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Cost of Stability and Least Core in Path-Disruption Games[1]

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Abstract Path-disruption games are a class of cooperative games in which agents try to block any path from an intruder's source vertex to a target vertex. We study algorithmic properties and characterizations of the least core and the cost of stability for these games.

Keywords. cooperative games, cost of stability, path-disruption games

Introduction

Princess Nectarine, adored ruler of the Shiitake Kingdom, has been living happily alongside her beloved husband and fellow-ruler Tanky… that is, only until recently when her ex-boyfriend Angelo suddenly appeared at the border of her realm, eager to "have a word" with Tanky and herself. Seemingly tired of all the arguments and quarrels she and Angelo were having in the past, Nectarine gathers all the guards, armed tortoises, fire traps, and whatever else she has at her disposal, places them at well-considered spots in her kingdom, and decides to sit out this whole mess of a situation, hoping that Angelo will eventually give up on his futile attempts to reach her home and castle. Even though her situation since the appearance of Angelo is far from being all fun and games, what Princess Nectarine is playing right now is essentially a *path-disruption game*.

Path-disruption games (PDGs, for short) form a class of cooperative games defined by Bachrach and Porat [7] and model scenarios in which a society of players, connected via links forming a network, aims to prevent an intruder from reaching a target node from some source node within the network. A subgroup of players wins a game if they are able to collectively block all paths leading from the source to the target. As to applications, the network—as well as the intruder's intention—may be of various kinds: a hacker trying to gain access to a particular computer in a digital network, a terroristic attacker (or any other suspect) trying

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to reach a certain building via a road network, or a person struggling her way through a social hierarchy so as to reach a certain rewarding level in the hierarchy.

Next to many other communities studying various forms of connectivity, reachability, and reliability in networks including security issues, the analyses of cooperative game representations on networks have developed recently, considering, for instance, network flow games [14,8] and reliability games [5,4]. Cooperative game theory allows us to treat single nodes as individuals, called *players* or *agents*, that can team up in order to achieve a common goal. In the scenario underlying path-disruption games, their goal is to disconnect a source node from a target node in a network by disrupting all possible paths. Being concerned with the roles that coalitions or single agents play in such a setting, it is common to study certain solutions concepts. For instance, one might ask which players are most useful in order to disrupt a path and what structural properties of the graph lead to them being useful. Path-disruption games have further been studied, e.g., by Aziz et al. [1] and Rey et al. [18]. Here, we are interested in stability concepts such as the core of a game. The core of a cooperative game is the set of all imputations stabilizing the grand coalition, inasmuch as no coalition has an incentive to leave the grand coalition when the players are paid according to such an imputation. In other words, it holds that the combined payoff of each coalition's members is equal to or greater than the value of the respective coalition.

To return to our initial example, Princess Nectarine might be interested in having all of her troops engaged in repelling Angelo, instead of only some of her armed forces, due to them expecting a greater reward. She can achieve this by paying every soldier at least as much as they would be able to earn in any other coalition than the great one—if this is possible at all.

The core of a PDG, however, is not guaranteed to be nonempty. Therefore, in the following we study the computational aspects of the least core and of the cost of stability for these games. Both concepts provide a means to stabilize the grand coalition in a game whose core is not empty. Informally speaking, the least core [15], on the one hand, is defined via charging a fee if a coalition has an incentive to deviate; this notion has been studied in relation to various games, for instance in the related threshold network flow games [2], as well as in parts for path-disruption games [7]. The cost of stability [3], on the other hand, provides subsidies if and only if the grand coalition is formed (see also [17,6,16]).

Path-disruption games as treated in this paper come in two flavors: with costs (henceforth *PDGCs*) and without (henceforth *PDGNs*). The latter of the two is the most basic case were coalitions are rewarded solely on the basis of their ability to separate the source from the target. The variant with costs, on the other hand, considers the participation of each player to be costly. In real-life applications, this cost may be interpreted as additional resources or risks that are associated with the occupation of strategic points in a network.

For both cases, and the three stability concepts, we obtain results on the complexity of computational problems regarding the core and its extensions. For the no-cost case, closed formulas will be derived, by means of which some desired properties can be calculated. If there are costs involved, we model finding the least core and the cost of stability, respectively, as linear optimization problems. Although this approach does not yield concise formulas, additional insight into

PDGCs is gained by restating the problems to be solved as geometrical ones. Moreover, we will investigate how likely it is for a path-disruption game without costs, played on a scale-free network, to have a nonempty core.

1. Preliminaries

The tool of analysis is cooperative game theory—the formal study of coalition formation by self-interested agents. Its main object of investigation is a *(cooperative) game*. A *(cooperative) game* is a tuple $\mathcal{G} = (N, v)$, where $N = \{1, \ldots, n\}$ is called a set of *players* or *agents* and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *valuation function*. A subset $C \subseteq N$ is called a *coalition* of players. N is called the *grand coalition*. Players are intended to be self-interested which, in the present context, means that they want to maximize their individual payoff. An n-tuple $(p_1, \ldots, p_n) \in \mathbb{R}^n_{\geq 0}$ is called a *payoff vector* if $\sum_{i \in N} p_i \leq v(N)$. The *payoff of coalition* $C \subseteq N$ is denoted by $p(C) = \sum_{i \in C} p_i$. Centered around the payoff there are several solution concepts providing criterions of stability or fairness for a cooperative game. This work will focus on the *core* of a game and its derivations. For a cooperative game $\mathcal{G} = (N, v)$, its *set of imputations* is defined by

$$\mathcal{I}(\mathcal{G}) = \left\{ p \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{i \in N} p_i = v(N) \;\wedge\; \forall i \in N : p_i \geq v(\{i\}) \right\}.$$

The *core of* $\mathcal{G}$ consists of those imputations that guarantee a stable grand coalition, that is, no coalition has an incentive to deviate from the grand coalition:

$$\text{Core}(\mathcal{G}) = \{p \in \mathcal{I}(\mathcal{G}) \mid \forall C \subseteq N : p(C) \geq v(C)\}.$$

Both the set of imputations and the core of a given game can be empty. In order to cope with games of such sort, the weaker notions of the *least core* [15] and the *cost of stability* [3] have been introduced. Both aim to make coalitions other than the grand coalition less attractive by imposing a fee on all coalitions or introducing an additional reward for joining the grand coalition, respectively.

The first idea is captured by the notion of the ε-core. For a cooperative game $\mathcal{G} = (N, v)$ and some ε the *ε-core of* $\mathcal{G}$ is defined as

$$\varepsilon\text{-Core}(\mathcal{G}) = \left\{ p \in \mathbb{R}^n_{\geq 0} \;\middle|\; \forall C \subseteq N : p(C) \geq v(C) - \epsilon \right\}.$$

Let $\epsilon(\mathcal{G})$ denote the smallest $\varepsilon \geq 0$ for which the ε-core of $\mathcal{G}$ is not empty. Then the ε-core of $\mathcal{G}$ is called the *least core of* $\mathcal{G}$.

The second approach is offered by the cost of stability. Instead of distributing the burden to stabilize the grand coalition among all coalitions, a subsidy is added to the value of the grand coalition, thus increasing the amount of what can be distributed among the players provided they stick to the grand coalition. The term *cost of stability* then refers to the smallest such subsidy which results in a nonempty core. For a cooperative game $\mathcal{G} = (N, v)$ and some $\Delta \geq 0$, $\mathcal{G}(\Delta) = (N, v_\Delta)$ denotes the *adjusted cooperative game* with

$$v_\Delta(C) = \begin{cases} v(C) & C \neq N \\ v(C) + \Delta & C = N. \end{cases}$$

The value of the smallest Δ such that $\text{Core}\,(\mathcal{G}(\Delta)) \neq \emptyset$ is called the *cost of stability of* $\mathcal{G}$ and will be denoted as $\text{CoS}(\mathcal{G})$.

Before we can eventually introduce path disruption games, some graph-theoretic definitions are required, as in these games the players are placed on the nodes of a graph. A *(directed) graph* is a pair $G = (V, E)$ such that V is a set of *nodes* and $E \subseteq V \times V$ is a set of *edges*. An *undirected graph* is a directed graph such that for all $v, w \in V$ it holds that $(v, w) \in E$ if and only if $(w, v) \in E$. Let $G = (V, E)$ be a graph. A *path* from v_1 to v_ℓ is a subgraph $P = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, such that there exists a sequence of pairwise distinct edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{\ell-1}, v_\ell) \in E'$. Two nodes $s, t \in V$ are *connected* if and only if there exists a path from s to t. Furthermore, G is connected if and only if every pair of nodes $v, w \in V$ is connected. A further graph-theoretic concept which is crucial in order to talk about path-disruption games is that of s-t cuts.

Definition 1 (s-t cut) *Let $G = (V, E)$ be a graph and $s, t \in V$ such that s and t are connected. Then a set $S \subseteq V \setminus \{s, t\}$ is called an s-t* cut *if and only if the removal of the nodes in S from V (and of their incident edges from E) disconnects s and t.*

We can now give a detailed definition of path disruption games. As already mentioned, these games come in two flavors: without costs and, more generally, with costs.

Definition 2 (path-disruption game without costs) *Let $G = (V, E)$ be an undirected graph. A* path-disruption game without costs (PDGN) *is a cooperative game represented by a structure $\mathcal{G} = (N, v, G, s, t)$ such that*

- *$s, t \in V$,*
- *$N = V \setminus \{s, t\}$ is a set of players, and*
- *for all coalitions $C \subseteq N$, we have* $v(C) = \begin{cases} 1 & \text{if } C \text{ is an } s\text{-}t \text{ cut} \\ 0 & \text{otherwise.} \end{cases}$

Example 3 *To clarify things, Figure 1 shows a simple example of a PDGN. Here, the players are the nodes 1 through 5. Winning coalitions are those subsets of players whose removal would result in a graph without any path from s to t left. For instance, $\{1, 2\}$, $\{3, 4, 5\}$, and $\{1, 2, 4\}$ are winning, but neither $\{1, 3\}$ nor $\{2, 4, 5\}$ are winning.*

Another observation about this particular game is that its core is empty. This results from the fact that the grand coalition trivially has a value of 1 and that there are two disjoint winning coalitions, namely $\{1, 2\}$ and $\{3, 4, 5\}$. Since these also have a value of 1, one cannot distribute the value of the grand coalition among the players of these coalitions such that both $p(\{1, 2\}) \geq 1$ and $p(\{3, 4, 5\}) \geq 1$. Furthermore, we have that the $\frac{1}{2}$-core of this game is its least core and that the cost of stability is 1. The analyses in Section 2 will shed more light on how to calculate these values and on how to determine emptiness of the core in the first place.

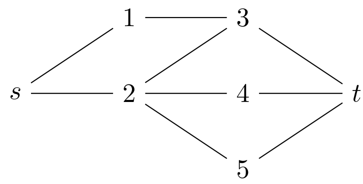

Figure 1. Path-disruption game without costs from Example 3

In real-life scenarios, however, the occupation of nodes may incur costs. To model such scenarios, PDGCs offer a cost-based generalization of PDGNs where the value of a coalition that blocks all paths from the source to the target accrues from the cost of the coalition deducted from a reward that is constant for all coalitions. The cost of a coalition, however, is not equal to the sum of the costs of its members but only to that of a cost-minimizing subset of the coalition.

Definition 4 (path-disruption game with costs) *Let $G = (V, E)$ be an undirected graph. A* path-disruption game with costs (PDGC) *is a cooperative game represented by a structure $\mathcal{G} = (N, v, G, s, t, c, r)$, such that*

- $s, t \in V$,
- $N = V \setminus \{s, t\}$,
- $c \in \mathbb{R}^n_{\geq 0}$ *is a* cost vector,
- $r \in \mathbb{R}_{\geq 0}$ *is a* reward, *and*
- *for all coalitions $C \subseteq N$, we have*

$$v(C) = \begin{cases} r - \min_{C' \subseteq C : \tilde{v}(C')=1} \sum_{i \in C'} c_i & \text{if } \tilde{v}(C) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{v}$ is the valuation of the underlying PDGN $\tilde{\mathcal{G}} = (N, \tilde{v}, G, s, t)$.

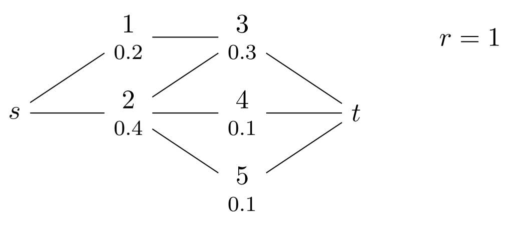

Figure 2. Path-disruption game with costs from Example 3

Example 5 *Figure 2 extends the example shown in Figure 1 from Example 3, now with costs assigned to each node and a reward of* 1*. The numbers under the players' names depict their respective cost. Here, the values of* $\{1, 2\}$ *and* $\{3, 4, 5\}$ *are* $1 - 0.6 = 0.4$ *and* $1 - 0.5 = 0.5$*, respectively. The value of* $\{1, 2, 3\}$*, however, is* 0.4 *as well, since* $\{1, 2\}$ *is a subset of this coalition that constitutes an s-t cut with minimal costs. Using the same reasoning, the value of the grand coalition is*

0.5*, since only the costs of the coalition* $\{3, 4, 5\}$ *(being the minimal-cost s-t cut contained in the grand coalition) contribute to its total value.*

As in the former example, here again the core is empty since there are two disjoint coalitions, namely $\{1, 2\}$ *and* $\{3, 4, 5\}$*, whose values are in sum greater than the value of the grand coalition, i.e.,* 0.9 *vs.* 0.5*. We will return to this matter in Section 4, where we will also see a method for how to infer the least core and the cost of stability of PDGCs.*

As seen above, there are several occasions where we need a method to compute an s-t cut with minimum weight with respect to some function that assigns a value to all nodes in V. The weight of such a cut, as well as the cut itself, can be computed by transforming the graph at hand into a flow network and running a max-flow algorithm on this derived graph, see, e.g., [1]. For more details on flow networks and max-flow algorithms see, e.g., [10].

Throughout this paper, two assumptions will be made: n will always, if not explicitly stated otherwise, denote the cardinality of the set of players N. Moreover, we will always assume that $v(N) > 0$.

2. Path-Disruption Games without Costs

It is well known that, for simple games (i.e., for games satisfying that $v(C) \in \{0, 1\}$ for each coalition C and that v is monotonic, which means that all subsets of a losing coalition must be losing as well), the concept of the core is closely related to that of *veto players*: These are agents whose removal from any otherwise winning coalition leads to its loss. Formally, a player $i \in C$ is called a *veto player* if she is critical for all winning coalitions $C \subseteq N$, that is, $v(C) = 1$ and $v(C \setminus \{i\}) = 0$. The core of a simple game $\mathcal{G}$ is nonempty if and only if there are veto players. And if there are veto players, Core$(\mathcal{G})$ consists precisely of those imputations p such that $p_i = 0$ if i is not a veto player. This is a powerful result, as it does not only provide us with a necessary and a sufficient condition for the nonemptiness of the core in simple games, it also tells us how to distribute the payoff among the players in order to construct a payoff vector in the core. Now, the existence of veto players in PDGNs can be shown in a similar manner.

Lemma 6 *Given a PDGN* $\mathcal{G} = (N, v, G, s, t)$*, an agent* $i \in N$ *is a veto player if and only if* (s, i) *and* (i, t) *are edges in* G*.*

Proof Clearly, every winning coalition C has to include i, as otherwise there would be a path from s to t that is not blocked. On the other hand, assume that there are only paths from s to t with length greater than 2 and that there exists a veto player i. Then $v(N) = 1$ and $v(N \setminus \{i\}) = 0$. This entails that there is a path from s to t via i that is not blocked by any player. But since $N \setminus \{i\}$ consists of every other player, it follows contradictorily that (s, i) and (i, t) must be edges of G. Hence, there cannot be veto players if there are no paths of length 2. ❑

As an immediate consequence, we have the following result.

Theorem 7 *The core of a PDGN $\mathcal{G} = (N, v, G, s, t)$ is nonempty if and only if there exists a player $i \in N$ such that (s, i) and (i, t) are edges in G.*

Since the nonemptiness of the core depends on the existence of paths of length two between s and t, the existence of an imputation in the core is not guaranteed. Here, we are interested in two weaker stability concepts, the least core, and the cost of stability.

Before we turn to the special case of PDGNs, let us first consider simple games in general. In these, the maximum payoff of the grand coalition is 1 and hence just as high as for any other winning coalition. Now, if there are two winning coalitions $C, D \subseteq N$, with $C \neq D$ and $C \cap D \neq \emptyset$, and the grand coalition is also winning, we can distribute the value of N in such a way that $p(C \cap D) = v(N)$ and thus $p(C) = p(D) = p(N) = 1 \geq v(C) = v(D)$. Therefore, given that C, D and N are the only winning coalitions, no fee needs to be imposed on either of them in order to prevent them from deviating from the grand coalition. However, if C and D were disjoint, there would consequently be no nonempty set with agents from both coalitions which the payoff could be distributed to. And because of this, if $p(C) = 1$ then $p(D) = 0$, and vice versa. From here we can already presume that the least ϵ is somehow bounded below by a function of the number of pairwise disjoint winning coalitions.

In order for us to arrive at an upper bound for ϵ, it is useful to observe that since a veto player i is a member of every winning coalition, it is possible to pay all the value of the grand coalition to her and to ensure this way that for all winning coalitions $C \subseteq N$ we have $p(C) = 1 \geq v(C)$, because $p(C) = p(\{i\})$. However, even if there aren't any veto players, it is still possible to find a set that nonvacuously intersects every winning coalition. By paying the minimum necessary value of a coalition, i.e., $1 - \epsilon$, to each of these players, $p(C) \geq v(C) - \epsilon$ is ensured for every $C \subseteq N$. This line of reasoning suggests the assumption that the least ϵ for the ϵ-core to be nonempty is bounded from above by a function of the cardinality of the least set that intersects all winning coalitions.

The intuition of the two preceding conjectures is captured in the following lemmas. Both are adaptations of Theorems 1 and 2, respectively, in the work of Resnick et al. [17], which give a lower and an upper bound for the cost of stability in simple games (see also Lemma 13 below).

Lemma 8 *Let $\mathcal{G} = (N, v)$ be a simple game with $v(N) = 1$. If there exist σ pairwise disjoint winning coalitions, then*

$$\epsilon(\mathcal{G}) \geq \frac{\sigma - 1}{\sigma}.$$

Proof Let $C_1, \dots, C_\sigma \subseteq N$ be pairwise disjoint winning coalitions and assume, for the sake of contradiction, $\epsilon(\mathcal{G}) < \frac{\sigma-1}{\sigma}$. Set $\epsilon = \epsilon(G)$. Then, for all k, $1 \leq k \leq \sigma$, we have that $v(C_k) = 1 - \epsilon > 1/\sigma$. But since the $C_1, \dots, C_\sigma$ are pairwise disjoint and $v(N) = 1$, there has to be at least one winning coalition C_k with $p(C_k) \leq 1/\sigma$, which is a contradiction. Hence, $\epsilon(\mathcal{G}) \geq \frac{\sigma-1}{\sigma}$. ❑

Lemma 9 *Let $\mathcal{G} = (N, v)$ be a simple game with $v(N) = 1$. For any two coalitions $S \subseteq N$ and $C \subseteq N$, if $v(C) = 1$ and $C \cap S \neq \emptyset$, then $\epsilon(\mathcal{G}) \leq \frac{|S|-1}{|S|}$.*

Proof Let $\epsilon = \frac{|S|-1}{|S|}$ and $p = (p_1, \dots, p_n)$ be such that for each $i \in N$, $p_i = \frac{1}{|S|}$ if $i \in S$, and $p_i = 0$ otherwise. Note that p is an imputation, since $p(N) = 1 = v(N)$ and $p_i \geq v(\{i\})$. Note also that p lies in the least ε-core of $\mathcal{G}$, because due to the properties of S, for every $C \subseteq N$ it is true that $p(C) \geq v(C) - \epsilon = {}^1/_{|S|}$. Therefore, $\epsilon(\mathcal{G}) \leq \frac{|S|-1}{|S|}$. ❑

Clearly, if we find a maximum set of pairwise disjoint winning coalitions and draw one element from each, we have $\sigma = |S|$ and hence know the value of the least ϵ for the ε-core to be nonempty. Luckily, in PDGNs this can be achieved easily because it suffices to compute the length of a shortest path from s to t, as will be shown by the following lemma.

Lemma 10 *Let $\mathcal{G} = (N, v, G, s, t)$ be a PDGN and let $\ell_{s,t}$ denote the length of a shortest path $\lambda_{s,t}$ from s to t. Then*

1. *the set of nodes of $\lambda_{s,t}$ nonvacuously intersects every winning coalition, and*
2. *there are $\ell_{s,t} - 1$ pairwise disjoint winning coalitions.*

Proof The first statement is true, since if there were a winning coalition without some node from $\lambda_{s,t}$, there would be a path from s to t that is not blocked. Hence, $\lambda_{s,t}$ and above all every path from s to t intersects any winning coalition in some node.

Now, it remains to be shown that there are at least $\ell_{s,t} - 1$ pairwise disjoint winning coalitions. For this purpose, consider the following procedure.

1. Select all nodes adjacent to s and call this set A.
2. Add an edge (s, x) to the edge set E of G, for every $x \neq s$ adjacent to any node in A.
3. Set N to $N \setminus A$ and remove all edges from E, that are incident to any element in A.
4. Repeat until $(s, t) \in A$.

In the beginning of each iteration, let λ be a shortest path from s to t, let λ_d be the node on λ with distance d from s, and let $\ell'_{s,t}$ be the length of λ. In the first step of the procedure above, $\lambda_1 \in A$ holds and for each $d > 1$, we have $\lambda_d \notin A$, because otherwise λ would not be a shortest path from s to t. In the second step, there are three kinds of nodes that are now being made adjacent to s: nodes in A, nodes in $N \setminus (A \cup \{\lambda_d\}_{1<d\leq\ell'_{s,t}})$, and d_2. It is crucial to note that there will be no edge from s to any λ_d other than λ_2, since s is only newly connected to nodes with former distance 2 to itself. This fact in particular ensures that (s, t) will not be added before the $\ell_{s,t}$-th iteration of the procedure, since $t = \lambda_{\ell_{s,t}}$.

It is obvious that the sets A, except for the one containing t, are s-t cuts that are disjoint from one another. Because the procedure is ensured to iterate $\ell_{s,t}$ times, it yields $\ell_{s,t} - 1$ disjoint s-t cuts, i.e., winning coalitions. ❑

From the application of Lemma 10 to Lemma 8 (lower bound) and to Lemma 9 (upper bound), we can directly infer the following theorem.

Theorem 11 *Let $\mathcal{G} = (N, v, G, s, t)$ be a PDGN and let $\ell_{s,t}$ be the length of the shortest s-t-path in G. It holds that*

$$\epsilon(\mathcal{G}) = \frac{\ell_{s,t} - 2}{\ell_{s,t} - 1}.$$

It might be of interest that the above discussion provides an alternative proof of Lemma 6, for in the case of a veto player being present, the length of the shortest path is 2, thus $\epsilon(\mathcal{G}) = \frac{2-2}{1} = 0$.

As an additional result, we obtain that the least ϵ for a PDGN to have a nonempty ϵ-core can be computed in polynomial time, as the shortest path length in an unweighted graph can be determined via breadth-first search in time linear in n. Given $\epsilon(\mathcal{G})$, a payoff vector that lies in the least core of $\mathcal{G}$ can be constructed by finding a minimal set S that nonvacuously cuts all winning coalitions and setting $p_i = 1 - \epsilon(\mathcal{G})$ for all i in S. The other way around, if we want to verify whether a given payoff p is in the least core, we can employ a mincut/maxflow algorithm to find a winning coalition C with the smallest $p(C)$, and if $p(C) - \epsilon(\mathcal{G}) \geq 0$ then p lies in the least core, otherwise it does not. The following result summarizes this discussion.

Theorem 12 *Let $\mathcal{G}$ be a PDGN. Then the following statements hold true.*

1. *The value $\epsilon(\mathcal{G})$ can be computed in polynomial time.*
2. *A payoff vector that lies in the least core of $\mathcal{G}$ can be found in polynomial time.*
3. *For a given payoff p, whether or not it lies in the least core of $\mathcal{G}$ can be verified in polynomial time.*

Next, we will focus on the related concept of the cost of stability of PDGNs. We will see that the cost of stability can be obtained in a way that is very similar to the computation of the least core.

Bachrach et al. [3] establish upper and lower bounds on the cost of stability in general coalitional games and for certain important subclasses, such as superadditive games. Resnick et al. [17] provide the following upper and lower bounds for the cost of stability in simple games. We will restrict these findings for our purposes to simple games with $v(N) = 1$, hence diminishing each bound by 1.

Lemma 13 (Resnick et al. [17]) *Let $\mathcal{G}$ be a simple game with $v(N) = 1$.*

1. *If there exist σ pairwise disjoint winning coalitions, then* $\mathrm{CoS}(\mathcal{G})) \geq \sigma - 1$.
2. *For any two coalitions $S \subseteq N$ and $C \subseteq N$, if $v(C) = 1$ and $C \cap S \neq \emptyset$, then* $\mathrm{CoS}(\mathcal{G}) \leq |S| - 1$.

From Lemma 2 it can be concluded that the cost of stability of a general simple game is bounded above by the cardinality of the smallest set S that nonvacuously intersects every winning coalition. On a side note, this is in particular true for sets containing exactly one veto player.

As we have already seen in Lemma 10, if $\ell_{s,t}$ denotes the length of a shortest s-t-path, $\ell_{s,t} - 1$ is equal to both the maximum number of pairwise disjoint winning coalitions and the cardinality of the smallest set of players that nonvacuously intersects each winning coalition.

Theorem 14 *Let $\mathcal{G} = (N, v, G, s, t)$ be a PDGN and let $\ell_{s,t}$ denote the length of a shortest path from s to t in G. Then* $\mathrm{CoS}(\mathcal{G}) = \ell_{s,t} - 2$.

We therefore have that the cost of stability can be computed simply by finding a shortest path and consequently in polynomial time. In order to construct a payoff p that lies in $\mathrm{Core}\,(\mathcal{G}(\mathrm{CoS}(\mathcal{G})))$, it suffices to set $p_i = 1$ for every $i \in S$, where S is a minimal set that nonvacuously intersects all winning coalitions. Verification of $p \in \mathrm{Core}\,(\mathcal{G}(\mathrm{CoS}(\mathcal{G})))$ is, as before, achieved by finding the coalition C with the least payoff by utilizing an algorithm that finds a minimum-weight s-t cut for a graph G and checking whether $p(C) \geq 1$.

Theorem 15 *Let $\mathcal{G}$ be a PDGN. Then the following statements hold true.*

1. *The value* $\mathrm{CoS}(\mathcal{G})$ *can be computed in polynomial time.*
2. *A payoff vector that lies in* $\mathrm{Core}\,(\mathcal{G}(\mathrm{CoS}(\mathcal{G})))$ *can be found in polynomial time.*
3. *For a given payoff p, whether or not it lies in* $\mathrm{Core}\,(\mathcal{G}(\mathrm{CoS}(\mathcal{G})))$ *can be verified in polynomial time.*

Furthermore, as a corollary, we can infer the following relation between $\epsilon(\mathcal{G})$ and $\mathrm{CoS}(\mathcal{G})$ in PDGNs.

Corollary 16 *For each PDGN $\mathcal{G}$,*

$$\epsilon(\mathcal{G}) = \frac{\mathrm{CoS}(\mathcal{G})}{\mathrm{CoS}(\mathcal{G}) + 1}.$$

3. Extract: Application to Scale-Free Networks

We apply the results of the preceding section to the special case of *scale-free networks*. This graph class has recently gained attention due to its supposed universality, in the sense that the existence of networks of this kind can be observed in domains as diverse as metabolic networks (see, e.g., [13]), human interactions (see, e.g., [11]), or natural language grammar (see, e.g., [12]). They are characterized by a degree distribution that follows a power law, resulting in many nodes having a small degree and only few—but all the more important—nodes having a very high degree. The latter nodes, called *hubs*, are responsible for very short path lengths between nodes in comparison to the overall network size. In particular, their degree distribution can be described by a power law $p_k \sim k^{-\gamma}$, where $\gamma \in \mathbb{R}_{\geqslant 0}$ and p_k denotes the probability of a randomly chosen node having a degree of k. Usually, scale-free-ness is only assumed if $2 < \gamma < 3$ (see, e.g., [9]).

We now briefly discuss our investigation on the implications that these properties of scale-free networks have on path-disruption games without costs.

By conducting numerical simulations, we have been able to show that the average path length, and hence the cost of stability, grows exponentially as a function of the exponent γ but is hardly affected by the number of nodes in the graph. Additionally, the probability of paths from the source to the target having length 2, and thus the probability of the game having a nonempty core, is with about 60% very likely on networks with $\gamma \approx 2$. It tends, however, exponentially towards 0, which means that nonemptiness of the core is in general very unlikely even if restricted to γ values with $2 < \gamma < 3$.

4. Extract: Path-Disruption Games With Costs

In order to show that the value $\epsilon(\mathcal{G})$ can be computed in polynomial time, Aziz and Sørensen [1] employ the ellipsoid method. Using a separation oracle similar to theirs, we obtain the following results for the cost of stability.

Theorem 17 *Let $\mathcal{G}$ be a PDGC. Then the following statements hold true.*

1. *The value* $\mathrm{CoS}(\mathcal{G})$ *can be computed in polynomial time.*
2. *A payoff vector that lies in* $\mathrm{Core}\,(\mathcal{G}(\mathrm{CoS}(\mathcal{G})))$ *can be found in polynomial time.*
3. *For a given payoff p, whether or not it lies in* $\mathrm{Core}\,(\mathcal{G}(\mathrm{CoS}(\mathcal{G})))$ *can be verified in polynomial time.*

5. Conclusion

First, we have studied three solution concepts with regard to path-disruption games without cost. The least core and the cost of stability are computable from the length of the shortest path between the source and the target. The verification problem for the cost of stability is also decidable in polynomial time. Second, we have investigated how these solution concepts behave under a scale-free regime. To sum up, despite their small average path length, scale-free networks can not generally be expected to cause a nonempty core. Third, we were concerned with the least core, and the cost of stability of path-disruption games with costs. For the nonemptiness of the core it is necessary that all coalitions with a value equal to the value of the grand coalition nonvacuously intersect all non-zero-valued coalitions. The cost of stability as well as the corresponding verification problem can be computed in polynomial time.

There are still open challenges for future treatments of PDGNs, considering that we only discussed three out of many solution concepts. For instance, we conjecture the significance of a player as measured by indices such as the Shapley value [19], to be closely related to the centrality of a player in a path-disruption game, e.g., players on hubs in a scale-free graph are likely to be significant.

Path-disruption games with costs provide a variety of interesting open questions. Furthermore, other classes of graphs (such as small-world graphs, which are typical for social networks) may be worth investigating.

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References

[1] H. Aziz and T. Sørensen. Path coalitional games. Technical Report arXiv:1103.3310v2 [cs.GT], ACM Computing Research Repository (CoRR), April 2011.

[2] Y. Bachrach. The least-core of threshold network flow games. In *Proceedings of the 36th International Symposium on Mathematical Foundations of Computer Science*, pages 36–47. Springer Berlin Heidelberg, 2011.

[3] Y. Bachrach, E. Elkind, R. Meir, D. Pasechnik, M. Zuckerman, J. Rothe, and J. Rosenschein. The cost of stability in coalitional games. In *Proceedings of the 2nd International Symposium on Algorithmic Game Theory*, volume 5814 in Lecture Notes in Computer Science, pages 122–134. Springer-Verlag, October 2009.

[4] Y. Bachrach, O. Lev, S. Lovett, J. Rosenschein, and M. Zadimoghaddam. Cooperative weakest link games. In *Proceedings of the 13th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 589–596. IFAAMAS, May 2014.

[5] Y. Bachrach, R. Meir, M. Feldman, and M. Tennenholtz. Solving cooperative reliability games. In *Proceedings of the 27th Conference on Uncertainty in Artificial Intelligence*, pages 27–34. AUAI Press, July 2011.

[6] Y. Bachrach, R. Meir, M. Zuckerman, J. Rosenschein, and J. Rothe. The cost of stability in weighted voting games. In *Proceedings of the 8th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 1289–1290. IFAAMAS, May 2009.

[7] Y. Bachrach and E. Porat. Path disruption games. In *Proceedings of the 9th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 1123–1130. IFAAMAS, May 2010.

[8] Y. Bachrach and J. Rosenschein. Power in threshold network flow games. *Journal of Autonomous Agents and Multi-Agent Systems*, 18(1):106–132, 2009.

[9] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, October 1999.

[10] T. Cormen, C. Leiserson, R. Rivest, and C. Stein. *Introduction to Algorithms.* The MIT Press, Cambridge, MA, USA, 3rd edition, 2009.

[11] D. de Solla Price. Networks of scientific papers. *Science*, 149(3683):510–515, 1965.

[12] R. Ferrer i Cancho, R. Solé, and R. Köhler. Patterns in syntactic dependency networks. *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, 69(5):051915, 2004.

[13] H. Jeong, B. Tombor, R. Albert, Z. Oltvai, and A. Barabási. The large-scale organization of metabolic networks. *Nature*, 407(6804):651–654, October 2000.

[14] E. Kalai and E. Zemel. Totally balanced games and games of flow. *Mathematics of Operations Research*, 7(3):476–478, 1982.

[15] M. Maschler, B. Peleg, and L. Shapley. Geometric properties of the kernel, nucleolus, and related solution concepts. *Mathematics of Operations Research*, 4(4):303–338, 1979.

[16] R. Meir, Y. Zick, E. Elkind, and J. Rosenschein. Bounding the cost of stability in games over interaction networks. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence*, pages 690–696, AAAI Press, July 2013.

[17] E. Resnick, Y. Bachrach, R. Meir, and J. Rosenschein. The cost of stability in network flow games. In *Proceedings of the 34th International Symposium on Mathematical Foundations of Computer Science*, volume 5734 in Lecture Notes in Computer Science, pages 636–650. Springer-Verlag, August 2009.

[18] A. Rey, J. Rothe, and A. Marple. Path-disruption games: Bribery and a probabilistic model. *Theory of Computing Systems*, 2016. To appear. Previous versions appeared at the conferences ADT'11, ECAI'12, STAIRS'12, and AAMAS'14.

[19] L. Shapley. A value for n-person games. In H. Kuhn and A. Tucker, editors, *Contributions to the Theory of Games*, volume II of *Annals of Mathematics Studies 40*. Princeton University Press, 1953.