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Foundational Ontologies for Units of Measure

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Abstract. Multiple ontologies for units of measure have been proposed within the Applied Ontology community, and all of these ontologies introduce an array of new classes based on supposed distinctions between quantities, quantity kinds, and measures. Units are combined using notions of dimensional analysis that often conflate the combination of units with algebraic operations on real numbers. In this paper we present an alternative approach that shifts the focus to the connection between the units of measure and the physical objects and processes that are being measured. One of the key features of this approach is that it makes minimal ontological commitments with respect to the TUpperWare upper ontology – the only new classes that are introduced are the classes for the units of measure. We propose correct and complete axiomatizations for combining units of measure, and the correct axiomatization of the relationship between the units of measure and the existing upper ontology.

Keywords. foundational ontology, units of measure, TUpperWare

1. Introduction

Given the widespread use units of measure in scientific and engineering applications, it is perhaps not surprising that numerous ontologies for units of measure already exist. The first, EngMath, dates back to the dawn of applied ontology, and since then several more ontologies have been proposed for use on the Semantic Web. Why do we need yet another ontology for units of measure? Consider the purpose of any such ontology – measuring physical objects and the processes that act upon such objects. This connection to physical objects indicates that any proper treatment of units of measures requires ontologies not only for the units of measure themselves, but also for the relationship to the physical objects that they are measuring.

All current ontologies for units of measure that are being used in the Semantic Web are specified in OWL, which lacks the expressiveness to axiomatize the intended semantics of both the units of measure as well as the things being measured. There has been an over-emphasis on producing some kind of taxonomy of quantities, rather than on axiomatizing the intended semantics of the quantities themselves. As an artefact of the representation in OWL, additional classes of entities are posited that are not necessary for the specification of the intended semantics of the units of measure.

Current ontologies for units of measure are cluttered with superfluous classes of entities that do not have sufficient ontological motivation, but rather seem to be artifacts of the style of ontology design in a particular representation language. In contrast, the ontologies introduced in this paper contain only two categories of entities – units of measure and the things being measured. All other concepts in the ontologies are represented as relations between entities in these two categories.

FOUnt (Foundational Ontologies for Units) includes a modular ontology for each unit of measure that includes an axiomatization addressing the following questions:

- How are the units of measure manipulated / combined?
- What is the nature of the entity being measured?
- What is the physical interpretation of the unit of measure, that is, what is the relationship between units of measure and the entity being measured?

For the first question, we introduce new ontologies that explicitly axiomatize how units can be added, subtracted, and multiplied. With the exception of EngMath [3], ontologies for units of measure that extend upper ontologies do not contain explicit axioms for specifying how units may be combined and manipulated. For the second, we utilize existing generic ontologies for time, mereotopology (space), location, and constitution to axiomatize concepts related to abstract and physical objects. All generic ontologies used in this paper are modules with the TUpperWare upper ontology [6]. One can view this work as being similar to the alignment of units of measure ontologies with an upper ontology [15]; however, it is more accurate that we begin with an upper ontology and then create an ontology for units of measure that is explicitly based upon the upper ontology.

We do not give an account of all units of measure in this paper, but rather we focus on the basic units of measure for duration (second), mass (kilogram), and spatial entities (length, area, volume), together with the derived units for density and volume.

2. Duration

Our approach to an ontology for units of duration has been motivated by considering the treatment of duration ontologies in earlier work within the knowledge representation community. In this section, we review this work, and see how it actually gives rise to a set of ontologies that can ultimately be used to serve as the axiomatization of the SI unit s of seconds.

The notion of duration presumes an underlying ontology of time. Ontologies of time have been thoroughly studied in the literature, both from a model-theoretic [21] and axiomatic [9] perspective. [5] uses the ontology whose models are a linear ordering over timepoints[1]. It is important to note that his axiomatization does not impose any additional assumptions about density or discreteness, since any axiomatization of duration should be independent of such conditions.

Following [9], we use the class **timeduration** for the class of entities that are the values for duration. The fundamental insight of [5] is that timedurations do not form a field,

[1]The axioms for T_{linear_time} in the Common Logic Interchange Format (CLIF) can be found at http://colore.oor.net/timepoints/linear_time.clif

such as the reals or rationals, as many approaches ([10, 17]) have assumed. Although we do want to be able to add durations together, the product of two timedurations is not a timeduration; thus, multiplication is not a function on the set of timedurations, and the underlying structure for timedurations cannot be a field. Nevertheless, we do want to be able to specify scalar multiples of timedurations (e.g. one task takes twice as long as another). If such scalars are elements of a field, then the intended models for timedurations must be vector spaces. We also want to define an ordering over timedurations (e.g. a week is longer than a day but shorter than a month). This leads to the claim that class of intended structures for timedurations is that of ordered vector spaces.

Timedurations alone are not sufficient for a duration ontology; we also need a function that assigns timedurations to time intervals or pairs of timepoints. Since earlier approaches have mistakenly assumed that timedurations formed a field, they have treated this function as metric [17]. Given that timedurations actually form a vector space, the duration function is no longer a metric, and we must find a suitable class of vector-valued functions to adequately capture the intended models. In particular, the mapping δ from $\mathbb{T} \times \mathbb{T}$ (i.e. pairs of timepoints) to the vector space $\mathbb{D}$, generalizes the notion of vector field from differential topology to the notion of a product order vector field, in which we associate a unique vector to each pair of elements in a linear ordering.

The axioms of $T_{duration}$ can be divided into two subtheories – those that axiomatize timedurations as elements of ordered vector spaces, and those that capture the formalization of the duration function as a product order vector field.

$T_{duration}$ does not mention specific constants that denote particular timedurations such as *second*, *hour*, *day*, or *year*. The axioms for such constants are not contained in the Duration Ontology, but rather are specified in a domain theory that extends $T_{duration}$, just as linear equations are domain theories for general vector spaces. Thus, equations such as $hour = mult(60, second)$, $day = mult(24, hour)$ define specific timedurations as the linear combinations of other timedurations. This allows for different time-keeping systems to share the same Duration Ontology.

Finally, there is the question of which entities have duration. Following the PSL Ontology [16], every object and every activity occurrence is associated with two timepoints, namely, the $beginof$ and $endof$ the object or occurrence. Thus, we can associate a timeduration with each object and each activity occurrence, and it is only the entities in these two classes that have a duration.

We can see from this discussion that six ontologies are required to fully axiomatize the notion of duration in the context of the “second” as an SI unit of measure. $T_{timeduration}$ serves as a quantity ontology that axiomatizes how timedurations can be combined. The time ontology T_{linear_time} is a generic object ontology that axiomatizes the underlying structure of timepoints, while $T_{duration}$ plays of role of a measure ontology that formalizes the relationship between timepoints and timedurations. $T_{pslcore}$ axiomatizes the classes of entities that have duration, and thus is the physical object ontology for duration, while $T_{psl\_duration}$ contains the explicit definition of the relation between an object or activity occurrence and its duration.

3. Methodology

Following the approach taken with the axiomatization of duration, FOUnt is a set of ontologies, in which each unit of measure has its own modular ontology that includes axioms for specifying how the unit is manipulated and combined, as well as an axiomatization of the unit's physical interpretation (the nature of the thing being measured). For each unit of measure, we therefore introduce a set of six ontologies:

1. The *quantity ontologies* axiomatize how units of measure can be combined and how they are related to other units of measure.
2. The *unit of measure ontologies* axiomatize specific units of measure.
3. The *generic object ontologies* axiomatize the generic ontologies that underly the quantity ontologies.
4. The *measure ontologies* introduce functions that map generic objects to their corresponding units of measure.
5. The *physical object ontologies* axiomatize the different kinds of physical objects that are measured.
6. The *physical measure ontologies* axiomatize the relationship between the measure ontologies and the physical object ontologies, so that one can speak about the units of measure for a class of physical objects.

Thus, notions such as quantity kinds, units, and dimensions are classified at the metalevel (as sets of axioms) rather than within a taxonomy in the ontology. The focus is on physical objects and the units that are used to measure them, rather than a focus on the units of measure themselves.

The next two sections apply this approach to units of mass and spatial measurement units. For mass, we detail the six ontologies, whereas for spatial units we primarily discuss suitable object ontologies that spatial measurements can be based upon.

4. Mass

In this section, we present the six ontologies that axiomatize this unit and the intended semantics the SI unit for mass is the kilogram (kg). We introduce the class **amount** for the quantity related to mass and for which the kilogram is an instance. As we saw with timedurations, amounts can be added or multiplied by an element of a field to get another amount, but multiplying amounts together does not give us an amount of matter; the theory T_{amount}[2] is therefore synonymous with the theory ordered vector spaces[3]. Finally, we present the two ontologies that axiomatize the physical interpretation – material objects are constituted by matter, and the mass of of a material object is equal to the mass of the matter that constitutes the object.

4.1. The Mass of Chunks of Matter

The underlying generic object ontology to represent the entities that have mass is T_{matter}[4], which axiomatizes a mereology for chunks of matter [19]. Approaches based on classical mereology [20] use a single parthood relation to specify parthood relationships, an approach known as mereological monism. This is not a viable approach if we are to support ontologies for units of measure related to notions as diverse as mass, length, area, and

[2] http://colore.oor.net/mass/amount.clif
[3] http://colore.oor.net/ordered_algebra/ordered_vectorspace.clif
[4] http://colore.oor.net/matter/matter.clif

volume (each of which will have a distinct mereology). We therefore adopt mereological pluralism [19], which is based on the idea that there are indeed multiple distinct parthood relations for different classes of entities.

The axioms in T_{matter} are tightly constrained by the requirements on the measure ontology T_{mass}[5]. We might at first think that the sum of two chunks of matter corresponds to adding the amounts that are the masses of the two chunks. However, if the two chunks overlap, then the mass of their sum should intuitively be less than adding the masses of the chunks. On the other hand, if the chunks are disjoint, then we should have equality of the sum of the masses, which is equivalent to saying that removing a chunk from some matter decreases the mass of the matter by the amount of the mass of the chunk that is removed. This requires that any chunkOf matter has a unique complement (corresponding to the chunkOf matter that can be removed), and hence the difference of two chunks can be defined (via the $chunk\_diff$ function). As a result, the mereology for chunks of matter must correspond to a boolean lattice.

4.1.1. Axiomatization of Mass

The measure ontology T_{mass} for amounts of matter introduces the function *mass* (see Figure 1) that maps chunks of matter to amounts. All chunks of matter have nonzero mass. The mass of some matter is the sum of the mass of a chunk of the matter and the mass of its difference. The last axiom of T_{mass} states that any chunk of matter contains two smaller chunks of equal mass. A consequence of this axiom is that all atoms in the mereology have equal mass:

$$(\forall x,y,z)\, proper\_chunk(y,x) \wedge proper\_chunk(z,x) \wedge$$
$$atomic\_chunk(y) \wedge atomic\_chunk(z) \supset mass(y) = mass(z)$$

and in finite mereologies these sentences are logically equivalent.

$$(\forall x)\, mat(x) \equiv amount(mass(x)) \tag{1}$$
$$(\forall x)\, mat(x) \supset lesser(zero, mass(x)) \tag{2}$$
$$(\forall x,y,z)\, chunk\_diff(x,y,z) \supset (mass(x) = add\_mass(mass(y), mass(z))) \tag{3}$$
$$(\forall x,y)\, proper\_chunk(y,x) \supset (\exists z)\, proper\_chunk(z,x) \wedge (y \neq z) \wedge (mass(y) = mass(z)) \tag{4}$$

Figure 1. T_{mass}: Axioms for the assignment of mass to chunks of matter.

The physical object ontology for units of mass requires an ontology $T_{constitution}$[6] for constitution that axiomatizes the relationship between matter and physical objects. The physical mass for an object is therefore equal to the mass of the matter that constitutes the object:

$$(\forall x,m)\, physical\_mass(x, mass(m)) \equiv constitutes(m,x)$$

[5] http://colore.oor.net/mass/mass.clif

[6] http://colore.oor.net/constitution/constitution.clif

4.2. Characterization of $Mod(T_{mass})$

The *mass* function is a vector-valued function that maps chunks of matter to a unique amount; to understand the models of T_{mass}, we therefore need to introduce a new class of mathematical structures.

Definition 1 $\langle \mathbb{P}, \mathbb{I}, \mathbb{V} \rangle$ *is a lattice vector field iff* [7]

1. $\mathbb{P} = \langle P, \leq \rangle$ *such that* $\mathbb{P} \in \mathfrak{M}^{boolean\_lattice\_ordering}$*;*
2. $\mathbb{V} = \langle V, \mathbf{0^V}, \mathbf{1^V}, +, \cdot, \prec \rangle$ *such that* $\mathbb{V} \in \mathfrak{M}^{ordered\_vectorspace}$*;*
3. $\mathbb{I} = \langle P, V, \mathbf{I} \rangle$ *such that* $\mathbb{I} \in \mathfrak{M}^{mapping\_bipartite}$*;*
4. $N^{\mathbb{I}}(\mathbf{x}) + N^{\mathbb{I}}(\mathbf{y}) = N^{\mathbb{I}}(inf(\mathbf{x},\mathbf{y})) + N^{\mathbb{I}}(sup(\mathbf{x},\mathbf{y}))$*, such that* $N^{\mathbb{I}}(\emptyset) = \mathbf{0}$*;*
5. $L^{\mathbb{P}}(\mathbf{x}) \subseteq N^{\mathbb{I}}(N^{\mathbb{I}}(L^{\mathbb{P}}(\mathbf{x})))$*, for each* $\mathbf{x} \in P$*.*

$\mathfrak{M}^{lattice\_vector\_field}$ *denotes the class of lattice vector fields.*

In a lattice vector field, we associate a unique vector in an ordered vector space (corresponding to a model of T_{amount}) to each element of a lattice (corresponding to the mereology of matter); the incidence structure represents this mapping. Conditions (4) and (5) constrain additional properties of this mapping, and correspond to Axioms (3) and (4) of T_{mass}, respectively. In what sense does $\mathfrak{M}^{lattice\_vector\_field}$ capture the essential intuitions about the mass of chunks of matter?

Theorem 1 *If* $\mathbb{L} = \langle \mathbb{P}, \mathbb{I}, \mathbb{V} \rangle$ *is a lattice vector field, then* $Aut(\mathbb{L}) \cong Aut(\mathbb{P})$*.*

Automorphisms of the mereology induce automorphisms of the lattice vector field (that is, mappings that preserve **chunkOf** preserve **mass**, so that symmetries of the structure for generic objects are equivalent to symmetries of the structure that maps generic objects to the unit of measure). If we remove any conditions in the definition of lattice vector fields (and hence any axioms in T_{mass}, this relationship fails.

The next theorem shows that T_{mass} axiomatizes the class of lattice vector fields.

Theorem 2 *There exists a bijection* $\varphi : \mathfrak{M}^{lattice\_vector\_field} \to Mod(T_{mass})$ *such that* $\varphi(\mathbb{P}) \in \mathfrak{M}^{boolean\_lattice\_ordering}$*,* $\varphi(\mathbb{V}) \in \mathfrak{M}^{ordered\_vectorspace}$*, and for any* $\mathbf{x} \in P$*,*
$\mathbf{mass}(\mathbf{x}) = \mathbf{y}$ *iff* $\mathbf{y} \in N^{\mathbb{I}}(\mathbf{x}) \cap V$*.*

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[7] $\mathfrak{M}^{ordered\_vectorspace}$ is the class of ordered vector spaces and $\mathfrak{M}^{mapping\_bipartite}$ is the class of bipartite incidence structures that axiomatize functions between two disjoint sets [8]. We also need the following notation: Suppose $\mathbb{P} \in \mathfrak{M}^{partial\_ordering}$ such that $\mathbb{P} = \langle V, \leq \rangle$.

The lower set for $\mathbf{x}$ in $\mathbb{P}$, denoted by $L^{\mathbb{P}}(\mathbf{x})$, is

$$L^{\mathbb{P}}(\mathbf{x}) = \{\mathbf{y} : \mathbf{y} \leq \mathbf{x}\} \qquad L^{\mathbb{P}}(X) = \bigcup_{\mathbf{x} \in X} L(\mathbf{x})$$

Suppose $\mathbb{I} \in \mathfrak{M}^{mapping\_bipartite}$, such that $\mathbb{I} = \langle P, L, \mathbf{I} \rangle$.

The neighbourhood of $\mathbf{x}$ in $\mathbb{I}$, denoted by $N^{\mathbb{I}}(\mathbf{x})$, is

$$N^{\mathbb{I}}(\mathbf{x}) = \{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in \mathbf{I}\} \qquad N^{\mathbb{I}}(X) = \bigcup_{\mathbf{x} \in X} N^{\mathbb{I}}(\mathbf{x})$$

5. Spatial Units of Measure

This section extends the general approach to spatial units of measure. Unlike duration or mass, multiple units for spatial measures exist: length, area, and volume, whose application depend on the spatial dimensionality of an object. The use of the term "dimension" is ambiguous. While many ontologies for units of measure emphasize dimensional analysis, wherein the term "dimension" is used to indirectly characterize which quantities are of the same kind (and hence are comparable). On the other hand, spatial ontologies represent entities of different dimension in a topological sense.

Although units for spatial measures are derived from the base SI unit *metre* (m) for length, length is assigned to 1D regions (i.e., linear features), while the units *squaremetre* (m^2) and *cubicmetre* (m^3) are used for area and volume of 2D and 3D regions, respectively. Thus, spatial units of measurement are dimension-dependent in the topological sense. We next discuss the spatial ontologies that accommodate entities of multiple dimensions and thus can be used in our approach to units of measure. We also outline how to extend with ontologies for spatial units of measures. Again, we apply a pluralist approach that distinguishes the mereotopology of spatial entities (e.g., points, curves, 2D regions and voluminal regions) described in Sec. 5.1 with associated spatial units of measure (Sec. 5.2) and the mereotopology of physical shapes (e.g., edges, surfaces, and boxes) described in Sec. 5.3. They are linked via the multidimensional occupy ontology in Sec. 5.4.

5.1. Multidimensional Mereotopology

The underlying generic object ontology for space and spatial regions is mereotopology, which captures the topological (i.e., contact) and mereological (i.e., parthood) relations between spatial regions. Only select few mereotopologies accommodate entities of multiple dimensions, we extend here the theory $CODI_{\updownarrow}$[9] from the CODI family [8]. It is based on two primitive notions: relative dimension and spatial containment. By restricting the relative dimension to a linear discrete bounded order, definitions for discrete classes of objects of specific dimensions can be added: points, curves, areal regions, and voluminal regions [7][10]. Spatial containment is the only primitive spatial relation. In its point-set interpretation, we say x is contained in y, i.e. $\mathrm{Cont}(x,y)$, if the set of points that x covers are a subset of the points that y covers. A region can contain not only (smaller) regions of the same dimension (equidimensional parthood), but also lower-dimensional entities. E.g., a 2D areal region can contain another 2D region, a linear region (e.g., a line or curve), or a point. A single axioms governs the relationship between containment and relative dimension: if y contains x, x must be of equal or lower dimension than y.

In $CODI_{\updownarrow}$, lower-dimensional spatial entities can exist not only as boundaries of higher-dimensional entities but also independently thereof. Moreover, while a higher-dimensional entity can contain infinitely many lower-dimensional entities (e.g., a line can contain infinitely many points), it is never defined as the sum of its contained lower-dimensional entities. With intersection, difference, and sum operations, the ontology defines a mereology (with a zero region of no particular dimension) over all entities of each dimension. The ontology does not require that atoms exist, but can be easily extended in

[9] http://colore.oor.net/multidim_mereotopology_codi/codi_updown.clif
[10] http://colore.oor.net/multidim_mereotopology_codi/codi_updown_3d.clif

that respect to the atomistic version $CODI_{\updownarrow}^{\text{at}}$ wherein the predicate $Min(x)$ picks out all atoms.

5.2. Length, Area, and Volume

The quantity ontologies for spatial units – m (spatial length[11]), m^2 (spatial area[12]), and m^3 (spatial volume[13]) – have axiomatizations that are synonymous with ordered vector spaces, just like the quantity ontologies for time durations and mass. It is important to note that FOUnt does not treat m^2 and m^3 as derived units – in FOUnt they are treated as basic with respect to their dimension. We do not derive area and volume from length – the mereologies for entities of different dimension are all distinct, and hence their units of measure are all distinct. In particular, it is *not* the case that multiplying metre units is ontologically justified, since m^2 is *not* a spatial length.

The measure ontologies for spatial units axiomatize the mapping from regions of different dimensions in the mereotopology to spatial length[14], area[15], and volume[16]. These measure ontologies are synonymous with T_{mass}, which we introduced earlier in the paper.

5.3. The Multidimensional Object Mereotopology

The Multidimensional Object Mereotopology is a qualitative representation of physical shapes and the spatial relationships between them. It extends the Shape Ontology [4], which is based on the sub-theory about incidence and betweenness relations from Hilbert's axiomatic theory of geometry. In the Shape Ontology, the *incident* predicate captures the incidence relation between four disjoint categories of entities: *phy_points*, *edges*, *surfaces*, and *boxes*, which correspond to zero-, one-, two-, and three-dimensional objects, respectively. In the Multidimensional Object Mereotopology, each of these categories has its own mereotopological theory, with the spatial relationships between entities within a category forming a Ground Mereotopology (MT) [2]. The four mereotopological theories are independent of one another.

5.4. Multidimensional Occupy Ontology

The Multidimensional Occupy Ontology[17] relates the shape entities (i.e., *phy_points*, *edges*, *surfaces*, and *boxes*) from the Multidimensional Object Mereotopology to the spatial regions from $CODI_{\updownarrow}$ they occupy. This results in the four modules T_{point_occupy}, T_{edge_occupy}, $T_{surface\_occupy}$, T_{box_occupy}, where each module axiomatizes a mapping of the mereotopology of all shape entities with a specific dimension to the mereotopology of regions in abstract space. This is exemplified in Figure 2 for the axioms of T_{box_occupy}. The other three theories are synonymous with T_{box_occupy}.

Multidimensional Occupy contains axioms that prescribes that incidence between an element x and an upper-dimensional element y requires the region occupied by x to

[11] http://colore.oor.net/size/spatial_length.clif
[12] http://colore.oor.net/size/spatial_area.clif
[13] http://colore.oor.net/size/spatial_volume.clif
[14] http://colore.oor.net/size/length.clif
[15] http://colore.oor.net/size/area.clif
[16] http://colore.oor.net/size/volume.clif
[17] http://colore.oor.net/multidim_occupy/multidim_occupy_root.clif

be contained in the region occupied by y. For example, edges and the surfaces they are incident with must satisfy the axiom:

$(\forall x,y)\, incident(x,y) \wedge edge\_occupies(x,r_1) \wedge surface\_occupies(y,r_2) \supset Cont(r_1,r_2)$.

$$(\forall x,y)\, box\_occupies(x,y) \supset box(x) \wedge VoluminousRegion(y)$$

$$(\forall x,y,z)\, box\_occupies(x,y) \wedge box\_occupies(x,z) \supset (y = z)$$

$$(\forall x)\, box(x) \supset (\exists y)\, box\_occupies(x,y)$$

$$(\forall x,y,r_1,r_2)\, box\_part(x,y) \wedge box\_occupies(x,r_1) \wedge box\_occupies(y,r_2) \supset P(r_1,r_2)$$

$$(\forall x,y,r_1,r_2)\, box\_C(x,y) \wedge box\_occupies(x,r_1) \wedge box\_occupies(y,r_2) \supset C(r_1,r_2)$$

Figure 2. T_{box_occupy}: Axioms for mappings from three-dimensional physical objects (i.e., boxes) into spatial regions.

The physical object ontology for spatial units further requires an ontology T_{bounds}[18] that axiomatizes the relationship between shapes and the physical objects that are bounded by their shapes; this theory is synonymous with the axiomatization of the relationship between matter and objects in $T_{constitution}$.

6. Derived Units

For derived units, we need to provide two different sets of axioms. The first axiomatizes the combinations of the derived units, while the second axiomatizes the relationship between the derived unit and the associated basic units. Once again, the illusion of multiplying real numbers obscures the true nature of the operation being performed. The dimensional analysis that is so prominent in existing ontologies for units of measure ignores the ontological distinctions that underly both the quantities and the objects being measured. Dimensional analysis as a technique for using the rules of algebra to convert among different units also ignores the physical interpretations of the relevant basic units. In this section, we consider two derived units – density (which is based on the relationship between the volume and mass of a physical object) and velocity (which is derived from the relationship between location and duration). We specify new ontologies for capturing the relationship between the different units, and show how new axiomatizations are needed to adequately represent how these units are combined.

6.1. Density

The SI unit for density is $kg \cdot m^{-3}$, indicating that density captures the relationship between mass and volume. Density is typically thought of as dividing the amount of mass (kg) by the volume quantity (m^3); however, we have already seen that amounts of mass and volume quantities are both represented by theories that are synonymous with ordered vector spaces, so the notion of division is not appropriate. Furthermore, density quantities

[18] http://colore.oor.net/shape/bounds.clif

themselves also need to be represented by a theory synonymous with an ordered vector space – we can add densities, multiply them by a scalar (one material is twice as dense as another), but multiplying densities does not give another density value. We therefore need to represent density as a relationship between three different ordered vector spaces – the vector space $\mathbb{V}$ of volumes, the vector space $\mathbb{M}$ of masses and a vector space of densities.

A key insight is to reconsider the understanding of density as a mapping rather than as some kind of quotient of other quantities. Each value of a density quantity associates a unique volume with a given amount; that is, each value of density maps the set of amounts to the set of volumes. For example, given a density of $2 \cdot kgm^{-3}$, an amount of $4 \cdot kg$ corresponds to a volume of $2 \cdot m^3$. Furthermore, this mapping is a bijection, since each value of density also maps the set of volumes to the set of amounts (e.g. given a density of $2 \cdot kgm^{-3}$, a mass amount of $4 \cdot kg$ corresponds to a volume of $2 \cdot m^3$). Since we are dealing with vector spaces, each mapping $\mu : \mathbb{V} \rightarrow \mathbb{M}$ is actually a linear mapping, in particular, a vector space isomorphism. It is well known that the set $Iso(\mathbb{V}, \mathbb{M})$ of vector space isomorphisms between $\mathbb{V}$ and $\mathbb{M}$ is itself a vector space. On the other hand, we have seen that each density value corresponds to a unique mapping, so the vector space $\mathbb{D}$ for densities must itself be isomorphic to $Iso(\mathbb{V}, \mathbb{M})$.

Definition 2 $\langle \mathbb{V}, \mathbb{M}, Iso(\mathbb{V}, \mathbb{M}) \rangle$ *is a linear bijection space iff* $\mathbb{V}, \mathbb{M}$, *and* $Iso(\mathbb{V}, \mathbb{M})$ *are ordered vector spaces. The class of linear bijection spaces is denoted by* $\mathfrak{M}^{linear\_bijection}$

To axiomatize this class of structures, the quantity ontology $T_{density}$[19] introduces a new class *density* for density values and a function *dmv* that associates a unique volume with a given amount (i.e. instead of computing density as the ratio of amount and volume). The two vector spaces in a linear bijection space correspond to the models of $T_{spatial\_volume}$ and T_{amount}, while the function $dmu(v, m)$ represents density as the linear bijective mapping between spatial volumes and amounts.

Theorem 3 *There exists a bijection* $\varphi : \mathfrak{M}^{linear\_bijection} \rightarrow Mod(T_{density})$ *such that* $\varphi(\mathbb{V}) \in Mod(T_{spatial\_volume})$, $\varphi(\mathbb{M}) \in Mod(T_{amount})$; *and for each* $\mu \in Iso(\mathbb{V}, \mathbb{M})$, $\varphi(\mu) = \mathbf{dmu}(\mathbf{v}, \mathbf{m})$ *iff* $\mu(\mathbf{v}) = \mathbf{m}$.

The physical object ontology $T_{physical\_density}$[20] for density combines the measure ontologies for matter and spatial volumes[21]:

$$(\forall x, m, r)\, physical\_density(dmv(mass(m), volume(r))) \equiv$$
$$(constitutes(m, x) \land occupies(m, r) \land VoluminousRegion(r)) \tag{5}$$

In other words, the physical density of an object is the density that corresponds to the mass of the matter that constitutes the object and the volume of the region occupied by that matter.

[19] http://colore.oor.net/density/density.clif

[20] http://colore.oor.net/density/definitions/physical_density.clif

[21] [13] claims that when density is ascribed the dimension ML^{-3}, the context makes it clear that M and L^{-3} are properties of one and the same thing. However, it is essential that any adequate ontology explicitly axiomatize this relationship

6.2. Velocity

The SI unit for velocity is $m \cdot s^{-1}$, indicating that velocity captures the relationship between length and duration. As we saw with density and its relationship to volume and mass, velocity should be represented as a mapping rather than as some kind of quotient of the quantities for length and duration. Each value of a velocity quantity associates a unique length with a given timeduration, that is, each value of velocity maps the set of spatial lengths to the set of timedurations. Since the combination of both spatial lengths and timedurations are synonymous with ordered vector spaces, the set of linear mappings between these two vector spaces also forms a vector space. The theory $T_{velocity}$[22] therefore also axiomatizes the class of linear bijection spaces $\mathfrak{M}^{linear\_bijection}$. The function *vld* is the operation that correctly specifies the relationship between spatial length and duration – using the ratio between real numbers for length and duration is ontologically flawed because the relationship holds among entities in three disjoint classes of the foundational ontology.

6.3. Velocity as a Vector Quantity

Unlike other units that have been considered so far in this paper, velocity is a vector quantity, since the physical interpretation of velocity includes both magnitude and direction. We therefore need additional axioms for the combination of velocities that are quite different than the ones we have used for the other units presented earlier in the paper. When we initially considered how to combine amounts of matter, we noticed amounts can be added together and multiplied by an element of a field. The axiomatization of the addition operation is synonymous with that of abelian groups. However, if velocity is considered to have both a magnitude and a direction, then such an approach is insufficient; instead, the addition operation must itself be that of a vector space rather than an abelian group. The combination of velocity units is therefore not axiomatized by a vector space, but must be some other class of mathematical structures. Since we want the units themselves to be vectors, the structures we need can be considered as a mapping that relates the vector space of the vector quantities and the vector space of velocity units [14].

Definition 3 *A bilinear map is a pair* $\mathbb{W}, \mathbb{V}$ *of vector spaces, together with a mapping* $\beta : \mathbb{W} \times \mathbb{V} \rightarrow \mathbb{V}$ *such that*

1. $\beta(\mathbf{x}+\mathbf{y},\mathbf{v}) = \beta(\mathbf{x},\mathbf{v}) + \beta(\mathbf{y},\mathbf{v})$;
2. $\beta(\mathbf{x},\mathbf{u}+\mathbf{v}) = \beta(\mathbf{x},\mathbf{u}) + \beta(\mathbf{x},\mathbf{v})$;
3. $\beta(\mathbf{a}\cdot\mathbf{x},\mathbf{v}) = \mathbf{a}\cdot\beta(\mathbf{x},\mathbf{v})$;
4. $\beta\mathbf{x},(\mathbf{a}\cdot\mathbf{v}) = \mathbf{a}\cdot\beta(\mathbf{x},\mathbf{v})$.

$\mathfrak{M}^{bilinear}$ *denotes the class of bilinear maps.*

The quantity ontology $T_{velocity\_vector}$[23] introduces the function *dmv* to represent the bilinear mapping and the combination of velocities as vector quantities. The next theorem shows that $T_{velocity\_vector}$ axiomatizes the class of bilinear maps.

[22] http://colore.oor.net/velocity/velocity.clif
[23] http://colore.oor.net/velocity/velocity_vector.clif

Theorem 4 *There exists a bijection* $\varphi : \mathfrak{M}^{bilinear} \rightarrow Mod(T_{velocity\_vector})$ *such that* $\varphi(\mathbb{W}) \in Mod(T_{vectorspace})$, $\varphi(\mathbb{V}) \in Mod(T_{velocity})$, *and for any* $\mathbf{x} \in P$,
$\mathbf{velq}(\mathbf{x}, \mathbf{y}) = \mathbf{z}$ *iff* $\mathbf{z} \in \beta(\mathbf{x}, \mathbf{y})$.

Since the combination of both spatial lengths and timedurations are synonymous with ordered vector spaces, the set of linear mappings between these two vector spaces (which represents the relationship between velocity, duration, and length) also forms a vector space. However, we have just seen above that as a vector quantity, velocity units form a bilinear map rather than a vector space – how can we reconcile these two viewpoints? Velocity is considered to be a vector quantity because it has both magnitude and direction, and this is reflected in $T_{velocity\_vector}$. What makes an entity a vector quantity is nothing intrinsic to the entity itself, but rather the operations that act upon it and the axioms that constrain these operations [24]. The axioms that capture the relationship between velocity, length, and duration treat velocity as a scalar quantity, since with these axioms velocity must form a vector space. In this sense, $T_{velocity}$ is an axiomatization of velocity as speed, which does not involve a notion of direction.

6.4. Velocity and Motion Ontologies

Although $m \cdot s^{-1}$ is commonly considered as a unit for measuring speed and velocity, it has other physical interpretations. For example, $m \cdot s^{-1}$ can be used in measuring the rate of change in the length of an object. In fact the rate of any one-dimensional spatial change can be measured by $m \cdot s^{-1}$. From the TUpperWare perspective, this is equivalent to say that $m \cdot s^{-1}$ is a unit for the rate of any change in the region that is occupied by an *edge* (see Section 5.4). Axiomatizing the physical interpretations of $m \cdot s^{-1}$, therefore, requires a motion ontology in which the relation *edge_occupies* becomes a fluent, and that includes a complete classification of all possible ways that this fluent can change. Part of the future work is to develop such an ontology based on the methodology presented in [1].

7. Existing Units of Measure Ontologies

QUDT [11] and OM [18] are two ontologies for units of measure that are widely used on the Semantic Web. These existing ontologies suffer from several drawbacks. Since they are specified in OWL, their axiomatization is too weak to capture their intended models, and there has been no effort to specify the relationships to existing upper ontologies. In fact, no semantic requirements are ever presented and hence no rigorous evaluation of the correctness and completeness of these ontologies is possible. Overall, there is a focus on conversion between systems of units rather than on *what* is being measured. This leads to an over-reliance on real numbers and algebraic manipulation related to dimensionality that masks the ontological distinctions that are required for the intended semantics of the units.

The original ontology for units of measure (with full first-order axiomatization) is EngMath [3], and is the closest in approach to FOUnt. EngMath reifies dimensions, which are defined to be properties (such as mass, length, and velocity) associated with

[24] [12] recognizes this distinction, but does not provide any axiomatization.

physical quantities. FOUnt does not reify dimensions; instead, dimensions are associated with sets of ontologies related to the particular unit of measure. Thus, length is considered to be a dimension, but this corresponds to the set of ontologies in FOUnt rather than a new class of thing. Similarly, FOUnt does not reify physical quantities, although specific units such as *metre* and *inch* are instances of the class corresponding to the physical quantities (e.g. *spatial_length*). There are also a few problems with the axiomatization of EngMath. First, units of measure do *not* form abelian groups with respect to multiplication. Physical dimensions are not composed from other dimensions using multiplication – each unit is axiomatized by its own set of ontologies, and these have nothing to do with the operation of multiplication. Second, although EngMath recognizes the distinction between scalar and vector quantities, this distinction is not reflected in the axioms.

All existing approaches lack axiomatizations for what we refer to as the Measure Ontologies that provide the physical interpretation of the units, although they often refer to the need for such axioms in their examples. In approaches that do axiomatize what we refer to as the Quantity Ontologies, their axiomatizations are incorrect – the combination of units is not preserved under multiplication.

Existing ontologies for units of measure are therefore not only incomplete (lacking axioms for physical interpretations of the units), but their axiomatizations contain fundamental ontological errors. The primary source of these errors is the treatment of so-called dimensional analysis. Units such as seconds, metres, and kilograms cannot be multiplied together, and algebraic operations cannot be used to represent the relationships between derived and basic units.

8. FOUnt: The Programme

FOUnt is the primary deliverable of a programme for designing ontologies for units of measure based on the TUpperWare foundational ontology. For each unit of measure, we introduce a set of ontologies that not only axiomatize how units of measure can be combined and how they are related to other units of measure, but that also axiomatize the relationship between the measure ontologies and the physical object ontologies, so that one can speak about the units of measure for a class of physical objects.

FOUnt imposes minimal ontological commitments with respect to TUpperWare – the only new classes that are introduced are the classes for the units of measure themselves. The ontologies provide a correct and complete axiomatization for combining units of measure, as well as a correct axiomatization of the relationship between the units of measure and the TUpperWare upper ontology.

This paper has presented the ontologies for units for duration, mass, spatial units, density, and velocity. Future work will explore the remaining SI units, beginning with the base units for temperature, electric current, and luminosity, and continuing through the SI derived units for acceleration, force, work, and power. The ultimate objective is a complete set of ontologies for all SI units. Future work will also design additional ontologies to formalize the physical interpretations of the units of measure. An example is a treatment of distances that relate points, but lead to a much more complex interaction between lengths and distances of endpoints. Other extensions will focus on specific application domains for units; for example, what is meant by "5 mm of rain"?

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