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On Decomposition Operations in a Theory of Multidimensional Qualitative Space

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Abstract. Mereotopological relations, such as contact, parthood and overlap, are central for representing spatial information qualitatively. While most existing mereotopological theories restrict models to entities of equal dimension (e.g., all are 2D regions), multidimensional mereotopologies are more flexible by allowing entities of different dimensions to co-exist. In many respects, they generalize traditional spatial data models based on geometric entities (points, simple lines, polylines, cells, polygon, and polyhedra) and algebraic topology that power much of the existing spatial information systems (e.g., GIS, CAD, and CAM). Geometric representations can typically be decomposed into atomic entities using set intersection and complementation operations, with non-atomic entities represented as sets of atomic ones. This paper accomplishes this for CODI, a first-order logic ontology of multidimensional mereotopology, by extending its axiomatization with the mereological closure operations intersection and difference that apply to pairs of regions regardless of their dimensions. We further prove that the extended theory satisfies important mereological principles and preserves many of the mathematical properties of set intersection and set difference.

This decomposition addresses implementation concerns about the ontology CODI by offering a simple mechanism for determining the mereotopological relations between complex spatial entities, similar to the operations used in algebraic topological structures. It further underlines that CODI accommodates both quantitative/geometric and qualitative spatial knowledge.

Keywords. spatial ontology, mereotopology, first-order logic, dimensionality

1. Introduction

Spatial information often plays a central in modeling the world, with virtually all upper ontologies including some spatial concepts and qualitative spatial relations between them. But these qualitative representations differ from how most information systems store spatial information using geometric representations, which encode the location of entities in an underlying metric space – typically Euclidean space. The entities are represented by geometric features, such as points, polylines, polygons, polyhedra, or collections thereof. Often this requires geometrically approximating entities, which may lead to artefacts such as accidental overlap (e.g., a road overlapping a lake because it is represented using a polygon and overlap is defined purely geometrically). Geometric representations (including algebraic topologic structures such as simplicial or other cell complexes, e.g. [3,12]) introduce metric precision that is often unwarranted when qualitative relations and broader types of features suffice to characterize a spatial scene. For example, for many queries it suffices to describe the relation between a linear entity (e.g., representing a road) and some 2D entity (e.g., representing a township) without the need to represent the entities' location and extent precisely using, for example, polygons and polylines. While geometric representations yield extra metric information, such as the length or size of an intersection, this precision can be misleading as it it derived from an approximation (with an implicit resolution) of the involved entities. More important might be the nature of the spatial relationship between the entities (e.g., does the road border the town or does it go into it?, is it entirely within the town or are parts outside?) that qualitative relations, especially mereotopological ones, can capture without the detour via geometric approximation of the entities.

Mereological and topological relations are deemed more intuitive to people—as evident by their predominance in natural language [24,22]— compared to geometric operations. Mereologically, spatial entities can be subdivided into smaller parts (i.e., subregions), and topologically they can be described by how they are connected to other entities. *Mereotopological relations* [20,31] arise from the combination of both kinds of relations, enabling the definition of relations such as *overlap* (i.e., connected via a shared part). Formalizations of spatial entities and the mereotopological relations between them can help bridge the conceptualization gap between geometry-based spatial information common in information systems and higher-level, intrinsically qualitative ways of expressing spatial information.

In the spirit of Whitehead's and Clarke's work [5,33], mereotopological theories are often tied to *region-based* representations of space that rely on extended regions of space, instead of points (as in geometry), as the most fundamental spatial entities. As such, interest in mereotopological relations has focused on *unidimensional* mereotopological theories, wherein all spatial entities, also called "regions" (we use the terms interchangably), must be of the same dimension, that is, all regions are two-dimensional or all are three-dimensional. However, this poses a significant barrier to integration with the wealth of existing geometric information stored in geographic information systems (GIS) and computer-aided design or manufacturing (CAD/CAM) systems. These systems use and relate geometric entities of varying dimensions: point features, linear features (e.g., polylines), areal features (e.g., polygons), and possibly voluminal features (e.g., polyhedra). In order to qualitatively describe the spatial relations between such objects and to fully interact with such systems, a suitable mereotopological theory must accommodate spatial entities of varying dimensions [15]. Towards this goal, we have developed a family of multidimensional spatial ontologies [18,19] based on insights from Galton's and Gott's earlier work [14,16]. A central role plays the ontology *CODI*, which formalizes the primitive notions of *spatial containment* and *relative spatial dimension* (the ontology's name stands for *CO*ntainment and *DI*mension) in first-order logic and defines a set of six intuitive mereotopological relations: *containment* and its refinement *parthood* together with *contact* and its refinements *partial overlap*, *incidence*, and *superficial contact*, which apply to entities of all dimensions but which operate largely independently of the precise dimensions of the involved spatial entities.

What is missing from [19] are suitable axiomatizations of the standard mereological operations product/intersection, difference/complement, and sum. The theories in [14,16] include such operations but limit them to entities of equal dimension. As result, some meaningful intersection and differences, for example between a linear feature (e.g., a road or river) and a 2D region, as shown in Fig. 1(c), are not captured, thereby limiting the use of the operations more than necessary. However, if we insist that these operations are closed for all spatial regions, that is, applying them to any pair of regions yields again a region, one can easily end up with mixed-dimensional entities, such as an entity consisting of a line segment and an isolated point (e.g., the intersection in Fig. 1(a)) or of a 2D region with a line segment removed (e.g., the difference $x - y$ in Fig. 1(c)). We develop an extension of *CODI* with intersection and difference operations that avoids these problems. We study the resulting axiomatization with respect to the following properties:

- Are the operations *well-defined*, i.e., do the intersection and difference of two arbitrary spatial entities always result in a unique spatial entity in every model?
- Do the operations exhibit the expected result when applied to entities of equal dimension (e.g., when compared to unidimensional mereotopologies)?
- Which mereological principles from unidimensional mereotopologies are preserved?
- Which mathematical properties (e.g., associativity, commutativity, idempotence) typically associated with set theoretic and unidimensional mereotopological notions of intersection and difference are preserved?

We show that the proposed intersection and difference operations are well-defined and that the resulting theory $CODI_{\downarrow}$ exhibits the expected unidimensional behavior and that it satisfies the most important mereological principles from unidimensional mereotopology. The operations satisfy most desired mathematical properties, though some require weakening (e.g., left- and right-alternative laws in place of associativity), while only few must be forfeited. Moreover, atomic models of $CODI_{\downarrow}$ can always be decomposed (i.e., partitioned) into a set of minimal (i.e., indivisible) parts, which simplifies the implementation of the theory on top of qualitative or geometric spatial representations wherein entities are represented as collections of minimal entities (e.g., line segments or atomic cells).

2. Related Work

Mereotopological theories originally emerged from interest in region-based theories—so-called *pointless topologies*—as alternative to point set topology and geometry. For information systems, three approaches for defining mereotopological relations between spatial entities are pursued: (1) as *qualitative calculi* of lattices of relations and associated composition operations, e.g., [9,27]; (2) as *9-intersection relations* based on matrices of the point-set intersections of the interior, boundary, and exterior of two entities, e.g., in [7,8,11,13,25]; or (3) as *axiomatic*, i.e., logic-based *theories/ontologies* with mereological and/or topological primitives, e.g., in [2,4,10,26,27,29]. The present work belongs to the last category, but we briefly review the most relevant related work in all three.

Qualitative Calculi of mereotopology, such as the Region Connection Calculus (RCC) [9,27], assume all regions in a single model to be of the same dimension, typically either all 2D areas or all 3D chunks of space. The presence of entities of lower dimensions would collapse important RCC relations (e.g. O and C) [6,15] and violate complementation principles, though lower-dimensional entities may exist implicitly (e.g., entities in external connection EC share points or curve segments), or be definable as higher-order constructs (e.g., a set of infinitely nested proper parts—a prime ideals—can approximate a point). The associated axiomatic theories permit straightforward mereological closure operations under which the entities form Boolean algebras with the null element removed, or generalizations thereof. To our knowledge, only [6] presents a multidimensional mereotopological calculus, but it does not permit explicit conclusions about the dimensionality of related entities nor are closure operations incorporated.

Matrices of Point-Set Intersections, such as used by the 9-intersection approach [11, 13,23], define mereotopological relations in terms of the point set intersections of the interior, boundary, and exterior of two regions. Extensions that take the dimensions of the participating entities into account, lead to a combinatorial increase in the number of distinct topological relations, ranging from 52 or 81 (for the dimension-extended and calculus-based methods, respectively [7]) to 300 relations [25] without even considered 3D entities. These approaches are restricted to 0D, 1D, and 2D entities and do not provide an axiomatization of mereological closure operations for decomposing entities.

Axiomatic Theories of mereotopology are surveyed more fully in [20]. Most use parthood and/or connection relations (with some varation, see [10]) as primitive terms. Most are directly tied to a *uni-dimensional* region-based conceptualization of space in the sense that all first-class entities therein must have the same dimension and, thus, mereotopological relations are only defined for entities of the same dimension. The theories that allow entities of different spatial dimensions to co-exist permit lower-dimensional entities only as they arise as boundaries of higher-dimensional entities [14,15,29,30], with other theories relying on higher-order constructs not definable in their first-order theories.

But the application of mereotopological and other qualitative relations is not limited to unidimensional spatial theories as evidenced by the dimensional extension of the 9-intersection approach [7]; the relations equally apply to spatial entities of different dimensions. Thus, the definition of mereotopological relations can be decoupled from a unidimensional conceptualization of space. Only Galton [14], Gott's INCH Calculus [16], and our own prior work [19] have developed full axiomatic theories that incorporate dimensionality into mereotopological relations and that permit spatial entities of different dimensions to co-exist[1]. But Galton and Gotts restrict intersections, differences, and sums to entities of equal dimension. The restriction of the sum operation is sensible to avoid entities of mixed dimensionality, but the same cannot be said about intersections or differences. For example, the intersection of the areal and linear features in Fig. 1(c) is itself a linear feature, but its existence is not entailed in either theory. This paper explores how to fill this gap by presenting and investigating an extension to our own multidimensional mereotopology *CODI* [19] with axioms that define intersections and differences for all pairs of entities regardless of their dimension.

3. A Modular Version of the Multidimensional Mereotopology CODI

This section reviews the multidimensional mereotopology *CODI* from [17,19] that forms the basis of the present work. *CODI*'s universe of discourse consists of spatial (and possibly spatio-temporal) regions of various dimensions located in a space $\mathbb{R}^n$. A spatial re-

[1] The RCC*-9 [6] provides logical definitions of mereotopological relations but no further axioms.

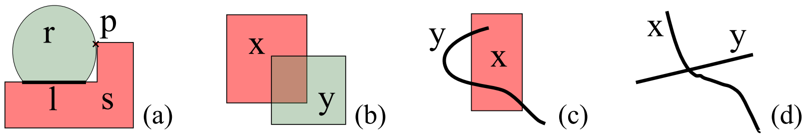

Figure 1. Models of *CODI* that illustrate intersections and differences involving regions of varying dimensions. Other mereotopologies only entail the existence of an entity for the intersection and difference in (b). In (a), r and s share two regions: the linear feature l and the point p. But both together cannot be the intersection, because such an entity would not be a dimensionally uniform region. Instead, $r \cdot s = l$. Then the intersection $\cdot$ is not associative because $r \cdot (s \cdot p) = r \cdot p = p \neq \emptyset = l \cdot p = (r \cdot s) \cdot p$.
In (b) intersection and the differences $x - y$ and $y - x$ are defined as expected between two regions of the same dimension. In (c) the intersection $x \cdot y$ is a proper part of y but not a proper part of x, though a lower-dimensional region contained in x. Hence, only the difference $y - x$ results in a new region, while $x - y$ loses the lower-dimensional artifact and results in x. In (d) $x - y = x$ and $y - x = y$ since the intersection $x \cdot y$ is of a lower dimension than both x and y.

gion can be *simple* or *complex*. Simple regions represent regular closed sets[2] of some dimension $m \leq n$ that can be embedded in $\mathbb{R}^m$. Complex regions are unions of finitely many regular closed sets—each set being called a component. Components of a single complex region can overlap only in their boundaries, that is, $x \cap y \subseteq (\mathrm{cl}(x) \setminus \mathrm{int}(x)) \cap (\mathrm{cl}(y) \setminus \mathrm{int}(y))$ holds for any two components x and y of a complex region. Each complex region also has an associated dimension m such that every component thereof is homeomorphically embeddable[3] in some space $\mathbb{R}^m$ and no component can be embedded in a space $\mathbb{R}^{m-1}$. Thus, each component is a bounded manifold in $\mathbb{R}^m$.

Examples of permissible regions are points and sets of points; 1D entities such as curve and line segments, infinite curves and lines, as well as complex linear features (e.g., finite sets of segments connected, if at all, only in their endpoints); 2D *areal regions* including polygons, cell complexes, and other bounded regions (e.g., a part of the surface of a sphere) as well as unbounded regions (e.g., planes, semi-planes, or the entire surface of a sphere); or 3D *voluminal regions*. Regions with lower-dimensional artefacts, for example, an areal region with a protruding or missing line segment or a missing boundary is not permitted as entity in the domain, but could be described as a "scene" (i.e., a model of *CODI*) consisting of two or more entities that happen to intersect. The theory can be limited to a specific number of dimensions if so desired as shown in [17,18].

CODI's formalization is based on two primitive relations: (1) a mereological predicate of spatial containment, denoted as $Cont(x,y)$ and interpreted point-set topologically as "x is a subset of y" (i.e., x is entirely contained within y), and (2) a predicate $\leq_{\dim}$—typically written in infix notation as $x \leq_{\dim} y$— that compares the dimension of two spatial regions and is interpreted as "if all of y's components can be embedded (as regular closed sets) in spaces of dimension m, then all of x's component can be embedded in spaces of dimension m or lower"[4]. Compact axiomatizations of both primitives and their interaction are presented next as basis for the subsequent axiomatization of the closure operations.

[2] A regular closed region x satisfies $x = \mathrm{cl}(x) = \mathrm{cl}(\mathrm{int}(x))$, that is, the closure is equivalent to the closure of its own interior. All regions being regular closed or regular open is a common requirement throughout mereotopologies [9,20] to avoid regions of mixed dimensions or with lower-dimensional artifacts.

[3] An arc, for example, is homeomorphic to a straight line segment and thus can be embedded in $\mathbb{R}^1$. Likewise, an entity consisting of three arcs radiating from a single point is treated as 1D feature because each arc is individually homeomorphically embeddable in $\mathbb{R}^1$, even though the entire entity cannot be embedded in $\mathbb{R}^1$.

[4] The original axiomatization in [19,17] uses $x <_{\dim} y$ as primitive, but is otherwise logically equivalent.

The Module of Relative Dimensionality The predicate $\leq_{\text{dim}}$ is a reflexive and transitive relation (D'-A1,3), with $<_{\text{dim}}$ and $=_{\text{dim}}$ as definitions (D'-D1,2). For mathematical simplicity, the theory introduces a unique *zero* (or null) *region*, indicated by the unary predicate *ZEX* and treated as having a lower dimension than all other regions. Additional notions of maximal and minimal dimension (apart from the zero region) *MaxDim* and *MinDim* and of covering dimension (i.e, next greater dimension) $\prec_{\text{dim}}$ (D-D5–7) are defined. The module DI'_{linear} ={D'-D1–D4, D-D6–D7, D'-A1–A6}[5] assumes that a lowest dimension exists (D-A6) but axioms that require a highest dimension to exist or the dimensions to be a discrete set are optional in the mereotopological theory [17].

(D'-D1) $x <_{\text{dim}} y \leftrightarrow x \leq_{\text{dim}} y \wedge y \not\leq_{\text{dim}} x$
(D'-D2) $x =_{\text{dim}} y \leftrightarrow x \leq_{\text{dim}} y \wedge y \leq_{\text{dim}} x$
(D-D5) $MaxDim(x) \leftrightarrow \forall y\,[y \leq_{\text{dim}} x]$ (maximal-dimensional entities)
(D-D6) $MinDim(x) \leftrightarrow \neg ZEX(x) \wedge \forall y\,[\neg ZEX(y) \rightarrow y \geq_{\text{dim}} x]$ (minimal nonzero dimensional ent.)
(D-D7) $x \prec_{\text{dim}} y \leftrightarrow x <_{\text{dim}} y \wedge \forall z\,[z \leq_{\text{dim}} x \vee y \leq_{\text{dim}} z]$ (next highest dimension)
(D'-A1) $x \leq_{\text{dim}} x$ ($\leq_{\text{dim}}$ reflexive)
(D'-A3) $x \leq_{\text{dim}} y \wedge y \leq_{\text{dim}} z \rightarrow x \leq_{\text{dim}} z$ ($\leq_{\text{dim}}$ transitive)
(D'-A4) $ZEX(x) \wedge ZEX(y) \rightarrow x = y$ (unique zero region)
(D'-A5) $ZEX(x) \rightarrow x \leq_{\text{dim}} y$ (zero region has lowest dimension)
(D'-A6) $\exists x\,[MinDim(x)]$ (a region of lowest dimension exists)

The Module of Spatial Containment The primitive relation of spatial containment, $Cont(x,y)$, is a reflexive, antisymmetric, and transitive relation (C-A1–A3) with the zero region being defined as the only region not containing itself (C-A4). Contact $C(x,y)$ becomes definable without any dimensional constraints (C-D) as expected (i.e., regions are in contact if they "share a common region") as long as the theory is mereologically closed, that is, for any two regions in contact there is a shared region in the domain. This further motivates the need for the intersection operation. In CO_{basic} ={C-A1–A4, C-D, D'-A4}[6], containment is provably extensional (C-T1), that is, any region is identifiable by its unique set of contained regions. It is also provable that contact is reflexive (except for the zero region, which is not is contact with anything), symmetric, and containment is monotone with respect to contact (C-T2–T5; omitted here).

(C-A1) $\neg ZEX(x) \leftrightarrow Cont(x,x)$ (*Cont* reflexive and definition of *ZEX*)
(C-A2) $Cont(x,y) \wedge Cont(y,x) \rightarrow x = y$ (*Cont* antisymmetric)
(C-A3) $Cont(x,y) \wedge Cont(y,z) \rightarrow Cont(x,z)$ (*Cont* transitive)
(C-A4) $ZEX(x) \rightarrow \forall y\,[\neg Cont(x,y) \wedge \neg Cont(y,x)]$ (zero region never in *Cont* relation)
(C-D) $C(x,y) \leftrightarrow \exists z\,[Cont(z,x) \wedge Cont(z,y)]$ (contact)
(C-T1) $\forall z\,[Cont(z,x) \leftrightarrow Cont(z,y)] \rightarrow x = y$ (*Cont* extensional)

The Combined Theory CODI A single axiom relates the primitive notions: if x is contained in y, then y must have at least the dimension of x (CD-A1). Parthood and proper parthood are defined as unidimensional variants of containment (EP-D,EPP-D) with parthood being extensional (EP-T9)[7]. *Min* and *Max* denote minimal (i.e., indivisible) and

[5] All modules presented here are provided in Common Logic format in COLORE: http://colore.oor.net in the folders multidim_mereotopology_XXX. DI'_{linear}, for example, is axiomatized in http://colore.oor.net/multidim_mereotopology_dim/dim_prime_linear.clif.

[6] http://colore.oor.net/multidim_mereotopology_cont/cont_basic.clif

[7] http://colore.oor.net/multidim_mereotopology_codi/theorems/ep_theorems.clif

maximal (i.e., not a proper part of another region) entities within a dimension (ME-D1,D2). To prove decomposibility later on, we need to restrict the theory to models that are *atomic*, wherein each region must contain some minimal part (ME-E1).

(CD-A1) $Cont(x,y) \rightarrow x \leq_{\text{dim}} y$ (a contained entity is of the same or a lesser dimension)
(EP-D) $P(x,y) \leftrightarrow Cont(x,y) \wedge x =_{\text{dim}} y$ (equidimensional parthood)
(EP-T9) $\forall z[P(z,x) \leftrightarrow P(z,y)] \rightarrow x = y$ (P extensional)
(EPP-D) $PP(x,y) \leftrightarrow P(x,y) \wedge x \neq y$ (equidimensional proper parthood)
(ME-D1) $Max(x) \leftrightarrow \neg ZEX(x) \wedge \forall y[\neg PP(x,y)]$ (maximal in a dimension)
(ME-D2) $Min(x) \leftrightarrow \neg ZEX(x) \wedge \forall y[\neg PP(y,x)]$ (minimal in a dimension)
(ME-E1) $\neg ZEX(x) \rightarrow \exists y[P(y,x) \wedge Min(y)]$ (nonzero regions have a minimal part)

Three specialized symmetric contact relations become definable: partial overlap PO when two regions of equal dimension share a region of the same dimension (e.g., the 2D regions in Fig. 1(b) share a 2D part; PO-D)[8]; incidence SC when a region shares a part with a higher-dimensional region (e.g., in Fig. 1(c) the curve shares a segment with the 2D region, and in Fig. 1(a) the 1D region l is contained in the 2D region r; Inc-D); and superficial contact SC when two regions share a lower-dimensional region (e.g., the 2D regions r and s share 1D region l and point p in Fig. 1(a); SC-D). Together, the three relations are an exhaustive and mutually exclusive set of subrelations of contact [19]. Refer to [17, Ch. 6] for a detailed discussion of the contact relations. We define $CODI = DI'_{\text{linear}} \cup CO_{\text{basic}} \cup$ {CD-A1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1,D2}[9] and $CODI^{\text{at}} = CODI \cup$ {ME-E1} as its atomic version[10].

(PO-D) $PO(x,y) \leftrightarrow \exists z[P(z,x) \wedge P(z,y)]$ (partial overlap)
(Inc-D) $Inc(x,y) \leftrightarrow \exists z[(Cont(z,x) \wedge P(z,y) \wedge z <_{\text{dim}} x) \vee (P(z,x) \wedge Cont(z,y) \wedge z <_{\text{dim}} y)]$ (incidence)
(SC-D) $SC(x,y) \leftrightarrow \exists z[Cont(z,x) \wedge Cont(z,y)] \wedge \forall z[Cont(z,x) \wedge Cont(z,y) \rightarrow z <_{\text{dim}} x \wedge z <_{\text{dim}} y]$ (superficial contact)

4. Extending *CODI* with Downwards Closure Operations

This section presents our technical contributions. We first propose axioms for the closure operations of intersection and relative complement (difference) as extension to *CODI* and $CODI^{\text{at}}$. We then verify that the operations are well-defined functions, i.e. for each pair of elements x, y, only one element qualifies as potential intersection and only one as potential difference, and show that the axioms suffice to guarantee that $CODI^{\text{at}}_{\downarrow}$, the extension of $CODI^{\text{at}}$, satisfies the *decomposability property*. The proposed axioms for the operations are guided by Gott's previous work [16] and Casati and Varzi's detailed study of mereological and topological closure operations in unidimensional mereotopologies [4].

If we want to ensure that the entities resulting from the operations are again of uniform dimension (i.e., each component of such a region is of the same dimension), the closure operations must be necessarily lossy in that they lose pieces of the intersection or difference that are of lower dimensions than other pieces. As the result, the operations violate a number of properties typically associated with product and difference operations in set theory, such as associativity of intersections: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

[8] Our use of *PO* differs from the use in most unidimensional mereotopologies, e.g., the RCC [9] or [4], where *PO* dissallows full containment, a notion expressable here as $PO(x,y) \wedge \neg Cont(x,y) \wedge \neg Cont(y,x)$.

[9] `http://colore.oor.net/multidim_mereotopology_codi/codi.clif`

[10] `http://colore.oor.net/multidim_mereotopology_codi/codi_atomic.clif`

4.1. Intersections

We already have distinguished contact relations depending on the relative dimensionality of the shared region. But this only considers the shared region of greatest dimension. Consider for example Figure 1(a): the two 2D regions r and s share l as a common piece of boundary (a 1D region) but also the point p (a 0D region), which is unrelated from l. Likewise, a 1D curve could properly intersect a 2D region in a curve segment, while also meeting the 2D region at a separate point. The problem with these cases is that the full intersection could consist of scattered pieces of varying dimensions. To yield a valid region we must ensure that the pieces are of uniform dimension. This is achieved by dropping extra lower-dimensional pieces from the intersection, thus defining the intersection $x \cdot y$ as the *maximal intersection of highest dimension* among all entities shared by the regions x and y (Int-A1 – Int-A4). This intersection could still consist of scattered parts (e.g, the intersection between x and y in Fig. 1(c)) as long as they are of equal dimension. All other, lower-dimensional entities are lost unless they are contained in the intersection of highest dimension, in which case they are still contained in the intersection by transitivity of containment.

(Int-A1) $\neg C(x,y) \rightarrow ZEX(x \cdot y)$ (empty intersection)
(Int-A2) $\neg ZEX(x \cdot y) \rightarrow Cont(x \cdot y, x)$ ($x \cdot y$ is contained in the intersecting entities)
(Int-A3) $Cont(z,x) \wedge Cont(z,y) \rightarrow z \leq_{\text{dim}} x \cdot y$
($x \cdot y$ has a dimension greater or equal to all entities contained in both x and y)
(Int-A4) $Cont(z,x) \wedge Cont(z,y) \wedge z =_{\text{dim}} x \cdot y \leftrightarrow P(z, x \cdot y)$ (every entity of the dimension of $x \cdot y$ that is contained in both x and y is a part of $x \cdot y$ and vice versa)

We can now verify that the intersection operation $\cdot$ is fully defined in $CODI \cup \{\text{Int-A1} - \text{Int-A4}\}$, that is, for every pair of elements x, y, one and only one element meets all requirements from Int-A1 – Int-A4 to serve as $x \cdot y$. We only sketch the proofs for Thms. 1 and 2 and refer to the full proofs in [17].

Theorem 1. *The operation $\cdot$ is fully defined in $CODI \cup$ {Int-A1 – Int-A4}.*

Proof Sketch. Let $\mathscr{M}$ be a model with x, y being arbitrary elements from the domain $\text{Dom}(\mathscr{M})$. We need to prove that some z exists such that it can serve as $x \cdot y$ and that this z is unique. Three exhaustive cases are considered: (1) $\langle x,y \rangle \notin \mathbf{C}_{\mathscr{M}}$[11] where $x \cdot y \in \mathbf{ZEX}_{\mathscr{M}}$; (2) $\langle x,y \rangle \in \mathbf{C}_{\mathscr{M}}$ and $\langle z, x \cdot y \rangle \in (\mathbf{<_{dim}})_{\mathscr{M}}$; and (3) $\langle x,y \rangle \in \mathbf{C}_{\mathscr{M}}$ and $\langle z, x \cdot y \rangle \in (\mathbf{=_{dim}})_{\mathscr{M}}$. These are exhaustive because $\langle x,y \rangle \in \mathbf{C}_{\mathscr{M}}$ and $\langle z, x \cdot y \rangle \in (\mathbf{>_{dim}})_{\mathscr{M}}$ is impossible. $\square$

Typical properties of intersection/product operations are: commutativity ($x \cdot y = y \cdot x$), associativity ($(x \cdot y) \cdot z = x \cdot (y \cdot z)$), idempotence ($x \cdot x = x$), existence of an identity 1 ($1 \cdot x = x \cdot 1 = x$), and existence of a zero element 0 ($0 \cdot x = x \cdot 0 = 0$). Commutativity and idempotence are easily proved (Int-T5,T10), but associativity fails in cases where one of the intersections $x \cdot y$ and $y \cdot z$ is nonuniform as in Fig. 1(a). However, the intersection operation satisfies a weaker form of associativity known as the left- and right-alternative laws (Int-T11,T12). The zero region serves as null element (Int-T13), while an identity element would require postulating that a universal region that contains every nonzero region exists: $\exists x \forall y [\neg ZEX(y) \rightarrow Cont(y,x)]$. Int-T6–T9 verify the correct dimensionality

[11]The bold symbols, such as $\mathbf{C}_{\mathscr{M}}$, $\mathbf{ZEX}_{\mathscr{M}}$, or $(\mathbf{<_{dim}})_{\mathscr{M}}$, denote the symbols' extensions in the model $\mathscr{M}$.

of the intersection for the four exhaustive cases of $\neg C$, PO, Inc, and SC. All of Int-T5–T13 are provable from $CODI \cup \{\text{Int-A1} - \text{Int-A4}\}$[12].

(Int-T5) $x \cdot y = y \cdot x$ (intersection commutative)[13]
(Int-T6) $ZEX(x \cdot y) \leftrightarrow \neg C(x,y)$ (zero intersection only for disconnected entities)
(Int-T7) $PO(x,y) \rightarrow x =_{\text{dim}} x \cdot y =_{\text{dim}} y$ (PO: $x \cdot y$ is of the same dimension as both intersecting entities)
(Int-T8) $[Inc(x,y) \wedge x <_{\text{dim}} y] \rightarrow [x \cdot y =_{\text{dim}} x \wedge x \cdot y <_{\text{dim}} y]$ (Inc: $x \cdot y$ has the same dimension as one of the intersecting entities and a lower dimension than the other)
(Int-T9) $SC(x,y) \rightarrow x \cdot y <_{\text{dim}} x \wedge x \cdot y <_{\text{dim}} y$ (SC: $x \cdot y$ is of a lower dim. than both x and y)
(Int-T10) $x \cdot x = x$ ($\cdot$ idempotent)
(Int-T11) $(x \cdot x) \cdot y = x \cdot (x \cdot y)$ ($\cdot$ left-alternative)
(Int-T12) $y \cdot (x \cdot x) = (y \cdot x) \cdot x$ ($\cdot$ right-alternative)
(Int-T13) $ZEX(y) \rightarrow ZEX(x \cdot y)$ (existence of a null element for $\cdot$)

4.2. Differences

Models of unidimensional mereotopologies are usually closed under complementation in the sense that some universal region U exists that contains all other nonzero regions, and every region y has a complement $y' = U - y$. But even without a universal entity, it is reasonable to require that *relative complements* exist: any region y that is a proper part of a region x has a nonzero relative complement $x - y$ if $y \neq x$. Inspired by set theory, we refer to $-$ as the *difference* operations.

To ensure that in a multidimensional mereotopology the difference $x - y$ has a uniform dimension, we again only take the pieces of largest dimension into account, similar to how the intersection operation drops lower-dimensional pieces in the intersection. Thus, whenever the difference is not empty, it has the same dimension as the minuend x (Dif-A1). For equidimensional regions, the difference matches the definitions from unidimensional mereotopology [14,16], as exemplified by the difference between the 2D regions in Fig. 1(b). Likewise, the difference between two 1D regions, e.g. between y and $x \cdot y$ in Fig. 1(c), is a 1D region. This idea extends to differences whose subtrahend is of a greater dimension than the minuend (Dif-A3), as exemplified by the difference $y - x$ between the line segment y and the 2D region x in Fig. 1(c), which is a line segment that is a proper part of y. But when the minuend is of a greater dimension than the subtrahend, the difference must be the minuend itself (Dif-A2), ignoring missing lower-dimensional artifacts (e.g., $x - y = x$ in Fig. 1(c)). This works in general because the intersection never has a greater dimension than either of the intersecting regions. Dif-T7 verifies that the difference between x and y is indeed equivalent to the difference between x and the intersection of x and y. Dif-A4 captures the exact conditions when the difference may be

[12] All theorems in Section 4—indicated by labels of the form XX-Tx—are proved using the automated theorem provers Prover9 and Vampire. The full proofs are provided in COLORE in `http://colore.oor.net/multidim_mereotopology_codi/theorems/` with proofs in the subfolder `output`. For example, the files `codi_int_theorems_goal5.p9.out` to `codi_int_theorems_goal24.p9.out` and similarly, the outputs ending with `.vam.out` contain proofs for Int-T5–T13. Some theorems are proved in multiple parts, for example, to treat the zero entity separately (e.g., Int-T5) or to prove the two directions of the biconditional in Dif-T9. Dif-T6–8 are re-expressed using extensionality of P (EP-T9). See [17] for more details.

[13] Commutativity is proven indirectly from multiple pieces: $\neg ZEX(x \cdot y) \rightarrow P(x \cdot y, y \cdot x))$, $\neg ZEX(y \cdot x) \rightarrow P(y \cdot x, x \cdot y))$, $ZEX(x \cdot y) \rightarrow ZEX(y \cdot x)$, and $ZEX(y \cdot x) \rightarrow ZEX(x \cdot y)$, which together entail commutativity with $\neg ZEX(x) \rightarrow [x = y \leftrightarrow P(x,y) \wedge P(y,x)]$ and D'-A4.

empty: either the minuend is zero or the minuend is contained in the subtrahend. From reflexivity of *Cont* for nonzero regions, it follows that $x - x$ is always the zero region.

We define the theory $CODI_{\downarrow} = CODI \cup \{$Int-A1 – Int-A4, Dif-A1 – Dif-A4$\}$[14] and its atomic variant as $CODI^{at}_{\downarrow} = CODI_{\downarrow} \cup$ ME-E1[15]. Because differences must exist for all pairs of entities, as proven later in Thmeorem 2, $CODI_{\downarrow}$ and $CODI^{at}_{\downarrow}$ both entail that some zero region must exist (Z-T1).

(Dif-A1) $\neg ZEX(x-y) \rightarrow x-y =_{\dim} x$	(dimension of the difference $x-y$)
(Dif-A2) $y <_{\dim} x \rightarrow x-y = x$	(difference $x-y$ for a lower-dimensional y)
(Dif-A3a) $x \leq_{\dim} y \rightarrow [Cont(z,x) \wedge z \cdot y <_{\dim} z \rightarrow Cont(z,x-y)]$	
(Dif-A3b) $x \leq_{\dim} y \rightarrow [Cont(z,x-y) \rightarrow Cont(z,x)]$	
(Dif-A3c) $x \leq_{\dim} y \rightarrow [P(z,x-y) \rightarrow z \cdot y <_{\dim} z]$	
	(Dif-A3a – Dif-A3c: constitution of $x-y$ when y has equal or greater dimension than x)
(Dif-A4) $ZEX(x-y) \leftrightarrow ZEX(x) \vee Cont(x,y)$	
	(zero difference $x-y$ only when x is contained in y or x is the zero region)
(Z-T1) $\exists x[ZEX(x)]$	(existence of a zero region)

We next prove theorems about the behavior of the difference operation that are prerequisites to prove that it is a well-defined function and satisfies the desired supplementation principles from unidimensional mereotopology. The first theorems confirm the relationship of $x-y$ to x and y (Dif-T1–T3), that the intersection $x \cdot y$ and the difference $x-y$ never overlap (Dif-T4), and that the difference can be described in terms of parthood alone (Dif-T5)[16]. Dif-T6,T7 capture the interaction between the intersection and difference operations, ensuring that $x-y = x-(x \cdot y) = x \cdot (x-y)$ holds as expected, where Dif-T6 confirms the *remainder principle* from mereology [28]. Dif-T8–T10 verify that the difference works correctly in borderline case, such as for parts, for equivalent regions, and for regions that only share a lower-dimensional region.

(Dif-T1) $\neg ZEX(x-y) \rightarrow P(x-y,x)$	(a nonempty difference $x-y$ is part of x)
(Dif-T2) $PP(y,x) \rightarrow PP(x-y,x)$	(for a proper part y of x, $x-y$ is also a proper part of x)
(Dif-T3) $\neg PO(x-y,y)$	(y and $x-y$ do not partially overlap)
(Dif-T4) $\neg PO(x-y,x \cdot y)$	($x \cdot y$ and $x-y$ do not partially overlap)
(Dif-T5) $P(z,x-y) \leftrightarrow P(z,x) \wedge \neg PO(z,x \cdot y)$	(parts of the difference $x-y$)
(Dif-T6) $x-y = x-(x \cdot y)$	($x-y$ and $x-(x \cdot y)$ are identical)
(Dif-T7) $x-y = x \cdot (x-y)$	($x-y$ and its intersection with x are identical)
(Dif-T8) $P(y,x) \rightarrow y = x-(x-y)$	($-$ involutory)
(Dif-T9) $x = y \leftrightarrow ZEX(x-y) \wedge ZEX(y-x)$	($-$ anticommutative)
(Dif-T10) $SC(x,y) \rightarrow x-y = x$	(difference between entities in superficial contact)

In set theory, the intersection and difference operations are interdefinable through the equivalence $x \cdot y = x-(x-y)$. Because of $CODI_{\downarrow}$'s lossy operations that ensures all regions have uniform dimensions, the same equivalence does not hold here as demonstrated by x and y in Fig. 1(c): $x \cdot y$ yields a segment of y, while $x-y = x$ because y has a lower dimension than x. Then $x-(x-y) = x-x$, yielding the zero region rather than the lower-dimensional intersection of x and y. Thus, intersections and differences must be axiomatized separately in *CODI* though many of the relationships between the operations from set theory are still preserved.

[14] http://colore.oor.net/multidim_mereotopology_codi/codi_down.clif

[15] http://colore.oor.net/multidim_mereotopology_codi/codi_down_atomic.clif

[16] Recall that *PO* denotes *overlap in a part* rather than *proper overlap* and thus includes full containment. Thus *PO* is the most general overlap relation between unidimensional entities similar to *O* in [4,9,27].

4.3. Supplementation and Extensionality Principles

Important criteria for evaluating unidimensional mereological and mereotopological theories are *supplementation principles* [4,28]. Its weakest form requires each part of a region to be complemented by some other non-overlapping part of the same region (EP-E1). The stronger variant requires every region that is not a part of x to have a proper part that does not overlap x (EP-E2) and thus is also not a part of x. Both are provable in $CODI_{\downarrow}$. EP-E2 can be generalized to multidimensional cases: every region y *not contained in* x has a part z such that the intersection $z \cdot x$ is of a lower dimension than z (EP-E3), meaning that z does not share a part with x. EP-E3 is also provable in $CODI_{\downarrow}$. In fact, all of three principles are consequences of the remainder principle (Dif-T6), which strengthens strong supplementation further. These supplementation principles verify that the axiomatized closure operations properly capture common intuitions about such operations. At the same time, they are a crucial step towards our ultimate goal of establishing *decomposability* for regions in models of $CODI_{\downarrow}^{\text{at}}$ such that every region is identified by a finite sum of minimal parts. Towards this goal, strong supplementation already entails that the partial overlap relation PO is extensional (PO-E1) in $CODI_{\downarrow}$, which ensures that every entity is uniquely identified by its atomic parts.

(EP-E1) $PP(y,x) \rightarrow \exists z\,[P(z,x) \wedge \neg PO(z,y)]$ (weak supplementation)
(EP-E2) $\neg ZEX(y) \wedge \neg P(y,x) \rightarrow \exists z\,[P(z,y) \wedge \neg PO(z,x)]$ (strong supplementation)
(EP-E3) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg Cont(y,x) \rightarrow \exists z\,[P(z,y) \wedge z \cdot x <_{\text{dim}} z]$
(strong supplementation of containment)
(PO-E1) $\forall z[PO(z,x) \leftrightarrow PO(z,y)] \rightarrow x = y$ (*PO* extensional)

PO-E1 and Dif-T5 now help prove that the difference $x - y$ is indeed fully defined for every pair of elements x, y in a model of $CODI_{\downarrow}$.

Theorem 2. *The operation $-$ is fully defined in $CODI_{\downarrow}$.*

Proof Sketch. The proof distinguishes the two exhaustive cases: (1) when $\langle x,y\rangle \in (\boldsymbol{>}_{\mathbf{dim}})_{\mathscr{M}}$ and (2) when $\langle x,y\rangle \in (\boldsymbol{\leq}_{\mathbf{dim}})_{\mathscr{M}}$. The case $\langle x,y\rangle \in (\boldsymbol{\leq}_{\mathbf{dim}})_{\mathscr{M}}$ considers each part z of the entity $x-y$ individually. If $\langle z,y\rangle \in \mathbf{PO}_{\mathscr{M}}$, then $\langle z, x-y\rangle \notin \mathbf{P}_{\mathscr{M}}$, otherwise $\langle z, x-y\rangle \in \mathbf{P}_{\mathscr{M}}$. By extensionality of PO, this uniquely determines the parts of $x-y$. $\square$

4.4. Decomposability of Atomic Models of $CODI_{\downarrow}$

In the atomic models of many unidimensional mereotopologies, each region can be decomposed into a set of minimal, covering and non-overlapping parts if extensionality of PO (PO-E1) and strong supplementation (EP-E2) are satisfied [32]. We now confirm that decomposability, as exemplified in Fig. 2, also works in the atomic version $CODI_{\downarrow}^{\text{at}}$ of our multidimensional mereotopology. Again, PO-E1 and EP-E2 are central to the proof.

In any model of $CODI_{\downarrow}^{\text{at}}$ all nonzero regions contain some minimal, i.e., indivisible, proper part. Proving decomposability then amounts to showing that the models are *atomistic*, that is, proving that every two partially overlapping regions share some *minimal part*. Then, any region is the sum of its minimal parts and those minimal parts uniquely define the region. This also means that each region in contact with a given region x shares a, possibly lower dimensional, region contained in some minimal part of x. In other words, the containment of lower-dimensional regions in models of the multidimensional mereotopology $CODI_{\downarrow}^{\text{at}}$ does not significantly alter the models' structure

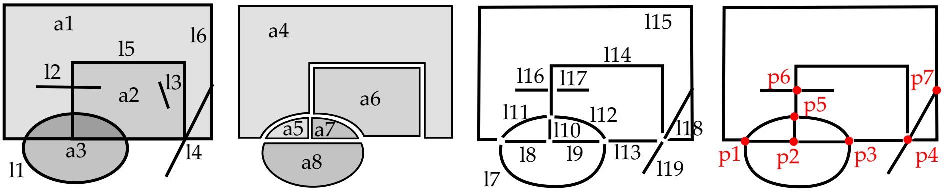

Figure 2. A model of $CODI_{\downarrow}^{\text{at}}$ (left) decomposed by intersections and differences into its atomic entities: points p1–p7, lines l7–l19, and areas a4–a8, shown in the three right figures. Non-atomic entities are sums of atomic ones, e.g., $a3 = a5 \cup a7 \cup a8$ and $l5$ (the boundary of a2) equals to $l9 \cup l10 \cup l14 \cup l13$. Because of the existence of complements, additional non-atomic entities are entailed to exist: $a3 \cdot a1 = a5 \cup a7$ and $l5 - l9 = l10 \cup l14 \cup l13$.

that is still defined—as in most unidimensional mereotopologies— by (equidimensional) parthood. Instead, the additional structure from containment of lower-dimensional works within the confines of the models' parthood structure.

Theorem 3. *Let $\mathscr{M}$ be a model of $CODI_{\downarrow}^{\text{at}}$ with domain* $\mathbf{M}$. *Then every entity $x \in \mathbf{M}$ is uniquely determined by its set of non-overlapping minimal parts that jointly cover x.*

Proof. Let $\mathscr{M}$ be a model of $CODI_{\downarrow}^{\text{at}}$ with domain $\mathbf{M}$.
By ME-E1, we have $\mathscr{M} \models \forall x[\neg ZEX(x) \rightarrow \exists y[P(y,x) \wedge Min(y)]]$

Recall that by PO-E1, every entity $x \in \mathbf{M}$ in the model is uniquely defined by the following extension of PO involving x: $\mathbf{PO}_{\mathscr{M}}(x) = \{\langle x,w\rangle \mid \langle x,w\rangle \in \mathbf{PO}_{\mathscr{M}}\}$.

Choose an arbitrary $x \in \mathbf{M}$. We need to prove that x is uniquely defined by the subset $\mathbf{PO}_{\mathscr{M}}^{Min}(x) \subseteq \mathbf{PO}_{\mathscr{M}}(x)$ relating x to minimal entities:

$$\mathbf{PO}_{\mathscr{M}}^{Min}(x) = \{\langle x,w\rangle \mid w \in \mathbf{Min}_{\mathscr{M}}\} \subseteq \mathbf{PO}_{\mathscr{M}}(x) = \{\langle x,w\rangle \mid \langle x,w\rangle \in \mathbf{PO}_{\mathscr{M}}\}$$

Now suppose there exists another entity $x' \in \mathbf{M}$ with $x' \neq x$ that partially overlaps the same set of minimal entities as x, i.e., $\mathbf{PO}_{\mathscr{M}}^{Min}(x') = \mathbf{PO}_{\mathscr{M}}^{Min}(x)$. From $x' \neq x$, either $\langle x',x\rangle \in \mathbf{PP}_{\mathscr{M}}$ or $\langle x',x\rangle \notin \mathbf{P}_{\mathscr{M}}$. In the former case, some minimal entity $w \in \mathbf{M}$ would exist by ME-E1 such that $w \in \mathbf{PO}_{\mathscr{M}}^{Min}(x)$ and $w \notin \mathbf{PO}_{\mathscr{M}}^{Min}(x)$, contradicting our assumption. In the second case, by EP-E2 some $w \in \mathbf{M}$ exists such that $\langle w,x'\rangle \in \mathbf{P}_{\mathscr{M}}$ and $\langle w,x\rangle \notin \mathbf{PO}_{\mathscr{M}}$. Such a w would—by ME-E1—contain a minimal entity $v \in \mathbf{M}$. Then $\langle x,v\rangle \notin \mathbf{PO}_{\mathscr{M}}^{Min}$ but $\langle x',v\rangle \in \mathbf{PO}_{\mathscr{M}}^{Min}$, which contradicts the assumption that x' and x partially overlap the same minimal entities.

Thus, any region x is uniquely defined by its minimal parts, i.e., by its extension $\mathbf{P}_{\mathscr{M}}^{Min}(x)$, in a model of $CODI_{\downarrow}^{\text{at}}$. Because distinct minimal entities cannot overlap and entities that partially overlap must share a minimal part, the minimal parts of a region x form a spatial partition: they are pairwise non-overlapping and they are exhaustive, i.e., any other region in contact with x must be in contact with one of its minimal parts. □

5. Conclusions

Mereotopology is at the core of representing space qualitatively, but with existing theories being either restricted to a unidimensional view of space wherein all entities must have the same dimension, or limited in their product/intersection and difference/complement operations to entities of equal dimension, which misses many viable inferred entities. This work proposes axiomatic extensions to the multidimensional mereotopology *CODI* that fully define an intersection and difference operation with nonzero results whenever regions are in contact. The operations' results agree with those of other mereotopologies when applied to regions of equal dimensions, corroborating earlier results that show *CODI* can be logically restricted to unidimensional mereotopologies as a special case [18].

The intersection and difference operations satisfy key properties from the set-theoretic intersection and difference operations as evidence that they behave as expected. But two properties cannot be preserved: (1) associativity of intersection and (2) inter-definability of intersection and differences, due to both operations being inherent lossy. However, lossy definitions are a necessary compromise to avoid creating entities of mixed dimensions (i.e. a region with a protruding or missing line segment).

It is further verified that the strong supplementation principle from mereology holds in the extended theory $CODI_{\downarrow}$ and that the partial overlap relation is extensional. This suffices to prove decomposibility of all atomic models—and thus all finite models—of $CODI_{\downarrow}$ with the consequence that any region in such a model can be uniquely represented as the set of its atomic parts. This property increases the theory's practical utility by allowing to represent complex entities of any dimension as sets of simple entities, similar to how geometric data models, e.g., ISO's Simple Features standard [21], define complex linear (e.g., MultiLineString) and areal (e.g., MultiPolygon) features. $CODI_{\downarrow}^{at}$ only requires information about its two primitives, relative dimension and containment, *between its minimal entities* (rather than all entities) to completely represent a model and query its mereotopological relations between simple or complex entities. This is due to the following inferred definitions, which can efficiently be implemented on existing geometric data models:

$$C(x,y) \equiv \exists z[Min(z) \wedge Cont(z,x) \wedge Cont(z,y)] \tag{Contact}$$

$$PO(x,y) \equiv \exists z[Min(z) \wedge P(z,x) \wedge P(z,y)] \tag{Overlap in a part}$$

$$Inc(x,y) \equiv \exists z\big[Min(z) \wedge [P(z,x) \wedge z <_{\text{dim}} y \wedge Cont(z,y)] \vee [P(z,y) \wedge z <_{\text{dim}} x \wedge Cont(z,x)]\big] \tag{Incidence}$$

$$SC(x,y) \leftrightarrow \exists z[Min(z) \wedge Cont(z,x) \wedge Cont(z,y)] \wedge \neg\exists z[Min(z) \wedge P(z,x) \wedge P(z,y))] \tag{Superficial contact}$$

CODI's approach avoids the combinatorial explosion of the number of mereotopological relations in related work [7,8,13,25] wherein each specific dimensional combination (e.g., 0D-1D, 0D-2D, 1D-1D, 1D-2D, etc.) is assigned a separate relation. Instead, *CODI* provides a small, more manageable set of relations. While the relative dimension relations $\leq_{\text{dim}}$ or $<_{\text{dim}}$ seem intuitive, their cognitive adequacy require further study. But the outcome does not affect the usability of $CODI_{\downarrow}$, as the dimension information is already implicitly available from geometric data sources, typically as a class of objects for each dimension, which permits implementing the mereotopological theory on top of or in lieu of geometric models. Future work must also axiomatize sums in a way that takes into account which sums are ontologically warranted or desirable [4,28].

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