Modeling Single-Peakedness for Votes with Ties

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Abstract. Single-peakedness is one of the most important and well-known domain restrictions on preferences. The computational study of single-peaked electorates has largely been restricted to elections with tie-free votes, and recent work that studies the computational complexity of manipulative attacks for single-peaked elections for votes with ties has been restricted to nonstandard models of single-peaked preferences for top orders. We study the computational complexity of manipulation for votes with ties for the standard model of single-peaked preferences and for single-plateaued preferences. We show that these models avoid the anomalous complexity behavior exhibited by the other models. We also state a surprising result on the relation between the societal axis and the complexity of manipulation for single-peaked preferences.

Keywords. computational social choice, single-peaked preferences, votes with ties, elections

1. Introduction

Elections are a general and widely used framework for preference aggregation in human and artificial intelligence applications. An important negative result from social choice theory, the Gibbard-Satterthwaite Theorem [26,35], states that every reasonable election system is manipulable. However, even though every election system is manipulable, it may be computationally infeasible to determine how to manipulate the outcome.

Bartholdi, Tovey, and Trick [2] introduced the computational study of the manipulation problem and this began an exciting line of research that explores the computational complexity of different manipulative attacks on elections (see, e.g., [17]).

The notion of single-peaked preferences introduced by Black [5] is the most important restriction on preferences from political science and economics and is naturally an important case to consider computationally. Single-peakedness models the preferences of a collection of voters with respect to a given axis (a total ordering of the candidates). Each voter in a single-peaked election has a single most-preferred candidate (peak) on the axis and the farther candidates are to the left/right from her peak the less preferred they are. Single-plateauedness extends this to model when each voter has multiple most-preferred candidates that appear sequentially on the axis, but are otherwise single-peaked [6].

This standard model of single-peaked preferences has many desirable social-choice properties. When the voters in an election are single-peaked, the majority relation is transitive [5] and there exist voting rules that are strategy-proof [33]. Single-peakedness for total orders can also lower the complexity of many different election problems when compared to the general case [19,7].

Most of the abovementioned research on the computational complexity of manipulation of elections, both for the general case and for single-peaked electorates, has been limited to the assumption that voters have tie-free votes. In many real-world scenarios voters have votes with ties, and this is seen in the online repository PREFLIB [30] that contains several different preference datasets that contain ties. There are also election systems defined for votes with ties, e.g., the Kemeny rule and the Schulze rule [28,36].

Recent work considers the complexity of manipulation for top-order votes (votes where all of the ties are between candidates ranked last) [34,31]. Fitzsimmons and Hemaspaandra [24] considered the complexity of manipulation, control, and bribery for more general votes with ties, and also the case of manipulation for a nonstandard model of single-peakedness for top-order votes. Menon and Larson [32] later examined the complexity of manipulation and bribery for an equivalent (for top orders) model of single-peakedness.

Fitzsimmons and Hemaspaandra [24] use the model of possibly single-peaked preferences from Lackner [29] where a preference profile of votes with ties is said to be single-peaked with respect to an axis if the votes can be extended to tie-free votes that are single-peaked with respect to the same axis. Menon and Larson [32] use a similar model for top orders that they state is essentially the model of single-peaked preferences with outside options [9]. Both Fitzsimmons and Hemaspaandra [24] and Menon and Larson [32] find that these notions of single-peakedness exhibit anomalous computational behavior where the complexity of manipulation can increase when compared with the case of single-peaked total orders.

We are the first to study the computational complexity of manipulation for the standard model of single-peaked preferences for votes with ties, and for single-plateaued preferences for votes with ties. In contrast to the recent related work using other models of single-peakedness with ties, we find that the complexity of weighted manipulation for m-candidate scoring rules and for m-candidate Copeland$^\alpha$ elections for all $0 \leq \alpha < 1$ does not increase when compared to the cases of single-peaked total orders, and that the complexity of weighted manipulation does not increase with respect to the general case of elimination veto elections. We also compare the social choice properties of these different models, and state a surprising result on the relation between the societal axis and the complexity of manipulation for single-peaked preferences.

Due to space constraints, most of our proofs have been omitted and can be found in the corresponding technical report of this work [25].

2. Preliminaries

An *election* consists of a finite set of candidates C and a finite collection of voters V. We will sometimes refer to this collection of voters as a *preference profile*. An *election system* $\mathscr{E}$ is a mapping from an election to a set of winners, which can be any subset of the candidate set (the nonunique winner model, our standard model), or at most a single candidate (the unique winner model).

Each voter in an election has a corresponding vote (or *preference order*) over the set of candidates. This is often assumed to be a *total order*, i.e., a strict ordering of the candidates from most to least preferred. Formally, a total order is a complete, reflexive, transitive, and antisymmetric binary relation. We use ">" to denote strict preference between two candidates.

Similarly, a *weak order* is a total order without antisymmetry, so each voter can rank candidates as tied (which we will sometimes refer to as indifference) as long as their ties are transitive. We use "∼" to denote a tie between two candidates. A *bottom order* is a weak order where all ties are between top-ranked candidates and a *top order* is a weak order where all ties are between bottom-ranked candidates. Notice that a total order is a weak order, a top order, and a bottom order, that a top order is a weak order, and that a bottom order is a weak order. Throughout this paper we will sometimes refer to weak orders as votes with ties.

For some of our results we consider weighted elections, where each voter has an associated positive integral weight and a voter with weight w counts as w unweighted voters all voting the same.

2.1. Election System Definitions

Our election systems include scoring rules, elimination veto, and Copeland$^\alpha$. We define each below and the extensions we use to properly consider votes with ties.

Given an election with m candidates, a scoring rule assigns scores to the candidates using its corresponding m-candidate scoring vector of the form $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$ and each $\alpha_i \in \mathbb{N}$. So, when the preferences of a voter are a total order, the candidate ranked in position i receives a score of α_i from that voter. Below we present examples of scoring rules and their corresponding m-candidate scoring vector.

Plurality: with scoring vector $\langle 1, 0, \ldots, 0 \rangle$.
Veto: with scoring vector $\langle 1, 1, \ldots, 1, 0 \rangle$.
Borda: with scoring vector $\langle m-1, m-2, \ldots, 1, 0 \rangle$.
Triviality: with scoring vector $\langle 0, 0, \ldots, 0 \rangle$.

To use a scoring rule to determine the outcome of an election containing votes with ties we must extend the above definition of scoring rules. We use the definitions of scoring-rule extensions for weak orders from our previous work [24], which generalize the extensions introduced for top orders from Baumeister et al. [4] and from Narodytska and Walsh [34] which in turn generalizes the extensions used by Emerson [13] for Borda.

Given a weak-order vote, we can write it as $G_1 > G_2 > \cdots > G_r$, where each G_i is a set of tied candidates, (so in the case of a total order vote each G_i is a singleton). For each G_i, let $k_i = \sum_{j=1}^{i-1} \|G_j\|$ be the number of candidates strictly preferred to the candidates in G_i. We now state the definitions of each of the four extensions. In Example 1 we present an example of how a given weak-order vote is scored using Borda and each of the scoring-rule extensions.

Min: each candidate in G_i receives a score of $\alpha_{k_i + \|G_i\|}$.
Max: each candidate in G_i receives a score of $\alpha_{k_i + 1}$.
Round down: each candidate in G_i receives a score of α_{m-r+i}.
Average: each candidate in G_i receives a score of $(\sum_{j=k_i+1}^{k_i+\|G_i\|} \alpha_j)/\|G_i\|$.

For top orders the scoring-rule extensions min, round down, and average are the same as round up, round down, and average used in the work by Menon and Larson [32].

Example 1 Given the candidate set $\{a,b,c,d,e\}$ and the weak order vote $(a \sim b > c \sim d > e)$ we show the scores assigned to each candidate using Borda and each of our extensions. We can write the vote as $\{a,b\} > \{c,d\} > \{e\}$, so $G_1 = \{a,b\}$, $G_2 = \{c,d\}$, and $G_3 = \{e\}$, and $k_1 = 0$, $k_2 = 2$, and $k_3 = 4$.

Recall that for total orders, the scoring vector for 5-candidate Borda is $\langle 4,3,2,1,0 \rangle$.
Borda using min: $\text{score}(a) = \text{score}(b) = 3$, $\text{score}(c) = \text{score}(d) = 1$, and $\text{score}(e) = 0$.
Borda using max: $\text{score}(a) = \text{score}(b) = 4$, $\text{score}(c) = \text{score}(d) = 2$, and $\text{score}(e) = 0$.
Borda using round down: $\text{score}(a) = \text{score}(b) = 2$, $\text{score}(c) = \text{score}(d) = 1$, and $\text{score}(e) = 0$.
Borda using average: $\text{score}(a) = \text{score}(b) = 3.5$, $\text{score}(c) = \text{score}(d) = 1.5$, and $\text{score}(e) = 0$.

For elimination veto for total orders, the veto scoring rule is used, the candidate with the lowest score is eliminated, and the rule is repeated on the remaining votes restricted to the remaining candidates until there is one candidate left [10]. We break ties lexicographically, and for comparison with related work our results for elimination veto use the unique winner model, and for votes with ties we use the min extension.

Pairwise election systems are one of the most natural cases for considering votes with ties. Copeland$^\alpha$ is an important and well-known election system that is defined using pairwise comparisons between candidates. In a Copeland$^\alpha$ election each candidate receives one point for each pairwise majority election with each other candidate she wins and α points for each tie (where $\alpha \in \mathbb{Q}$ and $0 \leq \alpha \leq 1$). For votes with ties we follow the obvious extension also used by Baumeister et al. [4] and Narodytska and Walsh [34].

For Copeland$^\alpha$ elections it will sometimes be easier to refer to the *induced majority graph* of an election. Given an election (C,V) its induced majority graph is constructed as follows. Each vertex in the induced majority graph corresponds to a candidate in C, and for all candidates $a,b \in C$ if $a > b$ by majority then there is an edge from a to b in the induced majority graph. We also will refer to the *weighted majority graph* of an election, where each edge from a to b in the induced majority graph is labeled with the difference between the number of voters that state $a > b$ and the number of voters that state $b > a$.

2.2. Election Problems

The computational study of the manipulation of elections was introduced by Bartholdi, Tovey, and Trick [2], and Conitzer, Sandholm, and Lang [11] extended this to the case for weighted voters and a coalition of manipulators. We define the constructive weighted coalitional manipulation (CWCM) problem below.
Name: $\mathscr{E}$-CWCM
Given: a set of candidates C, a collection of nonmanipulative voters S, a collection of manipulative voters T, and a preferred candidate $p \in C$.
Question: Does there exist a way to set the votes of T such that p is a winner of $(C, S \cup T)$ under election system $\mathscr{E}$?

For the case of CWCM for each of our models of single-peaked preferences we follow the model introduced by Walsh [37] where the societal axis is given as part of the input to the problem and the manipulators must state votes that are single-peaked with respect to this axis (for the corresponding model of single-peakedness). (See Section 3 for all of the definitions of single-peaked preferences that we use.)

3. Models of Single-Peaked Preferences

We consider four important models of single-peaked preferences for votes with ties. For the following definitions, for a given axis L (a total ordering of the candidates) and a given preference order v, we say that v is strictly increasing (decreasing) along a segment of L if each candidate is preferred to the candidate on its left (right) with respect to L. In Figure 1 we present an example of each of the four models of single-peaked preferences, and in Figure 2 we show how the four models relate to each other. We now give the definition of the standard model of single-peakedness from Black [5].

Definition 1 *Given a preference profile V of weak orders over a set of candidates C, V is single-peaked with respect to a total ordering of the candidates L (an axis) if for each voter $v \in V$, L can be split into three segments X, Y, and Z (X and Z can each be empty) such that Y contains only the most preferred candidate of v, v is strictly increasing along X and v is strictly decreasing along Z.*

Observe that for a preference profile of votes with ties to be single-peaked with respect to an axis, each voter can have a tie between at most two candidates at each position in her preference order since the candidates must each appear on separate sides of her peak. Otherwise the preference order would not be *strictly* increasing/decreasing along the given axis.

The model of single-plateaued preferences extends single-peakedness by allowing voters to have multiple most preferred candidates (an indifference plateau) [6]. This is defined by extending Definition 1 so that Y can contain multiple candidates.

Lackner [29] recently introduced another extension to single-peaked preferences, which we refer to as "possibly single-peaked preferences" throughout this paper. A preference profile is possibly single-peaked with respect to a given axis if there exists an extension of each preference order to a total order such that the new preference profile of total orders is single-peaked. This can be stated without referring to extensions, for votes with ties, in the following way.

Definition 2 *Given a preference profile V of weak orders over a set of candidates C, V is possibly single-peaked with respect to a total ordering of the candidates L (an axis) if for each voter $v \in V$, L can be split into three segments X, Y, and Z (X and Z can each be empty) such that Y contains the most preferred candidates of v, v is weakly increasing along X and v is weakly decreasing along Z.*

Notice that the above definition extends single-plateauedness to allow for multiple indifference plateaus on either side of the peak. So for votes with ties, possibly single-peaked preferences model when voters have weakly increasing and then weakly decreasing or only weakly increasing/decreasing preferences along an axis.

Another generalization of single-peaked preferences for votes with ties was introduced by Cantala [9]: the model of single-peaked preferences with outside options. When preferences satisfy this restriction with respect to a given axis, each voter has a segment of the axis where they have single-peaked preferences and candidates appearing outside of this segment on the axis are strictly less preferred and the voter is tied between them. Similar to how single-plateaued preferences extend the standard single-peaked model to allow voters to state multiple most preferred candidates, single-peaked preferences

with outside options extends the standard model to allow voters to state multiple least preferred candidates.

Definition 3 *Given a preference profile V of weak orders over a set of candidates C, V is single-peaked with outside options with respect to a total ordering of the candidates L (an axis) if for each voter $v \in V$, L can be split into five segments O_1, X, Y, Z, and O_2 (O_1, X, Z, and O_2 can each be empty) such that Y contains only the most preferred candidate of v, v is strictly increasing along X and v is strictly decreasing along Z, for all candidates $a \in X \cup Y \cup Z$ and $b \in O_1 \cup O_2$, v states $a > b$, and for all candidates $x, y \in O_1 \cup O_2$, v states $x \sim y$.*

Menon and Larson [32] state that the model of single-peakedness for top orders that they use is similar to single-peaked preferences with outside options. It is clear from their paper that for top orders these models are the same. So for the remainder of the paper we will refer to the model used by Menon and Larson [32] as "single-peaked preferences with outside options for top orders."

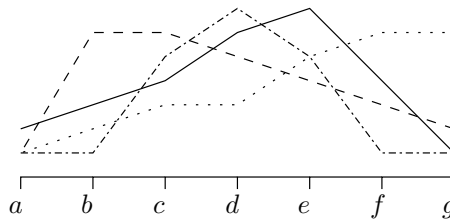**Figure 1.** Given the axis $L = a < b < c < d < e < f < g$, the solid line is the single-peaked order ($e > d > c \sim f > b > a > g$), the dashed line is the single-plateaued order ($b \sim c > d > e > f > g > a$), the dashed-dotted line is the single-peaked with outside options order ($d > c \sim e > a \sim b > f \sim g$), and the dotted line is the possibly single-peaked order ($f \sim g > e > d \sim c > b > a$). See Figure 2 for the relationships between the four models.

3.1. Social Choice Properties

We now state some general observations on single-peaked preferences with ties, including how the models relate to each other, as well as their social-choice properties.

It is easy to see that for total-order preferences each of the four models of single-peakedness with ties that we consider are equivalent. In Figure 2 we show how each model relates for weak orders, top orders, and bottom orders.

Given a preference profile of votes it is natural to ask how to determine if an axis exists such that the profile satisfies one of the above restrictions. This is referred to as the consistency problem for a restriction. Bartholdi and Trick [3] showed that single-peaked consistency for total orders can be determined in polynomial time (i.e., is in P), and Fitzsimmons [23] extended this result to show that single-peaked, single-plateaued, and possibly single-peaked consistency for weak orders is in P. This leaves the consistency problem for single-peaked preferences with outside options, which we will now show to be in P.

It is easy to see that given a preference profile of top orders, it is single-peaked with outside options if and only if it is possibly single-peaked. So it is immediate from
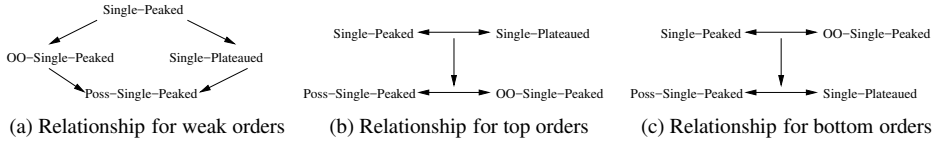
(a) Relationship for weak orders (b) Relationship for top orders (c) Relationship for bottom orders

Figure 2. Relationships between the four models of single-peaked preferences for different types of votes with ties, where $A \rightarrow B$ indicates that a preference profile satisfying model A also satisfies model B. OO-Single-Peaked and Poss-Single-Peaked refer to Single-Peaked with Outside Options and Possibly Single-Peaked respectively.

the result by Lackner [29] that shows that possibly single-peaked consistency for top orders is in P, that the consistency problem for single-peaked preferences with outside options for top orders is in P. For weak orders the construction used to show that single-peaked consistency for weak orders is in P by Fitzsimmons [23] can be adapted to hold for the case of single-peaked preferences with outside options for weak orders, so this consistency problem is also in P.

Theorem 1 *Given a preference profile V of weak orders it can be determined in polynomial time if there exists an axis L such that V is single-peaked with outside options with respect to L.*

One of the most well-known and desirable properties of an election with single-peaked preferences is that there exists a transitive majority relation [5]. A majority relation is transitive if for all distinct candidates $a, b, c \in C$ if $a > b$ and $b > c$ by majority then $a > c$ by majority. When the majority relation is transitive then candidates that beat-or-tie every other candidate by majority exist and they are denoted the weak Condorcet winners. This also holds when the voters have single-plateaued preferences [6].

It is not the case that the majority relation is transitive when preferences in an election are single-peaked with outside options or are possibly single-peaked. Fitzsimmons [23] points out that for possibly single-peakedness this was implicitly shown in an entry of Table 9.1 in Fishburn [22], where a preference profile of top orders that violates single-peaked preferences was described that does not have a transitive majority relation, and this profile is possibly single-peaked (and single-peaked with outside options). Cantala also provides an example of a profile that is single-peaked with outside options that does not have a weak Condorcet winner, and so does not have a transitive majority relation [9].

4. Computational Results

For total-order preferences, weighted manipulation for any fixed number of candidates is known to be NP-complete for every scoring rule that is not isomorphic to plurality or triviality [27]. For single-peaked total orders, Faliszewski et al. [19] completely characterized the complexity of weighted manipulation for 3-candidate scoring rules and this result was generalized by Brandt et al. [7] for any fixed number of candidates. These results both showed that the complexity of weighted manipulation for scoring rules often decreases when voters have single-peaked total orders.

For top-order preferences, dichotomy theorems for 3-candidate weighted manipulation for scoring rules using round down, min, and average were shown by Menon and

Larson for the general case [31] and for the case of single-peaked preferences with outside options for top orders [32]. Menon and Larson [32] found that, counterintuitively, the complexity of weighted manipulation for single-peaked preferences with outside options often increases when moving from total orders to top orders. We also mention that for the scoring-rule extension max, we earlier found that the complexity of 3-candidate weighted manipulation for Borda increases for possibly single-peaked preferences when moving from total orders to top orders [24].

We show that for the standard model of single-peakedness and for single-plateauedness that the complexity of weighted manipulation for m-candidate scoring rules using max, min, round down, and average does not increase when moving from total orders to top orders, bottom orders, or weak orders.

The following results are close analogs to the case for single-peaked total orders due to Brandt et al. [7].

Lemma 2 *If p can be made a winner by a manipulation of top-order, bottom-order, or weak-order votes that are single-peaked, single-plateaued, possibly single-peaked, or single-peaked with outside options for a scoring rule using max, min, round down, or average, then p can be made a winner by a manipulation (of the same type) in which all manipulators rank p uniquely first.*

Theorem 3 *Let $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ be a scoring vector. If α-CWCM is in P for single-peaked total orders, then α-CWCM is in P for single-peaked and single-plateaued top orders, bottom orders, and weak orders for all our scoring rule extensions.*

The most surprising result in the work by Menon and Larson [32] was that the complexity of 3-candidate CWCM for elimination veto for single-peaked preferences with outside options for top orders using min is NP-complete, whereas for single-peaked total orders [32] and even for total orders in the general case [10] it is in P.

Theorem 4 *[10] m-candidate elimination veto CWCM for total orders is in* P *in the unique-winner model.*

Theorem 5 *[32] m-candidate elimination veto CWCM for single-peaked total orders is in* P *in the unique-winner model.*

Theorem 6 *[32] 3-candidate elimination veto CWCM for top orders that are single-peaked with outside options is* NP*-complete in the unique-winner model.*

Menon and Larson [32] state this case as a counterexample to the conjecture by Faliszewski et al. [19], which states that the complexity for a natural election system will not increase when moving from the general case to the single-peaked case. (Though Menon and Larson do qualify that the conjecture from Faliszewski et al. concerned total orders.) However, for the standard model of single-peaked preferences and for single-plateaued preferences, elimination veto CWCM for top orders, bottom orders, and weak orders using min is in P for any fixed number of candidates, thus the counterexample crucially relies on using a nonstandard definition.

Theorem 7 *m-candidate elimination veto CWCM for single-peaked and for single-plateaued top orders, bottom orders, and weak orders using min is in* P *in the unique-winner model.*

The proof of the above theorem follows from a similar argument as the proof for the case of single-peaked total orders from Menon and Larson [32].

It is known that 3-candidate Copeland$^\alpha$ CWCM for all rational $\alpha \in [0, 1)$ is NP-complete for total orders in the nonunique-winner model [20], and when $\alpha = 1$ (also known as Llull) CWCM is in P for $m \leq 4$ [20,21] and the cases for $m \geq 5$ remain open. Fitzsimmons and Hemaspaandra [24] showed that the NP-completeness of the 3-candidate case holds for top orders, bottom orders, and weak orders, and Menon and Larson [31] independently showed the top-order case. Fitzsimmons and Hemaspaandra [24] also showed that 3-candidate Llull CWCM is in P for top orders, bottom orders, and weak orders.

Recall that weak Condorcet winners always exist when preferences are single-peaked with ties, and when they are single-plateaued. So the results that Llull CWCM for single-peaked total orders is in P from Brandt et al. [7] also holds for the case of single-peaked and single-plateaued top orders, bottom orders, and weak orders.

Copeland$^\alpha$ for $\alpha \in [0, 1)$ was shown to be in P by Yang [38] for single-peaked total orders.

Theorem 8 *[38] Copeland$^\alpha$ CWCM for $\alpha \in [0, 1)$ is in P for single-peaked total orders.*

In contrast, for top orders that are single-peaked with outside options, it is NP-complete even for three candidates [32].

Theorem 9 *[32] 3-candidate Copeland$^\alpha$ CWCM for $\alpha \in [0, 1)$ is NP-complete for top orders that are single-peaked with outside options.*

For single-peaked and single-plateaued weak orders, bottom orders, and top orders we again inherit the behavior of single-peaked total orders.

Theorem 10 *Copeland$^\alpha$ CWCM for $\alpha \in [0, 1)$ is in P for single-peaked and single-plateaued top orders, bottom orders, and weak orders.*

Proof. Let L be our axis. Consider a set of nonmanipulators with single-plateaued weak orders. Replace each nonmanipulator v of weight w with two nonmanipulators v_1 and v_2 of weight w. The first nonmanipulator breaks the ties in the vote in increasing order of L and the second nonmanipulator breaks the ties in the vote in decreasing order of L, i.e., if $a \sim_v b$ and aLb, then $a >_{v_1} b$ and $b >_{v_2} a$. Note that v_1 and v_2 are single-peaked total orders and that the weighted majority graph induced by the nonmanipulators after replacement can be obtained from the weighted majority graph induced by the original nonmanipulators by multiplying each weight in the graph by 2. When we also multiply the manipulator weights by 2, we have an equivalent Copeland$^\alpha$ CWCM problem, where all nonmanipulators are single-peaked total orders and all manipulators have even weight.

Suppose p can be made a winner by having the manipulators cast single-plateaued votes with ties. Now replace each manipulator of weight $2w$ by two weight-w manipulators. The first manipulator breaks the ties in the vote in increasing order of L and the second manipulator breaks the ties in the vote in decreasing order of L. Now the replaced manipulator votes are single-peaked total orders and p is still a winner. We need the following fact from the proof of Theorem 8: *if p can be made a winner in the single-peaked total order case, then p can be made a winner by having all manipulators cast the same*

P-time computable vote. It follows that p can be made a winner by having all replaced manipulators cast the same single-peaked total order vote. But then p can be made a winner by having all original manipulators cast the same single-peaked total order vote. Since this vote is P-time computable, it follows that Copeland$^\alpha$ CWCM for $\alpha \in [0, 1)$ is in P for single-plateaued weak orders, and this also holds for single-peaked weak orders since every single-peaked profile of weak orders is also single-plateaued.

It is clear to see that similar arguments hold for single-peaked top orders, and for single-plateaued top orders and bottom orders. The case for single-peaked bottom orders follows from the case for single-peaked total orders. ❑

4.1. Societal-Axis Results

Recall from Lemma 2 that for all our single-peaked models and all our scoring rule extensions, we can assume that all manipulators rank p uniquely first. When given an axis where the preferred candidate is in the leftmost or rightmost location, there is exactly one single-peaked total order vote that puts p first, namely, p followed by the remaining candidates on the axis in order. This is also the case for single-peaked and single-plateaued orders with ties, since p is ranked uniquely first and no two candidates can be tied on the same side of the peak. It follows that the weighted manipulation problems for scoring rules for single-peaked total orders and single-peaked and single-plateaued orders with ties are in P for axes where p is in the leftmost/rightmost position. However, this is not the case when preferences with ties are single-peaked with outside options or possibly single-peaked. In Case 1 of the proof of Theorem 1 in the work by Menon and Larson [32], an axis of *pLaLb* is used to show NP-hardness for their model.

It is attractive to conjecture that for single-peaked and single-plateaued preferences, the less symmetrical (with respect to p) the axis is, the easier the complexity of manipulation, but surprisingly this turns out to not be the case, even for total orders.

For the theorem stated below let m_1 and m_2 denote the number of candidates to the left and to the right on the axis with respect to the preferred candidate of the manipulators.

Theorem 11 *For single-peaked total orders, single-peaked weak orders, and single-plateaued weak orders, $\langle 4, 3, 2, 0, 0 \rangle$ CWCM is in P for $m_1 = m_2 = 2$ and NP-complete for $m_1 = 1$ and $m_2 = 3$ for all our scoring rule extensions.*

5. Related Work

The work by Menon and Larson [32] on the complexity of manipulation and bribery for single-peaked preferences with outside options for top orders is the most closely related to this paper. For manipulation, they show that for single-peaked preferences with outside options the complexity often increases when moving from total orders to top orders. They additionally considered a notion of nearly single-peakedness. We instead study the complexity of weighted manipulation for the standard model of single-peaked preferences with ties and for single-plateaued preferences with ties.

The focus of our paper is on the computational aspects of models of single-peaked preferences with ties. These models can also be compared based on which social-choice

properties that they have, such as the guarantee of a weak Condorcet winner. Barberá [1] compares such properties of the models of single-peaked, single-plateaued, and single-peaked with outside options for votes with ties.

Since single-peakedness is a strong restriction on preferences, in real-world scenarios it is likely that voters may only have nearly single-peaked preferences, where different distance measures to a single-peaked profile are considered. Both the computational complexity of different manipulative attacks [18,15] and detecting when a given profile is nearly single-peaked [14,8] have been considered.

An important computational problem for single-peakedness is determining the axis given a preference profile, this is known as its consistency problem. Single-peaked consistency for total orders was first shown to be in P by Bartholdi and Trick [3]. Doignon and Falmagne [12] and Escoffier, Lang, and Öztürk [16] independently found faster direct algorithms. Lackner [29] proved that possibly single-peaked consistency for top orders is in P (and for local weak orders and partial orders is NP-complete), and Fitzsimmons [23] later showed that single-peaked, single-plateaued, and possibly single-peaked consistency for weak orders is in P.

6. Conclusions

The standard model of single-peakedness is naturally defined for votes with ties, but different extensions have been considered. In contrast to recent work that studies the models of possibly single-peaked and single-peaked preferences with outside options and finds an anomalous increase in complexity compared to the tie-free case, we find that for scoring rules and other important natural systems, the complexity of weighted manipulation does not increase when moving from total orders to votes with ties in the standard single-peaked and in the single-plateaued cases. Single-peaked and single-plateaued preferences for votes with ties also retain the important social-choice property of the existence of weak Condorcet winners. This is not to say that possibly single-peaked and single-peaked preferences with outside options are without merit, since they both model easily understood structure in preferences.

References

[1] S. Barberà. Indifferences and domain restrictions. *Analyse & Kritik*, 29(2):146–162, 2007.
[2] J. Bartholdi, III, C. Tovey, and M. Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6(3):227–241, 1989.
[3] J. Bartholdi, III and M. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986.
[4] D. Baumeister, P. Faliszewski, J. Lang, and J. Rothe. Campaigns for lazy voters: Truncated ballots. In *AAMAS-12*, pages 577–584, June 2012.
[5] D. Black. On the rationale of group decision-making. *Journal of Political Economy*, 56(1):23–34, 1948.
[6] D. Black. *The Theory of Committees and Elections*. Cambridge University Press, 1958.
[7] F. Brandt, M. Brill, E. Hemaspaandra, and L. Hemaspaandra. Bypassing combinatorial protections: Polynomial-time algorithms for single-peaked electorates. *JAIR*, 53:439–496, 2015.

[8] R. Bredereck, J. Chen, and G. Woeginger. Are there any nicely structured preference profiles nearby? *Mathematical Social Sciences*, 79:61–73, 2016.

[9] D. Cantala. Choosing the level of a public good when agents have an outside option. *Social Choice and Welfare*, 22(3):491–514, 2004.

[10] T. Coleman and V. Teague. On the complexity of manipulating elections. In *CATS-07*, pages 25–33, January/February 2007.

[11] V. Conitzer, T. Sandholm, and J. Lang. When are elections with few candidates hard to manipulate? *JACM*, 54(3):Article 14, 2007.

[12] J.-P. Doignon and J.-C. Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994.

[13] P. Emerson. The original Borda count and partial voting. *Social Choice and Welfare*, 40(2):352–358, 2013.

[14] G. Erdélyi, M. Lackner, and A. Pfandler. Computational aspects of nearly single-peaked electorates. In *AAAI-13*, pages 283–289, July 2013.

[15] G. Erdélyi, M. Lackner, and A. Pfandler. Manipulation of k-approval in nearly single-peaked electorates. In *ADT-15*, pages 71–85, Sept. 2015.

[16] B. Escoffier, J. Lang, and M. Öztürk. Single-peaked consistency and its complexity. In *ECAI-08*, pages 366–370, July 2008.

[17] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. Using complexity to protect elections. *Commun. ACM*, 53(11):74–82, 2010.

[18] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. The complexity of manipulative attacks in nearly single-peaked electorates. *JAIR*, 207:69–99, 2014.

[19] P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. The shield that never was: Societies with single-peaked preferences are more open to manipulation and control. *Information and Computation*, 209(2):89–107, 2011.

[20] P. Faliszewski, E. Hemaspaandra, and H. Schnoor. Copeland voting: Ties matter. In *AAMAS-08*, pages 983–990, May 2008.

[21] P. Faliszewski, E. Hemaspaandra, and H. Schnoor. Weighted manipulation for four-candidate Llull is easy. In *ECAI-12*, pages 318–323, Aug. 2012.

[22] P. Fishburn. *The Theory of Social Choice*. Princeton University Press, 1973.

[23] Z. Fitzsimmons. Single-peaked consistency for weak orders is easy. In *TARK-15*, June 2015. To appear.

[24] Z. Fitzsimmons and E. Hemaspaandra. Complexity of manipulative actions when voting with ties. In *ADT-15*, pages 103–119, Sept. 2015.

[25] Z. Fitzsimmons and E. Hemaspaandra. Modeling single-peakedness for votes with ties. Tech. Rep. arXiv:1604.08191 [cs.GT], arXiv.org, Apr. 2016.

[26] A. Gibbard. Manipulation of voting schemes. *Econometrica*, 41(4):587–601, 1973.

[27] E. Hemaspaandra and L. Hemaspaandra. Dichotomy for voting systems. *JCSS*, 73(1):73–83, 2007.

[28] J. Kemeny. Mathematics without numbers. *Daedalus*, 88:577–591, 1959.

[29] M. Lackner. Incomplete preferences in single-peaked electorates. In *AAAI-14*, pages 742–748, July 2014.

[30] N. Mattei and T. Walsh. PREFLIB: A library for preferences. In *ADT-13*, pages 259–270, 2013.

[31] V. Menon and K. Larson. Complexity of manipulation in elections with top-truncated ballots. Tech. Rep. arXiv:1505.05900 [cs.GT], arXiv.org, July 2015.

[32] V. Menon and K. Larson. Reinstating combinatorial protections for manipulation and bribery in single-peaked and nearly single-peaked electorates. In *AAAI-16*, pages 565–571, Feb. 2016.

[33] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

[34] N. Narodytska and T. Walsh. The computational impact of partial votes on strategic voting. In *ECAI-14*, pages 657–662, Aug. 2014.

[35] M. Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

[36] M. Schulze. A new monotonic and clone-independent, reversal symmetric, and Condorcet-consistent single-winner election method. *Social Choice and Welfare*, 36(2):267–303, 2011.

[37] T. Walsh. Uncertainty in preference elicitation and aggregation. In *AAAI-07*, pages 3–8, July 2007.

[38] Y. Yang. Manipulation with bounded single-peaked width: A parameterized study. In *AAMAS-15*, pages 77–85, 2015.