Stackelberg Games with Fuzzy Random Variables Based on Simple Recourses

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Abstract. Stackelberg games with simple recourses are formulated, in which the coefficients of equality constraints are represented as continuous type fuzzy random variables or discrete type ones. In general, these problems cannot be solved by applying the conventional methods, since simple recourses involving fuzzy random variables have not been defined. To cope with such fuzzy random variables, a possibility measure concept is applied. Then, two kinds of Stackelberg games with simple recourses are converted into usual optimization problems for some fixed degrees of permissible possibility levels specified by the first level’s player. For the transformed optimization problems, the Stackelberg solution concept of the first level’s player is introduced. By using an example, the property of the proposed Stackelberg solution concept is explained.

Keywords. Stackelberg games, simple recourses, fuzzy random variables, possibility measure

1. Introduction

To cope with the process of making choices under uncertainty, many kinds of stochastic programming techniques have been developed. Such stochastic programming techniques can be largely separated into two kinds of methods, \(i.e.,\) the two-stage programming methods [1,2,3,4,5] and the chance constraints methods [6,7,8]. The two-stage programming techniques have been adopted to many kinds of stochastic programming problems such as water resource allocation problems and power systems capacity expansion problems [9,10,11,12,13]. As an extension, Sakawa and Matsui [14] formulated multicriteria programming problems with simple recourses, which can be regarded as multicriteria expression of simple recourse programming problems. They defined the corresponding nondominated solution concept and developed an interactive decision making method to get a satisfying solution from among a nondominated solution set.

However, in general, the source of uncertain decision situations depends on not only randomness but also fuzziness. From this kind of circumstance, it seems to be natural that the uncertainty decision situation is formulated as mathematical models with both

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fuzziness and randomness simultaneously. To cope with such an uncertainty, Kwakernaak [15] first introduced the concept of fuzzy random variables. In multiobjective programming problems, Katagiri, Sakawa, and the others [16,17] defined the coefficients as fuzzy random variables. They developed interactive decision making methods by adopting chance constraint technique and possibility theory [18] to cope with fuzzy random variables. In a similar point of view, Yano and Zhang [19] formulated multiobjective programming problems, in which simple recourse is expressed as not random coefficients but fuzzy random coefficients. They defined a Pareto solution concept which depends on the degree of a possibility measure, and developed an interactive algorithm to get a satisfying solution.

In the multiobjective programming problems, it is supposed that only one player can choose his or her strategy according to his or her preference. Conversely, in Stackelberg games [20,21], it is supposed that there exist two kinds of players called the first level’s player and the second level’s player. After the first level’s player chooses a strategy at the start, the second level’s player chooses his or her strategy considering the first player’s decision. Since each of two players optimizes his or her own objective functions independently in these problems, an optimal solution concept cannot be applied. Alternatively, a concept of Stackelberg equilibrium solutions is introduced in Stackelberg games. Unfortunately, it is very difficult to get Stackelberg equilibrium solutions because two-level programming problems are NP-hard [22,23]. In order to get a Stackelberg solution, various computational methods have been developed. They are expressed as three types [21]: the vertex-enumeration method [24], the Kuhn-Tucker method [24,25], and the penalty-function method [26]. To cope with two-level stochastic programming problems, the various mathematical models based on the chance constraint methods were formulated and the methods were developed to get Stackelberg solutions by adopting the vertex-enumeration method or the Kuhn-Tucker method [7,27,28,29].

For the energy decision making problems, Dempe, Kalashnikov, Perez-Valdes, and Kalashnykova [31] and Kalashnikov, Dempe, Perez-Valdes, Kalashnykova, and Camacho-Vallejo [30] coped with fossil fuel in the gaseous state supply chain problems, and formulated as Stackelberg games with random variables, in which various types of imbalance problems with respect to payment arises. In their formulation, the first level’s player is a fossil fuel transport company who purchases the fossil fuel in the gaseous state in advance by estimating the demand of gas on the basis of actual historical information, and the second level’s player is a company who operates pipeline network, controls the operations of pipeline, and transports the fossil fuel in the gaseous state to the specified consumption regions. In the business duties of the fossil fuel in the gaseous state between the first level’s player (a fossil fuel transport company) and the second level’s player (a pipe line operating company), an imbalance often arises since the booked capacity does not coincide with the really extracted amount. In their formulations, under the assumption that the price variations for whole contract period can be predicted [31,32,33], or the price variations for whole contract period can be expressed as a discrete random variable [34], a Stackelberg equilibrium solution is obtained by minimizing such an imbalance.

Until now, although Stackelberg games with random variables based on simple recourses and Stackelberg games with random variables based on the chance constraint methods have been formulated, Stackelberg games with fuzzy random variables based on simple recourses have not been formulated. In the following sections, Stackelberg games with fuzzy random variables based on simple recourses are formulated. The equal-
ity constraints with respect to simple recourses contain the coefficients which are represented as discrete type fuzzy random variable or continuous type ones. In section 2, Stackelberg games with fuzzy random variables based on simple recourses are formulated, in which the coefficients in the equality constraints are represented as continuous type fuzzy random variables [16,17]. By adopting a possibility measure concept [18] to Stackelberg games with fuzzy random variables based on simple recourses, continuous type fuzzy random coefficients are converted into random variables. Utilizing both the two-level programming technique and the simple recourse programming technique, such Stackelberg games with fuzzy random variables based on simple recourses are reduced to nonlinear programming problems. The corresponding Stackelberg solution concept is defined. In section 3, Stackelberg games with fuzzy random variables based on simple recourses are formulated, in which the coefficients in the equality constraints are represented as discrete type fuzzy random variables. Similar to section 2, utilizing both the two-level programming and the simple recourse programming techniques, such Stackelberg games with fuzzy random variables based on simple recourses are converted into a nonlinear programming problem, and the corresponding Stackelberg solution concept is introduced. It is shown that such a Stackelberg solution can be obtained by adopting the linear programming technique. In section 4, a numerical example illustrates the correspondence between the permissible possibility levels specified by the first level’s player and the Stackelberg solution.

2. Stackelberg games with continuous type fuzzy random variables based on simple recourses

In this section, Stackelberg games with continuous type fuzzy random variables based on simple recourses are formulated, in which the coefficients in the equality constraints are represented as continuous type fuzzy random variables as follows.

Problem 1

min \( x_1 \geq 0 \) \( c_1 x_1 + c_2 x_2 \)
subject to \( a_{1i} x_1 + a_{2i} x_2 = \tilde{g}_{1i}, i = 1, \cdots, m_1 \) (1)

where \( x_2 \) solves

min \( x_2 \geq 0 \) \( d_1 x_1 + d_2 x_2 \)
subject to \( b_{1i} x_1 + b_{2i} x_2 = \tilde{g}_{2i}, i = 1, \cdots, m_2 \) (2)

subject to

\( f_1 x_1 + f_2 x_2 \leq w_{0i}, i = 1, \cdots, m_0 \) (3)

where \( x_v \) define \( (x_{v1}, \cdots, x_{vn_v})^T, v = 1, 2 \) are vectors of decision variables of the first level’s player \((v = 1)\) and the second level’s player \((v = 2)\), respectively; \( c_v \) define \( (c_{v1}, \cdots, c_{vn_v}), v = 1, 2 \), \( d_v \) define \( (d_{v1}, \cdots, d_{vn_v}), v = 1, 2 \) are vectors of the coefficients in the objective functions for the first level’s player and the second level’s player, respectively; \( a_{vi} \) define \( (a_{vi1}, \cdots, a_{vin_v}), v = 1, 2, i = 1, \cdots, m_1 \), \( b_{vi} \) define \( (b_{vi1}, \cdots, b_{vin_v}), v = 1, 2, i = 1, \cdots, m_2 \).
1, \ldots, m_2 \) are vectors of the coefficients in the constraints for the first level’s player and the second level’s player, respectively; \( \overline{g}_{v_t}, i = 1, \ldots, m_v, v = 1, 2 \) are the corresponding right-hand side coefficients defined as continuous type LR fuzzy random variables \([16, 17]\), which is a special kind of fuzzy random variables \([15]\). For some elementary event \( \theta, \overline{g}_{v_t}(\theta) \) is a realization value of a continuous type LR fuzzy random variable \( \overline{g}_{v_t} \). \( \overline{g}_{v_t}(\theta) \) is an LR fuzzy number \([18]\) whose membership function is expressed as following form.

\[
\mu_{\overline{g}_{v_t}}(s) = \begin{cases} 
L \left( \frac{\overline{g}_{v_t}(\theta) - s}{\alpha_v} \right), & s \leq \overline{g}_{v_t}(\theta) \\
R \left( \frac{s - \overline{g}_{v_t}(\theta)}{\gamma_v} \right), & s > \overline{g}_{v_t}(\theta) 
\end{cases}
\]  

(4)

where \( L(\cdot) \) is a real-valued continuous function from \([0, \infty)\) to \([0, 1]\), and \( \psi(\cdot) \) is a strictly decreasing and continuous function which satisfies the condition \( \psi(0) = 1 \). Also, \( R(\cdot) \) satisfies the same conditions, \( \alpha_v(> 0) \) and \( \gamma_v(> 0) \) are left spread parameters and right spread parameters \([18]\). The values \( \overline{g}_{v_t}, i = 1, \ldots, m_v, v = 1, 2 \) are random variables. Their probability functions and their cumulative functions are expressed as \( F_{v_t}(\cdot) \) and \( F_{v_t}(\cdot) \) respectively. It is supposed that random variables \( \overline{g}_{v_t}, i = 1, \ldots, m_v, v = 1, 2 \) are independent one another.

As a simple example of Problem1, let us consider the case where two companies (\( C_1 \) and \( C_2 \)) produce their own products, each of them has to decide amount of production (\( P_1 \) or \( P_2 \)) in advance, and they are noncooperative each other. Because of uncertainty about future demand, the demands (\( D_1 \) and \( D_2 \)) for the amount of productions of two companies (\( C_1 \) and \( C_2 \)) are set as LR fuzzy random variables. The difference between amount of production \( P_1 \) or \( P_2 \) and the corresponding demand \( D_1 \) or \( D_2 \) directly affects the cost or the profit of each company. Moreover, amount of production \( P_1 \) or \( P_2 \) of one company affects the cost or the profit of another company. Such a noncooperative game may be formulated as Problem1.

To cope with the equality constraints in (1) and (2), we define a concept of a possibility measure \([18]\) and introduce permissible possibility levels \( g_{v}, v = 1, 2; 0 < g_v \leq 1 \) specified by the first level’s player in his or her subjective manner.

\[
\text{Pos}(a_1 x_1 + a_2 x_2 = \overline{g}_{v_1}(\theta)) \geq g_{v_1}, i = 1, \ldots, m_1
\]  

(5)

\[
\text{Pos}(b_1 x_1 + b_2 x_2 = \overline{g}_{v_2}(\theta)) \geq g_{v_2}, i = 1, \ldots, m_2
\]  

(6)

The above inequality conditions can be converted into the following closed intervals.

\[
\overline{g}_{v_1}(\theta) - L^{-1}(g_{v_1}) \alpha_{v_1} \leq a_1 x_1 + a_2 x_2 \leq \overline{g}_{v_1}(\theta) + R^{-1}(g_{v_1}) \gamma_{v_1}
\]

\[
\overline{g}_{v_2}(\theta) - L^{-1}(g_{v_2}) \alpha_{v_2} \leq b_1 x_1 + b_2 x_2 \leq \overline{g}_{v_2}(\theta) + R^{-1}(g_{v_2}) \gamma_{v_2}
\]

For the above relations, we define two vectors \( y_{v} = (y_{v_1}, \ldots, y_{v_m})^T \geq 0, y_{v} = (y_{v_1}, \ldots, y_{v_m})^T \geq 0, v = 1, 2 \), where \( y_{v_1}, y_{v_2} \), \( v = 1, 2 \) represent the shortages and the excesses for the above inequalities. Let \( q_{v} = (q_{v_1}, \ldots, q_{v_m}), q_{v} = (q_{v_1}, \ldots, q_{v_m}), v = 1, 2 \) denote the penalty costs for shortages \( y_{v} \) or excesses \( y_{v} \). respectively. Then, we can formulate a Stackelberg game with fuzzy random variables based on simple recourses.

**Problem2** \((\xi_1, \xi_2)\)
Theorem 1. Let \( (x_1^*, x_2^*) \) be a Stackelberg solution to Problem2(\( \xi_1, \xi_2 \)) with weighting vectors \( q_{v_i}^+, q_{v_i}^- > 0, i = 1, \cdots, m_v, v = 1, 2 \). Let \( (y_{v_i}^{+, v}, y_{v_i}^{-, v}) \), \( v = 1, 2 \) be the corresponding shortages and excesses. Then, the complementary conditions

\[
y_{v_i}^{+, v} \cdot y_{v_i}^{-, v} = 0, \ i = 1, \cdots, m_v, v = 1, 2
\]

are satisfied.

In the following, let us suppose that \( q_{v_i}^+ > 0, q_{v_i}^- > 0, i = 1, \cdots, m_v, v = 1, 2 \). Then, the penalty functions \( R_v(x_1, x_2), v = 1, 2 \) can be converted into the following forms.
\[ R_1(x_1, x_2) \stackrel{\text{def}}{=} \sum_{i=1}^{m_1} q_{i1}^+ \left( E^g_{[g_{1i}]} - (a_{1i}x_1 + a_{2i}x_2) - L^{-1}(\xi_1)\alpha_{1i} \right) \]

\[ + \sum_{i=1}^{m_1} q_{i1}^- \left( (a_{1i}x_1 + a_{2i}x_2 + L^{-1}(\xi_1)\alpha_{1i}) \cdot F_{1i}(a_{1i}x_1 + a_{2i}x_2 + L^{-1}(\xi_1)\alpha_{1i}) \right) 
- \int_{-\infty}^{a_{1i}x_1 + a_{2i}x_2 + L^{-1}(\xi_1)\alpha_{1i}} g_{1i}f_{1i}(g_{1i})dg_{1i} \]

\[ + \sum_{i=1}^{m_1} q_{i1}^- \left( (a_{1i}x_1 + a_{2i}x_2 - R^{-1}(\xi_1)\zeta_{1i}) \cdot F_{1i}(a_{1i}x_1 + a_{2i}x_2 - R^{-1}(\xi_1)\zeta_{1i}) \right) 
- \int_{-\infty}^{a_{1i}x_1 + a_{2i}x_2 - R^{-1}(\xi_1)\zeta_{1i}} g_{1i}f_{1i}(g_{1i})dg_{1i} \]  \( (7) \)

\[ R_2(x_1, x_2) \stackrel{\text{def}}{=} \sum_{i=1}^{m_2} q_{2i}^+ \left( E^g_{[g_{2i}]} - (b_{1i}x_1 + b_{2i}x_2) - L^{-1}(\xi_2)\alpha_{2i} \right) \]

\[ + \sum_{i=1}^{m_2} q_{2i}^- \left( (b_{1i}x_1 + b_{2i}x_2 + L^{-1}(\xi_2)\alpha_{2i}) \cdot F_{2i}(b_{1i}x_1 + b_{2i}x_2 + L^{-1}(\xi_2)\alpha_{2i}) \right) 
- \int_{-\infty}^{b_{1i}x_1 + b_{2i}x_2 + L^{-1}(\xi_2)\alpha_{2i}} g_{2i}f_{2i}(g_{2i})dg_{2i} \]

\[ + \sum_{i=1}^{m_2} q_{2i}^- \left( (b_{1i}x_1 + b_{2i}x_2 - R^{-1}(\xi_2)\zeta_{2i}) \cdot F_{2i}(b_{1i}x_1 + b_{2i}x_2 - R^{-1}(\xi_2)\zeta_{2i}) \right) 
- \int_{-\infty}^{b_{1i}x_1 + b_{2i}x_2 - R^{-1}(\xi_2)\zeta_{2i}} g_{2i}f_{2i}(g_{2i})dg_{2i} \]  \( (8) \)

Using penalty functions \( R_v(\cdot), v = 1, 2 \), Problem2(\( \xi_1, \xi_2 \)) can be equivalently converted into the following simple form.

**Problem3(\( \xi_1, \xi_2 \))**

\[ \min_{x_1 \geq 0} z_1(x_1, x_2) \stackrel{\text{def}}{=} (c_1x_1 + c_2x_2) + R_1(x_1, x_2) \]

where \( x_2 \) solves

\[ \min_{x_2 \geq 0} z_2(x_1, x_2) \stackrel{\text{def}}{=} (d_1x_1 + d_2x_2) + R_2(x_1, x_2) \]

subject to \( f_{1i}x_1 + f_{2i}x_2 \leq w_0, \ i = 1, \cdots, m_0 \)

For the fixed values \( x_1^* \geq 0 \), the optimization problem for the second level’s player can be formulated as follows.

**Problem4(\( \xi_1, \xi_2; x_1^* \))**
\[
\min_{x_2 \geq 0} z_2(x_1, x_2) \overset{\text{def}}{=} d_1 x_1^* + d_2 x_2 + R_2(x_1, x_2)
\]
subject to \(f_1 x_1^* + f_2 x_2 \leq w_0, \ i = 1, \ldots, m_0 \) \tag{9}

Using the Lagrange function for Problem4(\(\xi_1, \xi_2; x_1^*\)), the Kuhn-Tucker conditions to Problem4(\(\xi_1, \xi_2; x_1^*\)) can be formulated as the following form.

\[
d_{2j} - \sum_{i=1}^{m_2} q_{2}^{j} b_{2i} + \sum_{i=1}^{m_2} q_{2}^{j} b_{2i} \cdot f_{2i}(b_1 x_1^* + b_2 x_2 + L^{-1}(\xi_2) a_2) + \sum_{i=1}^{m_2} q_{2}^{j} b_{2i} \\
\cdot f_{2i}(b_1 x_1^* + b_2 x_2 - R^{-1}(\xi_2) a_2) + \sum_{i=1}^{m_0} \tau_i e_{2i} - \xi_{2j} = 0, \ j = 1, \ldots, n_2
\]

\[
f_1 x_1^* + f_2 x_2 \leq w_0, \ i = 1, \ldots, m_0 \\
\tau_i (f_1 x_1^* + f_2 x_2 - w_0) = 0, \ i = 1, \ldots, m_0 \\
\xi_{2j} \cdot x_{2j} = 0, \ j = 1, \ldots, n_2, \\
x_2 \geq 0, \ \tau_i \geq 0, \ i = 1, \ldots, m_0, \ \xi_{2j} \geq 0, \ j = 1, \ldots, n_2
\]

where \(\tau_i \geq 0, \ i = 1, \ldots, m_0, \ \xi_{2j} \geq 0, \ j = 1, \ldots, n_2\) are Lagrange multipliers for the constraints (9) and the nonnegative conditions \(x_2 \geq 0\).

Conversely, the partial differentiation of \(z_2(x_1^*, x_2)\) for \(x_{2j}\) and \(x_{2k}\) can be calculated as:

\[
\frac{\partial z_2(x_1^*, x_2)}{\partial x_{2j} \partial x_{2k}} = \sum_{i=1}^{m_2} q_{2}^{j} b_{2i} b_{2k} \cdot f_{2i}(b_1 x_1^* + b_2 x_2 + L^{-1}(\xi_2) a_2) \\
+ \sum_{i=1}^{m_2} q_{2}^{j} b_{2i} b_{2k} \cdot f_{2i}(b_1 x_1^* + b_2 x_2 - R^{-1}(\xi_2) a_2), j, k = 1, \ldots, n_2.
\]

The Hessian matrix for \(z_2(x_1^*, x_2)\) can be written as:

\[
\nabla^2 z_2(x_1^*, x_2) = \left( \sum_{i=1}^{m_2} q_{2}^{j} f_{2i}(b_1 x_1^* + b_2 x_2 + L^{-1}(\xi_2) a_2) \\
+ \sum_{i=1}^{m_2} q_{2}^{j} f_{2i}(b_1 x_1^* + b_2 x_2 - R^{-1}(\xi_2) a_2) \right) \cdot B_i,
\]

where \(B_i, \ i = 1, \ldots, m_2\) are \((n_2 \times n_2)\)-dimensional matrices defined as follows.

\[
B_i = \begin{pmatrix}
    b_{2i}^2 & \cdots & b_{2i} b_{2in_2} \\
    \vdots & \ddots & \vdots \\
    b_{2in_2} b_{2i} & \cdots & b_{2in_2}^2
\end{pmatrix}, \ i = 1, \ldots, m_2 \tag{10}
\]

Since the matrices \(B_i, \ i = 1, \ldots, m_2\) are positive semidefinite, \(f_{2i}(\cdot)\), \(i = 1, \ldots, m_2 \geq 0\), and \(q_{2i}^+ > 0, q_{2i}^- > 0, \ i = 1, \ldots, m_2, \ \nu = 1, 2\), Problem4(\(\xi_1, \xi_2; x_1^*\)) is a convex programming problem. Therefore, using the Kuhn-Tucker conditions, Problem3(\(\xi_1, \xi_2\)) can be equivalently transformed as follows.
Problem 5 ($\xi_1, \xi_2$)

\[
\min \ (c_1 x_1 + c_2 x_2) + \sum_{i=1}^{m_1} q_{ii}^{-1} \left( E[g_{ii}] - (a_{ii} x_1 + a_{2i} x_2) - L^{-1}(\xi_i) \alpha_{ii} \right) \\
+ \sum_{i=1}^{m_1} q_{ii}^{-1} \left\{ (a_{1i} x_1 + a_{2i} x_2 + L^{-1}(\xi_i) \alpha_{ii}) \cdot F_{1i}(a_{1i} x_1 + a_{2i} x_2 + L^{-1}(\xi_i) \alpha_{ii}) \right\} \\
\left. \int_{-\infty}^{a_{1i} x_1 + a_{2i} x_2 + L^{-1}(\xi_i) \alpha_{ii}} g_{1i} f_{1i}(g_{1i}) d g_{1i} \right\} \\
+ \sum_{i=1}^{m_1} q_{ii}^{-1} \left\{ (a_{1i} x_1 + a_{2i} x_2 - R^{-1}(\xi_i) \nu_{ii}) \cdot F_{1i}(a_{1i} x_1 + a_{2i} x_2 - R^{-1}(\xi_i) \nu_{ii}) \right\} \\
\left. \int_{-\infty}^{a_{1i} x_1 + a_{2i} x_2 - R^{-1}(\xi_i) \nu_{ii}} g_{1i} f_{1i}(g_{1i}) d g_{1i} \right\}
\]

subject to

\[
d_{2j} - \sum_{i=1}^{m_2} q_{2i}^{-1} b_{2ij} + \sum_{i=1}^{m_2} \frac{q_{2i}^{-1} b_{2ij}}{m_2} \cdot F_{2i}(b_{1i} x_1 + b_{2i} x_2 + L^{-1}(\xi_i) \alpha_{2i}) + \sum_{i=1}^{m_2} q_{2i}^{-1} b_{2ij} \cdot F_{2i}(b_{1i} x_1 + b_{2i} x_2 + L^{-1}(\xi_i) \alpha_{2i}) + \sum_{i=1}^{m_2} q_{2i}^{-1} b_{2ij} \\
\cdot \tau_i (f_{1i} x_1 + f_{2i} x_2 - \nu) = 0, \quad i = 1, \ldots, m_0 \\
\zeta_{2j} - \varphi_{2j} = 0, \quad j = 1, \ldots, n_2 \\
x_1 \geq 0, \quad x_2 \geq 0, \quad \tau_i \geq 0, \quad i = 1, \ldots, m_0 \\
\zeta_{2j} \geq 0, \quad j = 1, \ldots, n_2
\]

3. Stackelberg games with discrete type fuzzy random variables based on simple recourses

We cope with a special case of Problem 1, in which the $\tilde{p}_{ij}, i = 1, \ldots, m_v, v = 1, 2$ are discrete type LR fuzzy random variable coefficients. In discrete type LR fuzzy random variables $\tilde{p}_{ij}, i = 1, \ldots, m_v, v = 1, 2$ (see (4)), the mean values $\bar{g}_{vi}, i = 1, \ldots, m_v, v = 1, 2$ are defined by the following discrete random variables:

\[
(g_{1i1}, p_{1i1}), \quad i = 1, \ldots, m_1, \quad s_{1i} = 1, \ldots, L_{1i} \tag{11}
\]

\[
(g_{2i2}, p_{2i2}), \quad i = 1, \ldots, m_2, \quad s_{2i} = 1, \ldots, L_{2i} \tag{12}
\]

where $g_{1i1}, g_{2i2}$ are realized values for scenarios $s_{1i} = 1, \ldots, L_{1i}, s_{2i} = 1, \ldots, L_{2i}$ which occur with the probabilities $p_{1i1}, p_{2i2} \geq 0$, respectively, where $\sum_{s_{1i}=1}^{L_{1i}} p_{1i1} = 1, i = 1, \ldots, m_1, \sum_{s_{2i}=1}^{L_{2i}} p_{2i2} = 1, i = 1, \ldots, m_2.$
Let \( y_i^+, y_i^- \geq 0 \) and \( y_i^{vs} \geq 0 \) are the shortage’s realized values and the excess’s realized values for each scenario \( s_{vi}, v = 1, 2 \). Because of the definition of the expected value, the penalty functions for the objective functions \( z_v(x_1, x_2), v = 1, 2 \) can be expressed like these.

\[
\begin{align*}
R_1(x_1, x_2) &\overset{\text{def}}{=} \min_{y_i^+, y_i^-} \left( \sum_{i=1}^{m_1} \sum_{t_i=1}^{L_{1i}} p_1^{i_t} y_{i_t}^{+} + \sum_{i=1}^{m_1} \sum_{t_i=1}^{L_{1i}} p_1^{i_t} y_{i_t}^{-} \right) \\
\text{subject to} & \\
& a_1 x_1 + a_2 x_2 + y_i^{+} \geq g_1 s_{vi} - L^{-1}(\xi_1) \alpha_{1i}, \ i = 1, \ldots, m_1, s_{1i} = 1, \ldots, L_{1i} \\
& a_1 x_1 + a_2 x_2 - y_i^{-} \geq L^{-1}(\xi_1) \xi_{1i}, \ i = 1, \ldots, m_1, s_{1i} = 1, \ldots, L_{1i} \\
R_2(x_1, x_2) &\overset{\text{def}}{=} \min_{y_2^+, y_2^-} \left( \sum_{i=1}^{m_2} \sum_{t_2i=1}^{L_{2i}} p_2^{t_2} y_{2t_2}^{+} + \sum_{i=1}^{m_2} \sum_{t_2i=1}^{L_{2i}} p_2^{t_2} y_{2t_2}^{-} \right) \\
\text{subject to} & \\
& b_1 x_1 + b_2 x_2 + y_{2t_2}^{+} \geq g_2 s_{vi} - L^{-1}(\xi_2) \alpha_{2i}, \ i = 1, \ldots, m_2, s_{2i} = 1, \ldots, L_{2i} \\
& b_1 x_1 + b_2 x_2 - y_{2t_2}^{-} \geq g_2 s_{vi} + R^{-1}(\xi_2) \xi_{2i}, \ i = 1, \ldots, m_2, s_{2i} = 1, \ldots, L_{2i}
\end{align*}
\]

where \( y_v^+ = (y_{v1}^+, \ldots, y_{vm_v}^+) \) and \( y_v^- = (y_{v1}^-, \ldots, y_{vm_v}^-) \). This theorem shows the correspondence between Stackelberg solution to Problem3(\( \xi_1, \xi_2 \)) and \( y_i^+, y_i^- \) and \( y_2^+, y_2^- \).

**Theorem 2.** Suppose that \( q_{vi}^+ + q_{vi}^- > 0, i = 1, \ldots, m_v, v = 1, 2 \). Let \( (x_1^*, x_2^*) \) be a Stackelberg solution to Problem3(\( \xi_1, \xi_2 \)) and \( y_1^+, y_1^-, y_2^+, y_2^- \) be the shortages and the excesses. Then, the complementary conditions \( y_i^{vs}, v_i = 0, i = 1, \ldots, m_v, s_{vi} = 1, \ldots, L_{vi}, v = 1, 2 \) are satisfied.

For the fixed values \( x_1^* \geq 0 \), the optimization problem of the second level’s player can be converted into the following form.

**Problem6(\( \xi_1, \xi_2, x_1^* \))**

\[
\begin{align*}
\min_{x_2 \geq 0} z_2(x_1^*, x_2, y_2^+, y_2^-) &\overset{\text{def}}{=} (d_1 x_1^* + d_2 x_2) \\
&+ \left( \sum_{i=1}^{m_2} \sum_{t_2i=1}^{L_{2i}} p_2^{t_2} y_{2t_2}^{+} + \sum_{i=1}^{m_2} \sum_{t_2i=1}^{L_{2i}} p_2^{t_2} y_{2t_2}^{-} \right) \\
\text{subject to} & \\
& b_1 x_1^* + b_2 x_2 + y_{2t_2}^{+} \geq g_2 s_{vi} - L^{-1}(\xi_2) \alpha_{2i}, \ i = 1, \ldots, m_2, s_{2i} = 1, \ldots, L_{2i} \\
& b_1 x_1^* + b_2 x_2 - y_{2t_2}^{-} \geq g_2 s_{vi} + R^{-1}(\xi_2) \xi_{2i}, \ i = 1, \ldots, m_2, s_{2i} = 1, \ldots, L_{2i} \\
& f_1 x_1^* + f_2 x_2 \leq w_0, \ i = 1, \ldots, m_0
\end{align*}
\]

Using the Lagrange function for Problem6(\( \xi_1, \xi_2, x_1^* \)), the Kuhn-Tucker conditions to Problem6(\( \xi_1, \xi_2, x_1^* \)) can be formulated as the following constraints.
\begin{align*}
d_j &= \sum_{i=1}^{m_1} \sum_{s_2j=1}^{L_2j} \tau_{2is_2j}^+ b_{2ij} + \sum_{i=1}^{m_2} \sum_{s_2j=1}^{L_2j} \tau_{2is_2j}^- b_{2ij} + \sum_{i=1}^{m_0} \tau_{0j} - \zeta_{0j} = 0, \quad j = 1, \cdots, n_2 \\
q_{2i}^+ & \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
q_{2i}^- & \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
b_1 x_1^+ + b_2 x_2 \geq \gamma_{2is_2j}^+ \geq g_2 x_2 - L^{-1}(\xi_2) \alpha_{2i}, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
b_1 x_1^+ + b_2 x_2 - \gamma_{2is_2j}^- \leq g_2 x_2 + R^{-1}(\xi_2) \zeta_{2i}, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
f_1 x_1^+ + f_2 x_2 \leq w_0, \quad i = 1, \cdots, m_0 \\
\tau_{2is_2j}^+ \left( g_2 x_2 - L^{-1}(\xi_2) \alpha_{2i} - (b_1 x_1^+ + b_2 x_2 + \gamma_{2is_2j}^+) \right) = 0, \quad i = 1, \cdots, m_2, \\
s_2j = 1, \cdots, L_2j \\
\tau_{2is_2j}^- \left( (b_1 x_1^+ + b_2 x_2 - \gamma_{2is_2j}^-) - g_2 x_2 - R^{-1}(\xi_2) \zeta_{2i} \right) = 0, \quad i = 1, \cdots, m_2, \\
s_2j = 1, \cdots, L_2j \\
\tau_{0} (f_1 x_1^+ + f_2 x_2 - w_0) = 0, \quad i = 1, \cdots, m_0 \\
\zeta_{0j}, x_2 = 0, \quad j = 1, \cdots, n_2 \\
\gamma_{2is_2j}^+, \gamma_{2is_2j}^- = 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
\zeta_{2is_2j}^+, \zeta_{2is_2j}^- = 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
x_2 \geq 0, \quad \zeta_{0j} \geq 0, \quad j = 1, \cdots, n_2, \\
y_{2is_2j}^+, y_{2is_2j}^- \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
\zeta_{2is_2j}^+, \zeta_{2is_2j}^- \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
\tau_{2is_2j}^+, \tau_{2is_2j}^- \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j \\
\tau_{0j} \geq 0, \quad i = 1, \cdots, m_2 \\
\end{align*}

where \( \zeta_{0j} \geq 0, \quad j = 1, \cdots, n_2, \quad \zeta_{2is_2j}^+, \zeta_{2is_2j}^- \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j, \quad \tau_{2is_2j}^+, \tau_{2is_2j}^- \geq 0, \quad i = 1, \cdots, m_2, \quad s_2j = 1, \cdots, L_2j, \quad \tau_{0j} \geq 0, \quad i = 1, \cdots, m_0 \) are simple multipliers for the nonnegative conditions \( x_2 \geq 0, \quad y_z^+ \geq 0, \quad y_z^- \geq 0 \) and the constraints (13), (14) and (15).

Since the Kuhn-Tucker conditions mean the necessary and sufficient conditions for the optimal solution of Problem6(\( \xi_1, \xi_2, x_1^+ \)), Problem3(\( \xi_1, \xi_2 \)) can be equivalently converted into the following optimization problem.

**Problem7** \((\xi_1, \xi_2)\)

\[
\min \left( c_1 x_1 + c_2 x_2 \right) + \left( \sum_{i=1}^{m_1} q_{1i}^+ \sum_{s_1i=1}^{L_1i} p_{1is_1i}^+ y_{1is_1i}^+ + \sum_{i=1}^{m_1} q_{1i}^- \sum_{s_1i=1}^{L_1i} p_{1is_1i}^- y_{1is_1i}^- \right)
\]
subject to
\[ a_1 x_1 + a_2 x_2 + y_{1st2}^+ \geq g_{1st2} - L^{-1}(\xi_1)\alpha_1, \quad i = 1, \ldots, m_1, s_1 = 1, \ldots, L_1 \]
\[ a_1 x_1 + a_2 x_2 - y_{1st2}^- \leq g_{1st2} + R^{-1}(\xi_1)\xi_1, \quad i = 1, \ldots, m_1, s_1 = 1, \ldots, L_1 \]
\[ b_1 x_1 + b_2 x_2 + y_{2st2}^+ \geq g_{2st2} - L^{-1}(\xi_2)\alpha_2, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ b_1 x_1 + b_2 x_2 - y_{2st2}^- \leq g_{2st2} + R^{-1}(\xi_2)\xi_2, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ d_{2j} = \sum_{i=1}^{m_2} \sum_{s_2=1}^{L_2} \tau_{2st2}^+ b_{2ij} + \sum_{i=1}^{m_1} \sum_{s_1=1}^{L_1} \tau_{2st2}^- b_{2ij} + \sum_{i=0}^{m_0} \tau_{0e2ij} - \xi_0 = 0, \quad j = 1, \ldots, n_2 \]
\[ q_{2i}^+ p_{2st2}^+ - \tau_{2st2}^+ - \xi_{2st2}^+ = 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ q_{2i}^- p_{2st2}^- - \tau_{2st2}^- - \xi_{2st2}^- = 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ f_1 x_1 + f_2 x_2 \leq w_0, \quad i = 1, \ldots, m_0 \]
\[ \tau_{2st2}^+ (g_{2st2} - L^{-1}(\xi_2)\alpha_2 - (b_1 x_1 + b_2 x_2 + y_{2st2}^+)) = 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ \tau_{2st2}^- (b_1 x_1 + b_2 x_2 - y_{2st2}^- - g_{2st2} - R^{-1}(\xi_2)\xi_2) = 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ \xi_0 x_{2j} = 0, \quad j = 1, \ldots, n_2 \]
\[ \xi_{2st2}^+ y_{2st2}^+ = 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ \xi_{2st2}^- y_{2st2}^- = 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ x_1 \geq 0, \quad x_2 \geq 0, \quad \xi_0 \geq 0, \quad j = 1, \ldots, n_2, \]
\[ y_{2st2}^+ \geq 0, \quad y_{2st2}^- \geq 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ \xi_{2st2}^+ \geq 0, \quad \xi_{2st2}^- \geq 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ \tau_{2st2}^+ \geq 0, \quad \tau_{2st2}^- \geq 0, \quad i = 1, \ldots, m_2, s_2 = 1, \ldots, L_2 \]
\[ \xi_0 \geq 0, \quad i = 1, \ldots, m_2 \]

By eliminating the complementary conditions (16)-(21), Problem7(\(\xi_1, \xi_2\)) can be solved by adopting linear programming technique.

4. A numerical example

Consider the following Stackelberg fuzzy random game with simple recourses.

Problem8
According to (7) and (8), the penalty functions \( R_v(x_1, x_2) \), \( v = 1, 2 \) can be transformed as follows.

\[
R_v(x_1, x_2) = \frac{1}{2} (x_1 + L^{-1}(\xi_1) \alpha_{11})^2 + \frac{4}{5} (x_1 - R^{-1}(\xi_1) \xi_{11})^2 + \frac{7}{10} (x_2 + L^{-1}(\xi_1) \alpha_{11})^2 + \frac{6}{5} (x_2 - R^{-1}(\xi_1) \xi_{12})^2
\]
Table 1. Three kinds of Stackelberg solutions to P14($\xi_1, \xi_2$)

<table>
<thead>
<tr>
<th>$(\xi_1, \xi_2)$</th>
<th>$z_1(x^<em>_1,x^</em>_2)$</th>
<th>$z_2(x^<em>_1,x^</em>_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0,0.0)</td>
<td>-10.1018</td>
<td>21.8442</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>-6.90783</td>
<td>22.0262</td>
</tr>
<tr>
<td>(1.0,1.0)</td>
<td>-3.31408</td>
<td>22.2394</td>
</tr>
</tbody>
</table>

\[-5x_1 - 7x_2 + 30 - 5L^{-1}(\xi_1)\alpha_{11} - 7L^{-1}(\xi_1)\alpha_{12} \]
\[R_2(x_1,x_2) = \frac{1}{8}(x_1 + 2x_2 + L^{-1}(\xi_2)\alpha_{21})^2 + \frac{1}{8}(x_1 + 2x_2 - R^{-1}(\xi_2)\zeta_{21})^2 \]
\[-3(x_1 + 2x_2 + L^{-1}(\xi_2)\alpha_{21}) - 2(x_1 + 2x_2 - R^{-1}(\xi_2)\zeta_{21}) + 26 \]

Therefore, using the Kuhn-Tucker conditions, Problem9($\xi_1, \xi_2$) can be equivalently converted into the following form.

**Problem10($\xi_1, \xi_2$)**

\[
\min -3x_1 - 8x_2 + R_1(x_1,x_2)
\]
subject to
\[x_1 + 2x_2 - 6 + 6\tau_1 + 2\tau_2 + \tau_3 - \zeta_{21} + 0.5L^{-1}(\xi_2)\alpha_{21} - 0.5R^{-1}(\xi_2)\zeta_{21} = 0 \]
\[2x_1 + 6x_2 \leq 27, 3x_1 + 2x_2 \leq 16, 4x_1 + 1x_2 \leq 18 \]
\[\tau_1(2x_1 + 6x_2 - 27) = 0 \quad (22) \]
\[\tau_2(3x_1 + 2x_2 - 16) = 0 \quad (23) \]
\[\tau_3(4x_1 + 1x_2 - 18) = 0 \quad (24) \]
\[\zeta_{21}(-x_2) = 0 \quad (25) \]
\[x_1, x_2 \geq 0, \tau_1, \tau_2, \tau_3 \geq 0, \zeta_{21} \geq 0 \]

We can directly get Stackelberg solutions to Problem3($\xi_1, \xi_2$) by solving convex programming problems $2^4 (= 16)$ times corresponding to the number of combination for the complementarity conditions (22)-(25). Table 1 shows three kinds of Stackelberg solutions corresponding to permissible possibility levels ($\xi_1, \xi_2$). It is clear that two objective functions $z_v(x^*_1,x^*_2)$, $v = 1,2$ are improved simultaneously by the less values of permissible possibility levels ($\xi_1, \xi_2$).

5. Conclusion

Until now, although Stackelberg games based on simple recourses and Stackelberg games based on chance constraint method have been formulated, Stackelberg games with fuzzy random variables based on simple recourses have not been formulated. From this point of view, both Stackelberg games with continuous type fuzzy random variables based on simple recourses and Stackelberg games with discrete type fuzzy random variables based
on simple recourses have been formulated. By adopting a possibility measure and the simple recourse programming technique, the original problem is converted into the traditional two-level programming problem, and a Stackelberg solution concept is defined, which depends on the permissible possibility levels.

In Stackelberg games with continuous type fuzzy random variables based on simple recourses, a nonlinear programming problem is reduced to the set of convex programming problems to get the corresponding Stackelberg solution. Similarly, in Stackelberg games with discrete type fuzzy random variables based on simple recourses, a nonlinear programming problem is reduced to the set of linear programming problems to get the corresponding Stackelberg solution.

References


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