_i.*

Theorem 2 *For an oligopoly with at least three agents, i.e., $n \geq 3$, under the validity of assumptions A1, A2, and A3, the existence of an interior equilibrium is guaranteed. In the case of two agents, i.e., $n = 2$, assuming the validity of A1, A2, and A3, along with the existence of $\varepsilon > 0$ such that $D'(r) \leq -\varepsilon$ for all $r > 0$, an interior equilibrium also exists.*

To ensure the uniqueness of the interior equilibrium, stronger assumptions need to be made regarding the cost functions of the producers.

Assumption A4 *The cost function $c_i(s_i)$ for each producer $i \in I$ can be represented by a quadratic equation:*

$$c_i(s_i) = \frac{1}{2}\alpha_i s_i^2 + \beta_i s_i, \tag{13}$$

where $\alpha_i > 0$ and $\beta_i > 0$ are the coefficients associated with producer i.

By incorporating this new assumption, we can establish the uniqueness of the interior equilibrium in the case of duopoly.

Theorem 3 *For the duopoly case ($n = 2$), assuming A1, A3, and A4, along with the condition that there exists $\varepsilon > 0$ satisfying $D'(r) \le -\varepsilon$ for $r > 0$, we can establish both the existence and uniqueness of the interior equilibrium.*

5. The consistency of the conjectures as the solution of a two-stage game

Based on the analysis of the previous sections, we can formulate the two-stage game $G = (I, S, F)$, which we will refer to as the **meta-model**. Here, $I = \{1, \ldots, n\}$ represents the set of the same agents as in the previously described oligopoly model. $S = \mathbb{R}^n_+$ denotes the set of strategies, which simultaneously represent the feasible vectors of conjectures $\sigma = (\sigma_1, \ldots, \sigma_n)$ that cam be adopted by the agents in the original oligopoly. $F = F(\sigma) = (f_1, \ldots, f_n)$ represents the vector of payoff functions for the payers, where each coordinate $f_i = f_i(\sigma)$, for $i \in I$, is calculated using the formula (3), in which the values of the variables $s_i \ge 0$, $i \in I$, and r correspond to those that appear at the exterior equilibrium state.

In the introduction, we noted that the Cournot–Nash conjecture does not satisfy the property of consistency in our proposed market model. In contrast, in the constructed meta-model mentioned in this section, the vector σ_i, $i \in I$, the coordinates of which are calculated from the solution of system (12), represents consistent conjectures for the original oligopoly. These consistent conjectures provide the Nash equilibrium for the meta-model. This intriguing result can be viewed as an additional supporting argument for this notion of consistency regarding conjectural variations. The most significant findings of this work are summarized in the following theorems.

Theorem 4 *Assuming that Assumptions A1, A2, and A3 hold, any Nash equilibrium in the meta-model $G = (I, S, F)$ corresponds to an interior equilibrium in the original oligopoly model.*

Given that the feasible set of players' strategies in the meta-model is the nonnegative orthant from $\mathbb{R}^n$, it is not compact. Consequently, the existence of a Nash equilibrium in the meta-model is not guaranteed. To ensure the existence of a Nash equilibrium, additional assumptions need to be introduced.

By imposing these additional assumptions, we have obtained three results that establish the existence of an interior equilibrium in the original oligopoly, which in turn guarantees the existence of a Nash equilibrium in the meta-model.

Theorem 5 *Assuming that Assumptions A4, A1, and A3 hold, and that the function $D = D(r)$ is concave. Under these conditions, the consistency criterion for the original oligopoly model becomes a necessary and sufficient condition for the vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ to be a Nash equilibrium in the meta-model.*

Corollary 1 *Le us assume that A1, A3, and A4 hold, along with the additional assumption that the demand function is affine in the form of*

$$D(r) = -Ar + B, \tag{14}$$

where $A > 0$ and $B > 0$, we can state the following. Under these conditions, the consistency criterion for the original oligopoly model remains a necessary and sufficient condition for the vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ to constitute a Nash equilibrium in the meta-model.

Now, we provide an alternative to the overly restrictive requirement of the demand function's concavity, which is replaced with the requirement that the derivative $D'(r)$ is Lipschitz continuous.

Theorem 6 *Let that assumptions A1, A3 and A4 be fulfilled. In addition, for $n \geq 3$ assume that the following inequality holds for any $r_1 > 0$ and $r_2 > 0$:*

$$|D'(r_1) - D'(r_2)| \leq \frac{1}{2\alpha^2 D(r_0)} |r_1 - r_2|, \tag{15}$$

where r_0 is the price introduced in Assumption A3, and $\alpha = \max\{\alpha_1, \ldots, \alpha_n\}$. For the duopoly ($n = 2$), we also assume the existence of $\varepsilon > 0$ satisfying $D'(r) \leq -\varepsilon$ for $r > 0$, and that the demand function is continuous in the sense by Lipschitz, that is,

$$|D'(r_1) - D'(r_2)| \leq \frac{2}{\left(\dfrac{\alpha_1 + \alpha_2}{\varepsilon \min\{\alpha_1, \alpha_2\}} + 3\max\{\alpha_1, \alpha_2\} \right)^2 D(r_0)} |r_1 - r_2|, \ \forall r_1, r_2 > 0. \tag{16}$$

Under these conditions, the consistency criterion for the original oligopoly model remains a necessary and sufficient condition for the vector $\sigma = (\sigma_1 \ldots \sigma_n)$ to be a Nash equilibrium in the meta-model.

6. Conclusions

The presented paper serves as a logical extension to the authors' previously published works, namely [12] and [13]. The main focus of this paper is to establish three significant results that demonstrate, under certain relatively weak conditions, the relationship between the classical Nash equilibrium in the meta-model and the consistent conjectural variations equilibrium in the original oligopoly model. Here, the meta-model consists of the same agents as in the original oligopoly, with their strategies representing the agents' influence coefficients in the original oligopoly.

These results have significant implications in several aspects. Firstly, they provide a solid justification for the concept of consistent conjectural variations equilibrium

(CCVE) and its relation to the classical Nash equilibrium. Secondly, the paper introduces an alternative approach to establish the CCVE when the conventional verification procedure is not applicable. By formulating a meta-game, where players' conjectures serve as strategies, the Nash equilibrium in the meta-model leads to the consistent equilibrium in the original model. Importantly, this approach involves solving a bilevel programming problem, linking equilibrium concepts in game theory with bilevel programming.

Future research aims to extend these ideas to other models beyond oligopoly, such as financial models, human migration models, and those related to natural gas and oil.

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